S. JAFARI AND T. NOIRI

MORE ON WEAK $\delta s$-CONTINUITY IN TOPOLOGICAL SPACES

Abstract. The notion of weakly $\delta s$-continuous functions is introduced by Ekici in [6]. In this paper, we further investigate some more properties of weakly $\delta s$-continuous functions. This type of continuity is a generalization of super continuity [16].

Key words: continuous, $\delta$-semiopen, $\delta$-semi-continuous, rarely $\delta s$-continuous, super-continuous.

AMS Mathematics Subject Classification: 54B05, 54C08, 54D05.

1. Introduction and preliminaries

Levine [12] defined semiopen sets which are weaker than open sets in topological spaces. After Levine’s semiopen sets, mathematicians gave in several papers different and interesting new open sets as well as generalized open sets. In 1968, Veličko [24] introduced $\delta$-open sets, which are stronger than open sets, in order to investigate the characterization of $H$-closed spaces. In 1997, Park et al. [19] have introduced the notion of $\delta$-semiopen sets which are stronger than semiopen sets but weaker than $\delta$-open sets and investigated the relationships between several types of open sets. In 1979, Popa [20] introduced the useful notion of rare continuity as a generalization of weak continuity [11]. The class of rarely continuous functions has been further investigated by Long and Herrington [13] and Jafari [7] and [8]. The concept of rare $\delta s$-continuity in topological spaces as a generalization of super continuity is introduced by Caldas et al. [3].

The notion of weakly $\delta s$-continuous functions is introduced by Ekici in [6]. The purpose of the present paper is to further investigate some more properties of weakly $\delta s$-continuous functions. This type of functions is weaker than both super continuous functions and $\delta$-semi-continuous functions and stronger than rare $\delta s$-continuous functions.

Throughout this paper, $(X, \tau)$ and $(Y, \sigma)$ (or simply, $X$ and $Y$) denote topological spaces on which no separation axioms are assumed unless explic-
itly stated. If $A$ is any subset of a space $X$, then $Cl(A)$ and $Int(A)$ denote the closure and the interior of $A$, respectively.

A subset $A$ of $X$ is called regular open (resp. regular closed) if $A = Int(Cl(A))$ (resp. $A = Cl(Int(A))$). Recall that a subset $A$ of $X$ is called semi-open [12] if $A \subset Cl(Int(A))$. The complement of a semi-open set is called semi-closed. A subset $A$ of $X$ is called preopen [15] if $A \subset Cl(Int(A))$. A rare or codense set is a set $A$ such that $Int(A) = \emptyset$, equivalently, if the complement $X \setminus A$ is dense. A point $x \in X$ is called a $\delta$-cluster [24] of $A$ if $S \cap U \neq \emptyset$ for each regular open set $U$ containing $x$. The set of all $\delta$-cluster points of $A$ is called the $\delta$-closure of $A$ and is denoted by $Cl_\delta(A)$. A subset $A$ is called $\delta$-closed if $Cl_\delta(A) = A$. The complement of a $\delta$-closed set is called $\delta$-open. The $\delta$-interior of a subset $A$ of a space $(X, \tau)$, denoted by $Int_\delta(A)$, is the union of all regular open sets of $(X, \tau)$ contained in $A$.

A subset $A$ of a topological space $X$ is said to be $\delta$-semiopen sets [19] if there exists a $\delta$-open set $U$ of $X$ such that $U \subset A \subset Cl(U)$, equivalently if $A \subset Cl(Int_\delta(A))$. The complement of a $\delta$-semiopen set is called a $\delta$-semiclosed set. A point $x \in X$ is called the $\delta$-semicluster point of $A$ if $A \cap U \neq \emptyset$ for every $\delta$-semiopen set $U$ of $X$ containing $x$. The set of all $\delta$-semicluster points of $A$ is called the $\delta$-semiclosure of $A$, denoted by $sCl_\delta(A)$ and the $\delta$-semiinterior of $A$, denoted by $sInt_\delta(A)$, is defined as the union of all $\delta$-semiopen sets contained in $A$. We denote the collection of all $\delta$-semiopen (resp. $\delta$-semiclosed, $\delta$-open, regular open and open) sets by $\delta SO(X)$ (resp. $\delta SC(X)$, $\delta O(X)$, $RO(X)$ and $O(X)$). We set $\delta SO(X, x) = \{U \mid x \in U \in \delta SO(X)\}$, $\delta O(X, x) = \{U \mid x \in U \in \delta O(X)\}$, $RO(X, x) = \{U \mid x \in U \in RO(X)\}$ and $O(X, x) = \{U \mid x \in U \in O(X)\}$.

