THE NOTION OF INTUITIONISTIC FUZZY IC-BAGS

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Abstract. This paper presents an overview of the existing notions of Bags, Fuzzy Bags, IC-Bags, IF-Bags and Fuzzy IC-Bags. Based on these notions, the concept of Intuitionistic Fuzzy Bags with Interval Counts (Intuitionistic fuzzy IC-Bags) is proposed. Further, some characteristics of Intuitionistic fuzzy IC-Bags are studied, and some examples are furnished in this context.

Keywords: Bags, IF-sets, Fuzzy Bags, IF-Bags, IC-Bags, Fuzzy IC-Bags, Intuitionistic fuzzy IC-Bags.

1 Introduction

In [4], Chakrabarty introduced and characterized the notion of Bags with Interval Counts (IC-Bags). In [12], the methodologies of using IC-Bags in decision analysis were discussed. As it is observed that under certain real-life decision scenarios, the object counts in a collection may not be fixed, hence the concept of the interval counts are useful for modelling such scenarios. Interval counts can be incorporated with different fuzzy membership grades for each object in the collection.

In [10], the notion of Intuitionistic Fuzzy Bags (IF-Bags) was introduced which generalizes the notion of fuzzy bags [16]. This generalization was based on Atanassov's notion of intuitionistic fuzzy sets [1, 2]. Chakrabarty proposed the notion of Fuzzy Bags with Interval Counts (Fuzzy IC-Bags) in [8] which generalizes the notion of IC-Bags [4].

In the present paper, we introduce the notion of Intuitionistic Fuzzy Bags with Interval Counts (Intuitionistic Fuzzy IC-Bags) which is a generalization of the notion of Fuzzy IC-bags done on the basis of the notion of intuitionistic fuzzy sets. Some properties of Intuitionistic fuzzy IC-Bags are studied. The concept of intuitionistic fuzzy base sets of Intuitionistic fuzzy IC-Bags are proposed. The notion of Base-equivalent

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intuitionistic fuzzy IC-Bags are introduced and intuitionistic fuzzy l-equalities and u-equalities are defined.

2 Bags, Fuzzy Bags, and IC-Bags: An Overview

This section presents an overview of the notions of Bags, Fuzzy Bags, and IC-Bags as presented in [16, 14, 15, 13, 5, 4, 6, 9, 12, 17].

A bag B drawn from a set X is represented by a function C_B where

$$C_B : X \longrightarrow N,$$

N being the set of non-negative integers.

The notion of bag is useful if we consider the result of operations in a relational database, in which case the same tuple can appear more than once in the result. Hence the structure and operations on bags are useful for working with this kind of real world information systems.

For any two bags B_1 and B_2 drawn from a set X

- $B_1 = B_2$ if for all x in X , $C_{B_1}(x) = C_{B_2}(x)$.
- $B_1 \sqsubseteq B_2$, i.e B_1 is a sub bag of B_2 , if for all x in X , $C_{B_1}(x) \leq C_{B_2}(x)$.
- $B = B_1 \oplus B_2$ if for all x in X , $C_B(x) = C_{B_1}(x) + C_{B_2}(x)$.
- $B = B_1 \ominus B_2$ if for all x in X , $C_B(x) = \max[C_{B_1}(x) - C_{B_2}(x), 0]$.

Here ' $\oplus$ ' and ' $\ominus$ ' represents the 'bag addition' and 'bag removal' operations.

For a bag B drawn from a set X , the support set of B denoted as B^* is a subset of X where for all x in X , the characteristic function is given by $\phi_{B^*}(x) = \min[C_B(x), 1]$.

The union of two bags B_1 and B_2 drawn from a set X is a bag $B_1 \sqcup B_2$ such that for all x in X , $C_{B_1 \sqcup B_2}(x) = \max[C_{B_1}(x), C_{B_2}(x)]$.

The intersection of B_1 and B_2 is a bag denoted by $B_1 \sqcap B_2$ such that for all x in X , $C_{B_1 \sqcap B_2}(x) = \min[C_{B_1}(x), C_{B_2}(x)]$.

A fuzzy bag F drawn from a set X is represented by a function CM_F where $CM_F : X \longrightarrow \mathcal{B}$, $\mathcal{B}$ being the set of all bags drawn from $I = [0, 1]$. $CM_F(x)$ can be characterized by $C_{CM_F^x} : I \longrightarrow N$, N being the set of non-negative integers.

The fuzzy supporting set of any fuzzy bag F drawn from a set X can be denoted by F^* whose membership function is given by $\psi_{F^*}(x) = \max_{\alpha} [\min(C_{CM_F^x}(\alpha), \alpha)]$.

The union of two fuzzy bags F_1 and F_2 drawn from a set X is a fuzzy bag F denoted by $F = F_1 \sqcup F_2$, such that for each $x \in X$ and $\alpha \in I$,

$$C_{CM_F^x}(\alpha) = \max[C_{CM_{F_1}^x}(\alpha), C_{CM_{F_2}^x}(\alpha)].$$

The intersection of F_1 and F_2 is a fuzzy bag F denoted by $F = F_1 \sqcap F_2$ such that for each $x \in X$ and $\alpha \in I$,

$$C_{CM_F^x}(\alpha) = \min[C_{CM_{F_1}^x}(\alpha), C_{CM_{F_2}^x}(\alpha)].$$

For any non empty set X , an IC bag β drawn from X is characterized by a pair of functions C_l^β and C_u^β such that

$$C_l^\beta : X \longrightarrow N$$

$$C_u^\beta : X \longrightarrow N$$

where for all x in X , $C_l^\beta(x) \leq C_u^\beta(x)$, N being the set of non-negative integers.

IC-Bags are useful for representing the situations in which we have incomplete information regarding the number of times a certain tuple can appear.

For an IC-bag β drawn from X , a subset $\sigma(\beta)$ of X is said to be the support set of β if for all x in X ,

$$C_l^\beta(x) > 0 \implies x \in \sigma(\beta),$$

$$C_l^\beta(x) = 0 \implies x \notin \sigma(\beta).$$

Two IC bags β_1 and β_2 drawn from X are said to be equal if and only if the following conditions hold for all x in Ω ,

$$C_l^{\beta_1}(x) = C_l^{\beta_2}(x),$$

$$C_u^{\beta_1}(x) = C_u^{\beta_2}(x).$$

If for all x in X , $C_l^{\beta_1}(x) = C_l^{\beta_2}(x)$, but $C_u^{\beta_1}(x) \neq C_u^{\beta_2}(x)$, then β_1 is said to be l -equal to β_2 . If for all x in X , $C_u^{\beta_1}(x) = C_u^{\beta_2}(x)$, but $C_l^{\beta_1}(x) \neq C_l^{\beta_2}(x)$, then β_1 is said to be u -equal to β_2 .

β_1 is said to be an IC sub bag of β_2 if for all x in X ,

$$C_l^{\beta_1}(x) \leq C_l^{\beta_2}(x),$$

$$C_u^{\beta_1}(x) \leq C_u^{\beta_2}(x).$$

β_1 is said to be an l -IC sub bag of β_2 if for all x in X ,

$$C_l^{\beta_1}(x) \leq C_l^{\beta_2}(x), C_u^{\beta_1}(x) > C_u^{\beta_2}(x).$$

β_1 is said to be an u -IC sub bag of β_2 if for all x in X ,

$$C_l^{\beta_1}(x) > C_l^{\beta_2}(x), C_u^{\beta_1}(x) \leq C_u^{\beta_2}(x).$$

3 IF-Bags: An Overview

This section presents an overview of the notion of Intuitionistic Fuzzy Bags(IF-bags) as presented in [10, 7].

For any non-empty set X , an IF-bag A drawn from X is characterized by

$$CM_A : X \longrightarrow Q,$$

where Q is the set of all crisp bags drawn from the cartesian product $I \times I$, I being the continuum $[0,1]$.

Hence, for any $x \in X$, $CM_A(x)$ is a crisp bag drawn from $I \times I$ and

$$C_{CM_A(x)} : I \times I \longrightarrow N$$

which is the characterizing count function for $CM_A(x)$. Thus, for any $(\alpha, \beta) \in I \times I$, $C_{CM_A(x)}(\alpha, \beta)$ indicates the number of times x appears with the degree of membership α and the degree of non- membership β in the IF-bag A . This clearly means it is the number of $x/(\alpha, \beta)$ objects in the bag A .

An IF-bag A can be reduced to a fuzzy bag A if for each $x \in X$, $CM_A(x)$ happens to be a crisp bag in which, for each $(\alpha, \beta) \in CM_A(x)$, $\alpha + \beta = 1$.

For any non-empty set X , there exists an IF-bag drawn from X , denoted by ϕ such that for each $x \in X$, $CM_\phi(x)$ is an empty bag. That is, for each $x \in X$ and $(\alpha, \beta) \in I \times I$, $C_{CM_\phi(x)}(\alpha, \beta) = 0$. This IF-bag is called the null IF-bag.

Two IF-bags A and B drawn from a set X are called equal, denoted by $A = B$ if for all $x \in X$, $CM_A(x) = CM_B(x)$. That is, the bag characterizing x in A is equal to the bag characterizing x in B . The equality of two fuzzy bags A and B drawn from X requires that for all $x \in X$ and for all $(\alpha, \beta) \in I \times I$, $C_{CM_A(x)}(\alpha, \beta) = C_{CM_B(x)}(\alpha, \beta)$.

If A and B be two IF-bags drawn from the set X , then we say that A is a sub-bag of B , denoted by $A \sqsubseteq B$ if for all $x \in X$, $(\alpha, \beta) \in I \times I$,

$$C_{CM_A(x)}(\alpha, \beta) \leq C_{CM_B(x)}(\alpha, \beta).$$

If A and B be two IF-bags drawn from the set X , then:

- The addition of A and B results in a new IF-bag, denoted by $A \oplus B$ such that for all $x \in X$ and for all $(\alpha, \beta) \in I \times I$,

$$C_{CM_{A \oplus B}(x)}(\alpha, \beta) = C_{CM_A(x)}(\alpha, \beta) + C_{CM_B(x)}(\alpha, \beta).$$

- The removal of B from A results in a new IF-bag, denoted by $A \ominus B$ such that for all $x \in X$ and for all $(\alpha, \beta) \in I \times I$,

$$C_{CM_{A \ominus B}(x)}(\alpha, \beta) = \max\{C_{CM_A(x)}(\alpha, \beta) - C_{CM_B(x)}(\alpha, \beta), 0\}.$$

- The union of A and B results in a new IF-bag, denoted by $A \sqcup B$ such that for all $x \in X$ and for all $(\alpha, \beta) \in I \times I$,

$$C_{CM_{A \sqcup B}(x)}(\alpha, \beta) = \max\{C_{CM_A(x)}(\alpha, \beta), C_{CM_B(x)}(\alpha, \beta)\}.$$

- The intersection of A and B results in a new IF-bag, denoted by $A \sqcap B$ such that for all $x \in X$ and for all $(\alpha, \beta) \in I \times I$,

$$C_{CM_{A \sqcap B}(x)}(\alpha, \beta) = \min\{C_{CM_A(x)}(\alpha, \beta), C_{CM_B(x)}(\alpha, \beta)\}.$$