**Lemma 1.** The intersection (resp. union) of an arbitrary collection of $\delta$-semiclosed (resp. $\delta$-semiopen) sets in $(X, \tau)$ is $\delta$-semiclosed (resp. $\delta$-semiopen)

**Corollary 1.** Let $A$ be a subset of a topological space $(X, \tau)$. Then the following properties hold:

1. $sCl_\delta(A) = \cap \{F \in \delta SC(X, \tau) : A \subset F\}$.
2. $sCl_\delta(A)$ is $\delta$-semiclosed.
3. $sCl_\delta(sCl_\delta(A)) = sCl_\delta(A)$.

**Lemma 2** ([1]). For subsets $A$ and $A_i$ ($i \in I$) of a space $(X, \tau)$, the following hold:

1. $A \subset sCl_\delta(A)$.
2. If $A \subset B$, then $sCl_\delta(A) \subset sCl_\delta(B)$.
3. $sCl_\delta(\cap \{A_i : i \in I\}) \subset \cap \{sCl_\delta(A_i) : i \in I\}$.
4. $sCl_\delta(\cup \{A_i : i \in I\}) = \cup \{sCl_\delta(A_i) : i \in I\}$.
5. $A$ is $\delta$-semiclosed if and only $A = sCl_\delta(A)$. 
Lemma 3 ([19]). For a subset $A$ of a space $(X, \tau)$, the following hold:
(1) $A$ is a $\delta$-semiopen set if and only if $A = \text{sInt}_\delta(A)$.
(2) $X - \text{sInt}_\delta(A) = \text{sCl}_\delta(X - A)$ and $\text{sInt}_\delta(X - A) = X - \text{sCl}_\delta(A)$.
(3) $\text{sInt}_\delta(A)$ is a $\delta$-semiopen set.

Definition 1. A function $f : X \to Y$ is said to be:
(1) Weakly continuous [11] if for each $x \in X$ and each open set $V$ containing $f(x)$, there exists $U \in O(X, x)$ such that $f(U) \subseteq \text{Cl}(V)$.
(2) $\delta$-semi-continuous [18] if for each $x \in X$ and each open set $V$ containing $f(x)$, there exists $U \in \delta\text{SO}(X, x)$ such that $f(U) \subseteq V$.
(3) Rarely quasi continuous [21] (resp. rarely $\delta$s-continuous [3]) if for each $x \in X$ and each $V \in O(Y, f(x))$, there exist a rare set $R_V$ with $V \cap \text{Cl}(R_V) = \emptyset$ and $U \in \text{SO}(X, x)$ (resp. $U \in \delta\text{SO}(X, x)$) such that $f(U) \subseteq V \cup R_V$.
(4) Super-continuous [16] if the inverse image of every open set in $Y$ is $\delta$-open in $X$.
(5) semi-continuous [12] if for each $x \in X$ and each open set $V$ in $Y$ containing $f(x)$, there exists $U \in \text{SO}(X, x)$ such that $f(U) \subseteq V$.

Definition 2. A function $f : X \to Y$ is said to be:
(1) Weakly quasicontinuous [22] if for each $x \in X$ and for each open set $U$ containing $x$ and each open set $G$ containing $f(x)$, there exists a nonempty open set $V$ such that $V \subseteq U$ and $f(V) \subseteq \text{Cl}(G)$.
(2) Weakly-$\theta$-continuous [5] if for each $x \in X$ and each open set $V$ of $Y$ containing $f(x)$, there exists an open set $U$ of $X$ containing $x$ such that $f(\text{Int}(\text{Cl}(U))) \subseteq \text{Cl}(V)$.