If $X = \{a, b, c, d, e\}$ be any set and if

$$A = \{a/\{(0.5, 0.3)/2, (0.6, 0.1)/3\}, b/\{(0.9, 0)/1, (0.2, 0.3)/5\}, c/\{(0.8, 0.2)/8, (0.4, 0.4)/6, (0.3, 0.1)/5\}, d/\{(0.3, 0.3)/1\}, e/\{(0.9, 0)/1, (0.2, 0.1)/6\}\},$$

$$B = \{a/\{(0.5, 0.5)/3\}, b/\{(0.6, 0.4)/2\}, c/\{(0.4, 0.4)/8\}, d/\{(0.3, 0.1)/2, (0.4, 0.3)/5\}, e/\{(0.9, 0)/8, (0.2, 0.2)/1\}\}$$

be two IF-bags drawn from X , then

$$A \oplus B = \{a/\{(0.5, 0.3)/2, (0.6, 0.1)/3, (0.5, 0.5)/3\}, b/\{(0.6, 0.4)/2, (0.9, 0)/1, (0.2, 0.3)/5\}, c/\{(0.8, 0.2)/8, (0.4, 0.4)/14, (0.3, 0.1)/5\}, d/\{(0.3, 0.3)/1, (0.3, 0.1)/2, (0.4, 0.3)/5\}, e/\{(0.9, 0)/9, (0.2, 0.1)/6, (0.2, 0.2)/1\}\}$$

$$A \ominus B = \{a/\{(0.5, 0.3)/2, (0.6, 0.1)/3\}, b/\{(0.9, 0)/1, (0.2, 0.3)/5\}, c/\{(0.8, 0.2)/8, (0.3, 0.1)/5\}, d/\{(0.3, 0.3)/1\}, e/\{(0.2, 0.1)/6\}\}$$

$$A \sqcup B = \{a/\{(0.5, 0.3)/2, (0.6, 0.1)/3, (0.5, 0.5)/3\}, b/\{(0.6, 0.4)/2, (0.9, 0)/1, (0.2, 0.3)/5\}, c/\{(0.8, 0.2)/8, (0.4, 0.4)/8, (0.3, 0.1)/5\}, d/\{(0.3, 0.3)/1, (0.3, 0.1)/2, (0.4, 0.3)/5\}, e/\{(0.9, 0)/8, (0.2, 0.1)/6, (0.2, 0.2)/1\}\}$$

$$A \sqcap B = \{a/\{(0.5, 0.3)/2, (0.6, 0.1)/3, (0.5, 0.5)/3\}, b/\{(0.6, 0.4)/2, (0.9, 0)/1, (0.2, 0.3)/5\}, c/\{(0.8, 0.2)/8, (0.4, 0.4)/8, (0.3, 0.1)/5\}, d/\{(0.3, 0.3)/1, (0.3, 0.1)/2, (0.4, 0.3)/5\}, e/\{(0.9, 0)/1, (0.2, 0.1)/6, (0.2, 0.2)/1\}\}$$

In an information system, let U be the universal bag and X be any set. Then the universal intuitionistic fuzzy bag (Universal IF-bag) denoted by $IF(U)$ for this information system is an IF-bag in which for all $x \in X$, $(\alpha, \beta) \in I \times I$, the following holds for each IF-Bag I drawn from X :

- $\sum_{(\alpha, \beta)} C_{CM_{IF(U)}(x)}(\alpha, \beta) = C_U(x)$

$$\bullet \ C_{CM_I(x)}(\alpha, \beta) \leq C_{CM_{IF(U)}(x)}(\alpha, \beta)$$

For an IF-bag A drawn from a set X , the complement of A is another IF-bag denoted by A^c such that for all $x \in X$, $(\alpha, \beta) \in I \times I$,

$$C_{CM_{A^c}(x)}(\alpha, \beta) = C_{CM_{F(U)}(x)}(\alpha, \beta) - C_{CM_A(x)}(\alpha, \beta).$$

For an IF-bag A drawn from a set X , the intuitionistic fuzzy supporting set of A is an IF-set of X denoted by A^* whose membership function μ and non-membership function ν are given as below:

$$\begin{aligned} \mu_{A^*}(x) &= \max_{\alpha} [\min \{ \max_{\beta} (C_{CM_A(x)}(\alpha, \beta)), \alpha \}] \\ \nu_{A^*}(x) &= \min_{\beta} [\min \{ \max_{\alpha} (C_{CM_A(x)}(\alpha, \beta)), \beta \}] \end{aligned}$$

4 Fuzzy IC Bags: An Overview

In this section, we briefly present the notion of Fuzzy IC Bags as presented in [8].

A Fuzzy IC-Bag ϕ drawn from a non-empty set $\mathcal{R}$ is characterized by the function ξ^ϕ such that $\xi^\phi : \mathcal{R} \rightarrow \mathcal{U}_{\mathcal{I}}$, where $\mathcal{U}_{\mathcal{I}}$ represents the set of all IC-Bags drawn from $\mathcal{I} = [0, 1]$.

Thus for any $\omega \in \mathcal{R}$, $\xi^\phi(\omega)$ is an IC-Bag drawn from $\mathcal{I}$. But since any IC-Bag can itself be characterized by a pair of functions over its set, hence $\xi^\phi(\omega)$ can be characterized by the pair of functions

$$\begin{aligned} C_l^{\xi^\phi(\omega)} : \mathcal{I} &\rightarrow \mathcal{N} \\ C_u^{\xi^\phi(\omega)} : \mathcal{I} &\rightarrow \mathcal{N} \end{aligned}$$

where for all $\alpha \in \mathcal{I}$, $C_l^{\xi^\phi(\omega)}(\alpha) \leq C_u^{\xi^\phi(\omega)}(\alpha)$, $\mathcal{N}$ being the set of non-negative integers.

For any fuzzy IC-Bag ϕ drawn from $\mathcal{R}$, a fuzzy subset $\beta(\phi)$ of $\mathcal{R}$ is called a fuzzy base set of ϕ , if and only if for all $x \in \mathcal{R}$,

$$\mu_{\beta(\phi)}(x) = \{ \omega \in \mathcal{I} : C_l^{\xi^\phi(x)}(\omega) \geq C_l^{\xi^\phi(x)}(\alpha) \forall \alpha \in \mathcal{I} \}$$

where $\mu_{\beta(\phi)}$ represents the fuzzy membership function of $\beta(\phi)$.

Two fuzzy IC-Bags ϕ_1 and ϕ_2 drawn from $\mathcal{R}$ are called *fuzzy IC-equal* if and only if for all $x \in \mathcal{R}$, $\alpha \in \mathcal{I}$, the following conditions hold:

$$\begin{aligned} C_l^{\xi^{\phi_1}(x)}(\alpha) &= C_l^{\xi^{\phi_2}(x)}(\alpha), \\ C_u^{\xi^{\phi_1}(x)}(\alpha) &= C_u^{\xi^{\phi_2}(x)}(\alpha). \end{aligned}$$

If for all $x \in \mathcal{R}$, $\alpha \in \mathcal{I}$, $C_l^{\xi^{\phi_1}(x)}(\alpha) = C_l^{\xi^{\phi_2}(x)}(\alpha)$, but for all $x \in \mathcal{R}$, $\alpha \in \mathcal{I}$, $C_u^{\xi^{\phi_1}(x)}(x)(\alpha) \neq C_u^{\xi^{\phi_2}(x)}(x)(\alpha)$, then ϕ_1 and ϕ_2 are called *fuzzy l-equal* to each other.

If for all $x \in \mathcal{R}$, $\alpha \in \mathcal{I}$, $C_u^{\xi^{\phi_1}(x)}(\alpha) = C_u^{\xi^{\phi_2}(x)}(\alpha)$ but for all $x \in \mathcal{R}$, $\alpha \in \mathcal{I}$, $C_l^{\xi^{\phi_1}(x)}(x)(\alpha) \neq C_l^{\xi^{\phi_2}(x)}(x)(\alpha)$, then ϕ_1 and ϕ_2 are called *fuzzy u-equal* to each other.

If any two fuzzy IC-Bags ϕ_1 and ϕ_2 drawn from $\mathcal{R}$ are both l-equal and u-equal to each other, then they are called *equal*.

For any two fuzzy IC-Bags ϕ_1 and ϕ_2 drawn from $\mathcal{R}$:

- ϕ_1 is called a *fuzzy IC-subbag* of ϕ_2 , denoted by $\phi_1 \sqsubseteq \phi_2$ if and only if for all $x \in \mathcal{R}$, $\alpha \in \mathcal{I}$, $C_u^{\xi^{\phi_1}(x)}(\alpha) \leq C_l^{\xi^{\phi_2}(x)}(\alpha)$.
- ϕ_1 is called a *fuzzy l-IC-subbag* of ϕ_2 if and only if for all $x \in \mathcal{R}$, $\alpha \in \mathcal{I}$,

$$\begin{aligned} C_l^{\xi^{\phi_1}(x)}(\alpha) &\leq C_l^{\xi^{\phi_2}(x)}(\alpha), \\ C_u^{\xi^{\phi_1}(x)}(x)(\alpha) &> C_u^{\xi^{\phi_2}(x)}(x)(\alpha). \end{aligned}$$

- ϕ_1 is called a *fuzzy u-IC-subbag* of ϕ_2 if and only if for all $x \in \mathcal{R}$, $\alpha \in \mathcal{I}$,

$$\begin{aligned} C_u^{\xi^{\phi_1}(x)}(\alpha) &\leq C_u^{\xi^{\phi_2}(x)}(\alpha), \\ C_l^{\xi^{\phi_1}(x)}(x)(\alpha) &> C_l^{\xi^{\phi_2}(x)}(x)(\alpha). \end{aligned}$$

A fuzzy IC-Bag ϕ_o drawn from $\mathcal{R}$ is called a *null fuzzy IC-Bag* if and only if for all $x \in \mathcal{R}$, $\alpha \in \mathcal{I}$, $C_l^{\xi^{\phi_o}(x)}(\alpha) = C_u^{\xi^{\phi_o}(x)}(x)(\alpha) = 0$.

The addition of two fuzzy IC-Bags ϕ_1 and ϕ_2 drawn from $\mathcal{R}$ results in the fuzzy IC-Bag $\phi_1 \oplus \phi_2$ such that for all $x \in \mathcal{R}$, $\alpha \in \mathcal{I}$,

$$\begin{aligned} C_l^{\xi^{\phi_1 \oplus \phi_2}(x)}(\alpha) &= C_l^{\xi^{\phi_1}(x)}(\alpha) + C_l^{\xi^{\phi_2}(x)}(\alpha), \\ C_u^{\xi^{\phi_1 \oplus \phi_2}(x)}(x)(\alpha) &= C_u^{\xi^{\phi_1}(x)}(x)(\alpha) + C_u^{\xi^{\phi_2}(x)}(x)(\alpha). \end{aligned}$$

The removal of the fuzzy IC-Bag ϕ_2 from the fuzzy IC-Bag ϕ_1 results in the fuzzy IC-Bag $\phi_1 \ominus \phi_2$ such that for all $x \in \mathcal{R}$, $\alpha \in \mathcal{I}$,

$$\begin{aligned} C_l^{\xi^{\phi_1 \ominus \phi_2}(x)}(\alpha) &= \max(C_l^{\xi^{\phi_1}(x)}(\alpha) - C_u^{\xi^{\phi_2}(x)}(\alpha), 0), \\ C_u^{\xi^{\phi_1 \ominus \phi_2}(x)}(x)(\alpha) &= \max(C_u^{\xi^{\phi_1}(x)}(x)(\alpha) - C_l^{\xi^{\phi_2}(x)}(x)(\alpha), 0). \end{aligned}$$