Definition 3. A function $f : X \to Y$ is said to be $I$-.$\delta$s-continuous [3] at $x \in X$ if for each $V \in O(Y, f(x))$, there exists $U \in \delta\text{SO}(X, x)$ such that $\text{Int}[f(U)] \subseteq V$. If $f$ has this property at each point $x \in X$, then we say that $f$ is $I$-.$\delta$s-continuous on $X$.

Remark 1 ([3]). It should be noted that super-continuity implies $I$-.$\delta$s-continuity and $I$-$\delta$s-continuity implies rare $\delta$s-continuity. But the converses are not true as shown by the following examples.

Example 1 ([3]). Let $X = Y = \{a, b, c\}$ and $\tau = \sigma = \{X, \emptyset, \{a\}\}$. Then a function $f : (X, \tau) \to (Y, \sigma)$ defined by $f(a) = f(b) = a$ and $f(c) = c$, is $I$-$\delta$s-continuous. Since $f$ is not continuous, then it is not super continuous.

Example 2 ([3]). Let $(Y, \sigma)$ be the same spaces as in the above Example. Then the identity function $f : (X, \tau) \to (Y, \sigma)$ is rare $\delta$s-continuous but it is not $I$-$\delta$s-continuous.
Remark 2. The following diagram \cite[Remark 4.1]{17} holds:

\[
\begin{array}{ccc}
\text{super -- continuity} & \rightarrow & \text{continuity} \\
\downarrow & & \downarrow \\
\delta -- \text{semi -- continuity} & \rightarrow & \text{semi -- continuity}
\end{array}
\]

2. Weakly-$\delta s$-continuous and some properties

In \cite{3}, unaware of the paper of Ekici \cite{6}, the notion of weakly-$\delta s$-continuous under the name of almost weakly-$\delta s$-continuous was defined. In this paper, we used the name weakly-$\delta s$-continuous functions.

**Definition 4.** A function $f : X \rightarrow Y$ is called weakly-$\delta s$-continuous \cite{6} if for each $x \in X$ and each open set $V$ containing $f(x)$, there exists $U \in \delta SO(X, x)$ such that $f(U) \subset Cl(V)$.

The following diagram holds:

\[
\begin{array}{ccc}
\text{super C.} & \rightarrow & \text{weak $\theta$ -- C.} & \rightarrow & \text{weak C.} \\
\downarrow & & \downarrow & & \downarrow \\
\delta -- \text{semi -- C.} & \rightarrow & \text{weak $\delta s$ -- C.} & \rightarrow & \text{weak quasi C.} \\
\downarrow & & \downarrow & & \downarrow \\
I. \delta s -- C. & \rightarrow & \text{rare $\delta s$ -- C.} & \rightarrow & \text{rare quasi C.}
\end{array}
\]

It should be mentioned that in the above diagram C. means continuity.

**Example 3.** Let $X = \{a, b, c\}$, $\tau = \{X, \{a\}, \{b\}, \{a, b\}, \emptyset\}$ and $\sigma = \{X, \{a\}, \{b, c\}, \emptyset\}$. Then the identity function $f : (X, \tau) \rightarrow (Y, \sigma)$ is $\delta$-semi-continuous but it is not weakly continuous.

**Example 4.** Let $X, \tau$ and $\sigma$ be the same as in Example 3. Let $f : (X, \tau) \rightarrow (Y, \sigma)$ be defined as $f(a) = b, f(b) = c$ and $f(c) = a$. Then $f$ is I. $\delta s$-continuous and not weakly quasicontinuous.