If ϕ_1 and ϕ_2 be two fuzzy IC-Bags drawn from $\mathcal{R}$, then their union is the fuzzy IC-Bag $\phi_1 \sqcup \phi_2$ such that for all $x \in \mathcal{R}$, $\alpha \in \mathcal{I}$,

$$\begin{aligned} C_l^{\xi^{\phi_1 \sqcup \phi_2}(x)}(\alpha) &= \max\{C_l^{\xi^{\phi_1}(x)}(\alpha), C_l^{\xi^{\phi_2}(x)}(\alpha)\}, \\ C_u^{\xi^{\phi_1 \sqcup \phi_2}(x)}(x)(\alpha) &= \max\{C_u^{\xi^{\phi_1}(x)}(x)(\alpha), C_u^{\xi^{\phi_2}(x)}(x)(\alpha)\}. \end{aligned}$$

The intersection of ϕ_1 and ϕ_2 is the fuzzy IC-Bag $\phi_1 \sqcap \phi_2$ such that for all $x \in \mathcal{R}, \alpha \in \mathcal{I}$,

$$\begin{aligned} C_l^{\xi^{\phi_1 \sqcap \phi_2}(x)}(\alpha) &= \min\{C_l^{\xi^{\phi_1}(x)}(\alpha), C_l^{\xi^{\phi_2}(x)}(\alpha)\}, \\ C_u^{\xi^{\phi_1 \sqcap \phi_2}(x)}(\alpha) &= \min\{C_u^{\xi^{\phi_1}(x)}(\alpha), C_u^{\xi^{\phi_2}(x)}(\alpha)\}. \end{aligned}$$

Clearly for any fuzzy IC-Bag ϕ drawn from $\mathcal{R}$, we have $\phi \sqcup \phi = \phi, \phi \sqcap \phi = \phi$.

5 Intuitionistic Fuzzy IC-Bags

In this section we introduce the notion of Intuitionistic Fuzzy IC-Bags and study some properties of them.

Definition 5.1

An Intuitionistic Fuzzy IC-Bag ι drawn from a non-empty set $\mathcal{R}$ is characterized by the function ζ^ι such that

$$\zeta^\iota : \mathcal{R} \longrightarrow \theta_{\mathcal{I}}$$

where $\theta_{\mathcal{I}}$ is the set of all IC-Bags drawn from the cartesian product $\mathcal{I} \times \mathcal{I}$, $\mathcal{I}$ representing the continuum $[0, 1]$.

Thus for any $\omega \in \mathcal{R}$, $\zeta^\iota(\omega)$ is an IC-Bag drawn from $\mathcal{I} \times \mathcal{I}$. However, any IC-Bag can itself be characterized by a pair of functions over its set, hence $\zeta^\iota(\omega)$ can be characterized by the pair of functions

$$\begin{aligned} C_l^{\zeta^\iota(\omega)} : \mathcal{I} \times \mathcal{I} &\longrightarrow \mathcal{N} \\ C_u^{\zeta^\iota(\omega)} : \mathcal{I} \times \mathcal{I} &\longrightarrow \mathcal{N} \end{aligned}$$

where for all $\alpha, \beta \in \mathcal{I}$, $C_l^{\zeta^\iota(\omega)}(\alpha, \beta) \leq C_u^{\zeta^\iota(\omega)}(\alpha, \beta)$, $\mathcal{N}$ being the set of non-negative integers.

As example, fuzzy IC-Bags can be suitable for representing real-life knowledge associated with object collections where the objects can occur with various membership and non-membership grades (their sums are not necessarily equal to one in each case) and the object counts are also interval valued in nature.

Example 5.1

Let $R = \{a, b\}$ be any non-empty set. Then an example of Intuitionistic Fuzzy IC-Bag ι drawn from R is furnished below:

$$\iota = \{a/\{(0.3, 0.1)/(2, 4), (0.4, 0.3)/(3, 9)\}, b/\{(0.1, 0.8)/(6, 9)\}\}$$

Definition 5.2

For any intuitionistic fuzzy IC-Bag ι drawn from a set $\mathcal{R}$, an intuitionistic fuzzy set $\eta(\iota)$ in $\mathcal{R}$ is called an *intuitionistic fuzzy base set* of ι , if and only if for all $x \in \mathcal{R}$,

$$\mu_{\eta(\iota)}(x) = \{(\alpha, \beta) \in \mathcal{I} \times \mathcal{I} : C_l^{\zeta^\iota(x)}(\alpha, \beta) \geq C_l^{\zeta^\iota(x)}(m, n) \forall (m, n) \in \mathcal{I} \times \mathcal{I}\}$$

where $\mu_{\eta(\iota)}$ represents the concerned degrees of membership and non-membership for each element in $\eta(\iota)$.

In case of Example 5.1, the intuitionistic fuzzy base set of ι is $\{a/(0.4, 0.3), b/(0.1, 0.8)\}$.

More than one intuitionistic fuzzy IC-Bags can have the same intuitionistic fuzzy base set.

Base-equivalent intuitionistic fuzzy IC-Bags are the intuitionistic fuzzy IC-Bags having the same intuitionistic fuzzy base set.

Definition 5.3

The intuitionistic fuzzy IC-Bags ι_1 and ι_2 drawn from a set $\mathcal{R}$ are called *intuitionistic fuzzy IC-equal* if and only if for all $x \in \mathcal{R}$, $(\alpha, \beta) \in \mathcal{I} \times \mathcal{I}$,

$$\begin{aligned} C_l^{\zeta^{\iota_1}(x)}(\alpha, \beta) &= C_l^{\zeta^{\iota_2}(x)}(\alpha, \beta), \\ C_u^{\zeta^{\iota_1}(x)}(\alpha, \beta) &= C_u^{\zeta^{\iota_2}(x)}(\alpha, \beta). \end{aligned}$$

If for all $x \in \mathcal{R}$, $(\alpha, \beta) \in \mathcal{I} \times \mathcal{I}$, $C_l^{\zeta^{\iota_1}(x)}(\alpha, \beta) = C_l^{\zeta^{\iota_2}(x)}(\alpha, \beta)$, but $C_u^{\zeta^{\iota_1}(x)}(\alpha, \beta) \neq C_u^{\zeta^{\iota_2}(x)}(\alpha, \beta)$, then ι_1 and ι_2 are called *intuitionistic fuzzy l-equal* to each other.

On the other hand, if for all $x \in \mathcal{R}$, $(\alpha, \beta) \in \mathcal{I} \times \mathcal{I}$, $C_u^{\zeta^{\iota_1}(x)}(\alpha, \beta) = C_u^{\zeta^{\iota_2}(x)}(\alpha, \beta)$, but $C_l^{\zeta^{\iota_1}(x)}(\alpha, \beta) \neq C_l^{\zeta^{\iota_2}(x)}(\alpha, \beta)$, then ι_1 and ι_2 are called *intuitionistic fuzzy u-equal* to each other. If ι_1 and ι_2 are l-equal as well as u-equal to each other, then they are said to be *equal*.

Example 5.2

Let $R = \{a, b\}$ be any non-empty set and let $\iota_1, \iota_2, \iota_3$, and ι_4 be the four intuitionistic Fuzzy IC-Bags drawn from R such that:

$$\begin{aligned} \iota_1 &= \{a/\{(0.2, 0.1)/(1, 2), (0.4, 0.3)/(2, 8)\}, b/\{(0.6, 0.2)/(3, 9), (0.7, 0.1)/(8, 9), (0.2, 0.2)/(3, 4)\}\} \\ \iota_2 &= \{a/\{(0.2, 0.1)/(1, 2), (0.4, 0.3)/(7, 8)\}, b/\{(0.6, 0.2)/(3, 9), (0.7, 0.1)/(8, 9), (0.2, 0.2)/(3, 4)\}\} \\ \iota_3 &= \{a/\{(0.2, 0.1)/(1, 3), (0.4, 0.3)/(7, 9)\}, b/\{(0.6, 0.2)/(3, 4), (0.7, 0.1)/(8, 10), (0.2, 0.2)/(3, 4)\}\} \\ \iota_4 &= \{a/\{(0.2, 0.1)/(1, 3), (0.4, 0.3)/(6, 9)\}, b/\{(0.6, 0.2)/(3, 4), (0.7, 0.1)/(9, 10), (0.2, 0.2)/(2, 4)\}\} \end{aligned}$$

Here ι_1 and ι_2 are intuitionistic fuzzy IC-equal; ι_2 and ι_3 are intuitionistic fuzzy l-equal; ι_3 and ι_4 are intuitionistic fuzzy u-equal.

Definition 5.4

An intuitionistic fuzzy IC-Bag ι_o drawn from a set $\mathcal{R}$ is called a null intuitionistic fuzzy IC-Bag if and only if for all $x \in \mathcal{R}$, $(\alpha, \beta) \in \mathcal{I} \times \mathcal{I}$,

$$C_l^{\zeta^{\iota_o}(x)}(\alpha, \beta) = C_l^{\zeta^{\iota_o}(x)}(\alpha, \beta) = 0.$$

Proposition 5.1

Associated with any intuitionistic fuzzy IC-Bag ι there can exist more than one intuitionistic fuzzy base set if and only if for atleast one $x \in \mathcal{R}$, there exists atleast two distinct (α_i, β_j) in $\mathcal{I} \times \mathcal{I}$ with the same value of $C_l^{\zeta^{\iota}(x)}(\alpha_i, \beta_j)$ such that for all $(m, n) \in \mathcal{I} \times \mathcal{I}$,

$$C_l^{\zeta^{\iota}(x)}(\alpha_i, \beta_j) \geq C_l^{\zeta^{\iota}(x)}(m, n).$$

Proof: If ι be any intuitionistic fuzzy IC-Bag drawn from $\mathcal{R}$ such that for atleast one $x \in R$, there exists $(\alpha_1, \beta_1), (\alpha_2, \beta_2) \in \mathcal{I} \times \mathcal{I} [(\alpha_1, \beta_1) \neq (\alpha_2, \beta_2)]$ for which

$$(\forall (m, n) \in \mathcal{I} \times \mathcal{I}) (C_l^{\zeta^{\iota}(x)}(\alpha_1, \beta_1) = C_l^{\zeta^{\iota}(x)}(\alpha_2, \beta_2) \geq C_l^{\zeta^{\iota}(x)}(m, n)).$$

Consider $\eta_1(\iota)$ and $\eta_2(\iota)$ be such that for all $(m, n) \in \mathcal{I} \times \mathcal{I}$,

$$\begin{aligned} \mu_{\eta_1(\iota)}(x) &= \{(\alpha_1, \beta_1) \in \mathcal{I} \times \mathcal{I} : C_l^{\zeta^{\iota}(x)}(\alpha_1, \beta_1) \geq C_l^{\zeta^{\iota}(x)}(m, n)\} \\ \mu_{\eta_2(\iota)}(x) &= \{(\alpha_2, \beta_2) \in \mathcal{I} \times \mathcal{I} : C_l^{\zeta^{\iota}(x)}(\alpha_2, \beta_2) \geq C_l^{\zeta^{\iota}(x)}(m, n)\} \end{aligned}$$

Thus, clearly $\eta_1(\iota)$ and $\eta_2(\iota)$ are two distinct intuitionistic fuzzy base sets of ι .