**Example 5.** Let $X = \{a, b, c\}$ and $\tau = \sigma = \{X, \{a\}, \emptyset\}$. Then the identity function $f : (X, \tau) \rightarrow (Y, \sigma)$ is continuous and hence weakly $\theta$-continuous. But it is not I. $\delta s$-continuous.

**Example 6.** Let $X = \{a, b, c\}$, $\tau = \{X, \{a\}, \emptyset\}$ and $\sigma = \{X, \{a\}, \{c\}, \{a, c\}, \emptyset\}$. Define a function $f : (X, \tau) \rightarrow (Y, \sigma)$ as follows: $f(a) = b$, $f(b) = a$ and $f(c) = c$. Then $f$ is weakly quasicontinuous (= weakly semi-continuous) \cite[Example 2]{9} but it is not rarely $\delta s$-continuous.
Theorem 1. For a function \( f : X \to Y \), the following are equivalent:

1. \( f \) is weakly \( \delta \)-continuous,
2. \( sCl_\delta(f^{-1}(\text{Int}(\text{Cl}(V)))) \subset f^{-1}(\text{Cl}(V)) \) for every subset \( V \subset Y \),
3. \( sCl_\delta(f^{-1}(\text{Int}(F))) \subset f^{-1}(F) \) for every regular closed subset \( F \subset Y \),
4. \( sCl_\delta(f^{-1}(U)) \subset f^{-1}(\text{Cl}(U)) \) for every open subset \( U \subset Y \),
5. \( f^{-1}(U) \subset s\text{Int}_\delta(f^{-1}(\text{Cl}(U))) \) for every open subset \( U \subset Y \),
6. \( sCl_\delta(f^{-1}(U)) \subset f^{-1}(\text{Cl}(U)) \) for each preopen subset \( U \subset Y \),
7. \( f^{-1}(U) \subset s\text{Int}_\delta(f^{-1}(\text{Cl}(U))) \) for each preopen subset \( U \subset Y \).

Proof. (1) \( \Rightarrow \) (2): Let \( V \) be a subset of \( Y \) and \( x \in X \setminus f^{-1}(\text{Cl}(V)) \). Then \( f(x) \in Y \setminus \text{Cl}(V) \). There exists an open set \( U \) containing \( f(x) \) such that \( U \cap V = \emptyset \). We have \( \text{Cl}(U) \cap \text{Int}(\text{Cl}(V)) = \emptyset \). Since \( f \) is weakly \( \delta \)-continuous, then there exists a \( \delta \)-semiopen set \( W \) containing \( x \) such that \( f(W) \subset \text{Cl}(U) \). Then \( W \cap f^{-1}(\text{Int}(\text{Cl}(V))) = \emptyset \) and \( x \in X \setminus sCl_\delta(f^{-1}(\text{Int}(\text{Cl}(V)))) \). Hence, \( sCl_\delta(f^{-1}(\text{Int}(\text{Cl}(V)))) \subset f^{-1}(\text{Cl}(V)) \).

(2) \( \Rightarrow \) (3): Suppose \( F \) is any regular closed set in \( Y \). Then \( sCl_\delta(f^{-1}(\text{Int}(F))) = sCl_\delta(f^{-1}(\text{Int}(\text{Cl}(F)))) \subset f^{-1}(\text{Cl}(F)) = f^{-1}(F) \).

(3) \( \Rightarrow \) (4): Suppose \( U \) is an open subset of \( Y \). Since \( \text{Cl}(U) \) is regular closed in \( Y \), then \( sCl_\delta(f^{-1}(U)) \subset sCl_\delta(f^{-1}(\text{Cl}(U))) \subset f^{-1}(\text{Cl}(U)) \).

(4) \( \Rightarrow \) (5): Suppose \( U \) is any open set of \( Y \). Since \( Y \setminus \text{Cl}(U) \) is open in \( Y \), then \( X \setminus s\text{Int}_\delta(f^{-1}(\text{Cl}(U))) = sCl_\delta(f^{-1}(Y \setminus \text{Cl}(U))) \subset f^{-1}(\text{Cl}(Y \setminus \text{Cl}(U))) \subset X \setminus f^{-1}(U) \). Hence, \( f^{-1}(U) \subset s\text{Int}_\delta(f^{-1}(\text{Cl}(U))) \).