Conversely, if we consider that there exists two distinct intuitionistic fuzzy base sets of ι , then we have for all $x \in R$, $(m, n) \in \mathcal{I} \times \mathcal{I}$,

$$\begin{aligned} \mu_{\eta_1(\iota)}(x) &= \{(\alpha, \beta) \in \mathcal{I} \times \mathcal{I} : C_l^{\zeta^{\iota}(x)}(\alpha, \beta) \geq C_l^{\zeta^{\iota}(x)}(m, n)\} \\ \mu_{\eta_2(\iota)}(x) &= \{(\alpha, \beta) \in \mathcal{I} \times \mathcal{I} : C_l^{\zeta^{\iota}(x)}(\alpha, \beta) \geq C_l^{\zeta^{\iota}(x)}(m, n)\} \end{aligned}$$

such that for atleast one $x \in R$, $\mu_{\eta_1(\iota)}(x) \neq \mu_{\eta_2(\iota)}(x)$.

By substituting (α_1, β_1) for $\mu_{\eta_1(\iota)}(x)$ and (α_2, β_2) for $\mu_{\eta_2(\iota)}(x)$, we have

$(\alpha_1, \beta_1), (\alpha_2, \beta_2) \in \mathcal{I} \times \mathcal{I}[(\alpha_1, \beta_1) \neq (\alpha_2, \beta_2)]$ such that for all $(\alpha, \beta) \in \mathcal{I} \times \mathcal{I}$,

$$C_l^{\zeta^{\iota}(x)}(\alpha_1, \beta_1) \geq C_l^{\zeta^{\iota}(x)}(\alpha, \beta)$$

$$C_l^{\zeta^{\iota}(x)}(\alpha_2, \beta_2) \geq C_l^{\zeta^{\iota}(x)}(\alpha, \beta)$$

Assuming $C_l^{\zeta^{\iota}(x)}(\alpha_1, \beta_1) \neq C_l^{\zeta^{\iota}(x)}(\alpha_2, \beta_2)$, if $C_l^{\zeta^{\iota}(x)}(\alpha_1, \beta_1) < C_l^{\zeta^{\iota}(x)}(\alpha_2, \beta_2)$, then it is a contradiction to the above. On the other hand, if $C_l^{\zeta^{\iota}(x)}(\alpha_1, \beta_1) > C_l^{\zeta^{\iota}(x)}(\alpha_2, \beta_2)$, then also it is a contradiction to the above. Hence proved.

Proposition 5.2

The two intuitionistic fuzzy IC-Bags ι_1 and ι_2 drawn from $\mathcal{R}$ can have the same intuitionistic fuzzy set in $\mathcal{R}$ as their intuitionistic fuzzy base set if and only if for each $x \in \mathcal{R}$, there exists some $(\alpha, \beta) \in \mathcal{I} \times \mathcal{I}$ such that

$$C_l^{\zeta^{\iota_1}(x)}(\alpha, \beta) \geq C_l^{\zeta^{\iota_2}(x)}(m, n) \quad \forall (m, n) \in \mathcal{I} \times \mathcal{I}$$

$$C_l^{\zeta^{\iota_2}(x)}(\alpha, \beta) \geq C_l^{\zeta^{\iota_1}(x)}(m, n) \quad \forall (m, n) \in \mathcal{I} \times \mathcal{I}$$

Proof: Consider two intuitionistic fuzzy IC-Bags ι_1 and ι_2 drawn from a set $\mathcal{R}$ have the same intuitionistic fuzzy base set η such that for all $(m, n) \in \mathcal{I} \times \mathcal{I}$,

$$\mu_{\eta(\iota_1)}(x) = \{(\alpha_i, \beta_j) \in \mathcal{I} \times \mathcal{I} : C_l^{\zeta^{\iota_1}(x)}(\alpha_i, \beta_j) \geq C_l^{\zeta^{\iota_1}(x)}(m, n)\} \quad (1)$$

$$\mu_{\eta(\iota_2)}(x) = \{(\alpha_p, \beta_q) \in \mathcal{I} \times \mathcal{I} : C_l^{\zeta^{\iota_2}(x)}(\alpha_p, \beta_q) \geq C_l^{\zeta^{\iota_2}(x)}(m, n)\} \quad (2)$$

Here (1) and (2) being equal, we consider $(\alpha, \beta) = (\alpha_i, \beta_j) = (\alpha_p, \beta_q)$, and for each $x \in \mathcal{R}$, $(m, n) \in \mathcal{I} \times \mathcal{I}$,

$$C_l^{\zeta^{\iota_1}(x)}(\alpha, \beta) \geq C_l^{\zeta^{\iota_2}(x)}(m, n) \quad (3)$$

$$C_l^{\zeta^{\iota_2}(x)}(\alpha, \beta) \geq C_l^{\zeta^{\iota_1}(x)}(m, n) \quad (4)$$

which proves the necessary part.

Conversely, we assume that for each $x \in \mathcal{R}$ there exists some $(\alpha, \beta) \in \mathcal{I} \times \mathcal{I}$ such that (3) and (4) holds true. If η_1 and η_2 be the intuitionistic fuzzy base sets of ι_1 and ι_2 respectively and we assume that η_1 is not equal to η_2 , then (3) and (4) holds true. Hence $(\alpha_i, \beta_j) \in \mathcal{I} \times \mathcal{I}$ and $(\alpha_p, \beta_q) \in \mathcal{I} \times \mathcal{I}$ may be replaced by $(\alpha, \beta) \in \mathcal{I} \times \mathcal{I}$. This proves the sufficient part.

Proposition 5.3

For any two intuitionistic fuzzy IC-Bags ι_1 and ι_2 which are intuitionistic fuzzy l-equal, $\eta(\iota_1) = \eta(\iota_2)$.