(5) \( \Rightarrow \) (1): Suppose \( x \in X \) and \( U \) is any open subset of \( Y \) containing \( f(x) \). Then \( x \in f^{-1}(U) \subset s\text{Int}_\delta(f^{-1}(\text{Cl}(U))) \). Take \( W = s\text{Int}_\delta(f^{-1}(\text{Cl}(U))) \). Thus \( f(W) \subset \text{Cl}(U) \) and hence \( f \) is weakly \( \delta \)-continuous at \( x \) in \( X \).

(1) \( \Rightarrow \) (6): Suppose \( U \) is any preopen set of \( Y \) and \( x \in X \setminus f^{-1}(\text{Cl}(U)) \). There exists an open set \( O \) containing \( f(x) \) such that \( O \cap U = \emptyset \). We have \( \text{Cl}(O \cap U) = \emptyset \). Since \( U \) is preopen, then \( U \cap \text{Cl}(O) \subset \text{Int}(\text{Cl}(U)) \cap \text{Cl}(O) \subset \text{Cl}(\text{Int}(\text{Cl}(U)) \cap O) \subset \text{Cl}(\text{Int}(\text{Cl}(U) \cap O)) \subset \text{Cl}(\text{Int}(\text{Cl}(U) \cap O)) \subset \text{Cl}(U \cap O) = \emptyset \). Since \( f \) is weakly \( \delta \)-continuous and \( O \) is an open set containing \( f(x) \), there exists a \( \delta \)-semiopen set \( W \) in \( X \) containing \( x \) such that \( f(W) \subset \text{Cl}(O) \). Then \( f(W) \cap U = \emptyset \) and \( W \cap f^{-1}(U) = \emptyset \). This implies that \( x \in X \setminus sCl_\delta(f^{-1}(U)) \) and then \( sCl_\delta(f^{-1}(U)) \subset f^{-1}(\text{Cl}(U)) \).

(6) \( \Rightarrow \) (7): Suppose \( U \) is any preopen set of \( Y \). Since \( Y \setminus \text{Cl}(U) \) is open in \( Y \), then \( X \setminus s\text{Int}_\delta(f^{-1}(\text{Cl}(U))) = sCl_\delta(f^{-1}(Y \setminus \text{Cl}(U))) \subset f^{-1}(\text{Cl}(Y \setminus \text{Cl}(U))) \subset X \setminus f^{-1}(U) \). This shows that \( f^{-1}(U) \subset s\text{Int}_\delta(f^{-1}(\text{Cl}(U))) \).

(7) \( \Rightarrow \) (1): Suppose \( x \in X \) and \( U \) is any open set of \( Y \) containing \( f(x) \). We have \( x \in f^{-1}(U) \subset s\text{Int}_\delta(f^{-1}(\text{Cl}(U))) \). Take \( W = s\text{Int}_\delta(f^{-1}(\text{Cl}(U))) \). Then \( f(W) \subset \text{Cl}(U) \). This means that \( f \) is weakly \( \delta \)-continuous at \( x \) in \( X \).