Proof: Let us assume that the two intuitionistic fuzzy IC-Bags ι_1 and ι_2 are intuitionistic fuzzy l-equal. For all $x \in \mathcal{R}$, $(m, n) \in \mathcal{I} \times \mathcal{I}$, we have

$$\begin{aligned}\mu_{\eta(\iota_{1a_1})}(x) &= \{(\alpha, \beta) \in \mathcal{I} \times \mathcal{I} : C_l^{\zeta^{\iota_1}(x)}(\alpha, \beta) \geq C_l^{\zeta^{\iota_1}(x)}(m, n)\} \\ \mu_{\eta(\iota_{2a_2})}(x) &= \{(\alpha, \beta) \in \mathcal{I} \times \mathcal{I} : C_l^{\zeta^{\iota_2}(x)}(\alpha, \beta) \geq C_l^{\zeta^{\iota_2}(x)}(m, n)\}\end{aligned}$$

Again since ι_1 and ι_2 are intuitionistic fuzzy l-equal, hence we have for all $x \in \mathcal{R}$, $(m, n) \in \mathcal{I} \times \mathcal{I}$, $C_l^{\zeta^{\iota_1}(x)}(m, n) = C_l^{\zeta^{\iota_2}(x)}(m, n)$. Hence proved.

Definition 5.5

For intuitionistic fuzzy IC-Bags ι_1 and ι_2 drawn from $\mathcal{R}$, ι_1 is called an intuitionistic fuzzy IC-subbag of ι_2 , denoted by $\iota_1 \sqsubseteq \iota_2$ if and only if for all $x \in \mathcal{R}$, $(\alpha, \beta) \in \mathcal{I} \times \mathcal{I}$,

$$C_u^{\zeta^{\iota_1}(x)}(\alpha, \beta) \leq C_l^{\zeta^{\iota_2}(x)}(\alpha, \beta).$$

Note that for any intuitionistic fuzzy IC-Bag ι drawn from $\mathcal{R}$, $\iota_o \sqsubseteq \iota$.

ι_1 is called an intuitionistic fuzzy l-IC-subbag of ι_2 , denoted by $\iota_1 \sqsubseteq_l \iota_2$ if and only if for all $x \in \mathcal{R}$, $(\alpha, \beta) \in \mathcal{I} \times \mathcal{I}$,

$$\begin{aligned}C_l^{\zeta^{\iota_1}(x)}(\alpha, \beta) &\leq C_l^{\zeta^{\iota_2}(x)}(\alpha, \beta) \\ C_u^{\zeta^{\iota_1}(x)}(\alpha, \beta) &\neq C_u^{\zeta^{\iota_2}(x)}(\alpha, \beta)\end{aligned}$$

ι_1 is called an intuitionistic fuzzy u-IC-subbag of ι_2 , denoted by $\iota_1 \sqsubseteq_u \iota_2$ if and only if for all $x \in \mathcal{R}$, $(\alpha, \beta) \in \mathcal{I} \times \mathcal{I}$,

$$\begin{aligned}C_u^{\zeta^{\iota_1}(x)}(\alpha, \beta) &\leq C_u^{\zeta^{\iota_2}(x)}(\alpha, \beta) \\ C_l^{\zeta^{\iota_1}(x)}(\alpha, \beta) &\neq C_l^{\zeta^{\iota_2}(x)}(\alpha, \beta)\end{aligned}$$

Example 5.3

Let $R = \{a, b\}$ be any non-empty set and let ι_1 , ι_2 , ι_3 , and ι_4 be the four intuitionistic Fuzzy IC-Bags drawn from R such that:

$$\begin{aligned}\iota_1 &= \{a/\{(0.8,0.9)/(2,3), (0.6,0.8)/(8,9)\}, b/\{(0.6,0.7)/(8,10), (0.1,0.1)/(6,7)\}\} \\ \iota_2 &= \{a/\{(0.8,0.9)/(3,4), (0.6,0.8)/(10,12)\}, b/\{(0.6,0.7)/(12,14), (0.1,0.1)/(7,8)\}\} \\ \iota_3 &= \{a/\{(0.8,0.9)/(2,3), (0.6,0.8)/(7,9)\}, b/\{(0.6,0.7)/(9,10), (0.1,0.1)/(7,9)\}\} \\ \iota_4 &= \{a/\{(0.8,0.9)/(1,2), (0.6,0.8)/(7,8)\}, b/\{(0.6,0.7)/(7,10), (0.1,0.1)/(7,7)\}\}\end{aligned}$$

Here clearly, $\iota_1 \sqsubseteq \iota_2$, $\iota_3 \sqsubseteq_l \iota_2$, $\iota_4 \sqsubseteq_u \iota_1$.

Proposition 5.4

For intuitionistic fuzzy IC-Bags ι_1 and ι_2 drawn from $\mathcal{R}$, the following holds:

- (a) If ι_1 is intuitionistic fuzzy l-equal to ι_2 , then ι_1 is an intuitionistic l-IC-subbag of ι_2 .
- (b) If ι_1 is intuitionistic fuzzy u-equal to ι_2 , then ι_1 is an intuitionistic u-IC-subbag of ι_2 .

6 Conclusion

In the present paper, we have extended the concept of fuzzy IC-Bags to intuitionistic fuzzy IC-Bags by applying the notion of Atanassov's intuitionistic fuzzy sets which is a generalization of the notion of fuzzy sets. We further did some characterization of intuitionistic fuzzy IC-Bags which includes the introduction of intuitionistic fuzzy base sets of Intuitionistic fuzzy IC-Bags. The notions of base-equivalent intuitionistic fuzzy IC-Bags and intuitionistic fuzzy l-equalities and u-equalities are also proposed. In future, we will study possible real-world applications of this proposal. Further, we would like to formalize the notions of cardinalities of intuitionistic fuzzy IC-Bags and explore their characteristics during our future work.

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