Theorem 2. If \( f : X \to Y \) is a weakly \( \delta \)-continuous function and \( Y \) is Hausdorff, then \( f \) has \( \delta \)-semiclosed point inverses.
**Proof.** Let \( y \in Y \) and \( x \in \{ x \in X : f(x) \neq y \} \). Since \( f(x) \neq y \) and \( Y \) is Hausdorff, there exist disjoint open sets \( G_1, G_2 \) such that \( f(x) \in G_1 \) and \( y \in G_2 \). Since \( G_1 \cap G_2 = \emptyset \), then \( Cl(G_1) \cap G_2 = \emptyset \). We have \( y \notin Cl(G_1) \). Since \( f \) is weakly \( \delta \)-continuous, there exists a \( \delta \)-semiopen set \( U \) containing \( x \) such that \( f(U) \subset Cl(G_1) \). Assume that \( U \) is not contained in \( \{ x \in X : f(x) \neq y \} \). There exists a point \( u \in U \) such that \( f(u) = y \). Since \( f(U) \subset Cl(G_1) \), we have \( y = f(u) \in Cl(G_1) \). This is a contradiction. Hence, \( U \subset \{ x \in X : f(x) \neq y \} \) and \( \{ x \in X : f(x) \neq y \} \) is \( \delta \)-semiopen in \( X \). This shows that \( \{ x \in X : f(x) \neq y \} \) is \( \delta \)-semiopen in \( X \), equivalently \( f^{-1}(y) = \{ x \in X : f(x) = y \} \) is \( \delta \)-semiclosed in \( X \).

Recall that a point \( x \in X \) is said to be in the \( \theta \)-closure \([17]\) of a subset \( A \) of \( X \), denoted by \( Cl_\theta(G) \), if \( Cl(G) \cap A \neq \emptyset \) for each open set \( G \) of \( X \) containing \( x \). \( A \) is called \( \theta \)-closed if \( A = Cl_\theta(A) \). The complement of a \( \theta \)-closed set is called \( \theta \)-open. \[\square\]

**Theorem 3.** For a function \( f : X \to Y \), the following are equivalent:

(1) \( f \) is weakly \( \delta \)-semiopen-
(2) \( f(sCl_\delta(V)) \subset Cl_\theta(f(V)) \) for each subset \( V \subset X \),
(3) \( sCl_\delta(f^{-1}(G)) \subset f^{-1}(Cl_\theta(G)) \) for each subset \( G \subset Y \),
(4) \( sCl_\delta(f^{-1}(Int(Cl_\theta(G)))) \subset f^{-1}(Cl_\theta(G)) \) for every subset \( G \subset Y \).

**Proof.**

(1) \( \Rightarrow \) (2) : Let \( V \subset X \), \( x \in sCl_\delta(V) \), and \( U \) be any open set of \( Y \) containing \( f(x) \). There exists a \( \delta \)-semiopen set \( W \) containing \( x \) such that \( f(W) \subset Cl(U) \). Since \( x \in sCl_\delta(V) \), then \( W \cap V \neq \emptyset \). This implies that \( \emptyset \neq f(W) \cap f(V) \subset Cl(U) \cap f(V) \) and \( f(x) \in Cl_\theta(f(V)) \). Hence, \( f(sCl_\delta(V)) \subset Cl_\theta(f(V)) \).

(2) \( \Rightarrow \) (3) : Let \( G \subset Y \). Then \( f(sCl_\delta(f^{-1}(G))) \subset Cl_\theta(G) \) and hence \( sCl_\delta(f^{-1}(G)) \subset f^{-1}(Cl_\theta(G)) \).

(3) \( \Rightarrow \) (4) : Let \( G \subset Y \). Since \( Cl_\theta(G) \) is closed in \( Y \), then \( sCl_\delta(f^{-1}(Int(Cl_\theta(G)))) \subset f^{-1}(Cl_\theta(Int(Cl_\theta(G)))) = f^{-1}(Cl(Cl_\theta(G))) \subset f^{-1}(Cl_\theta(G)) \).

(4) \( \Rightarrow \) (1) : Let \( U \) be any open set of \( Y \). We have \( U \subset Int(Cl(U)) = Int(Cl_\theta(U)) \). Thus, \( sCl_\delta(f^{-1}(U)) \subset Cl_\delta(f^{-1}(Int(Cl_\theta(U)))) \subset f^{-1}(Cl_\theta(U)) = f^{-1}(Cl(U)) \). This implies from Theorem 1 that \( f \) is weakly \( \delta \)-continuous. \[\square\]

**Theorem 4.** If \( f^{-1}(Cl_\theta(V)) \) is \( \delta \)-semiclosed in \( X \) for every subset \( V \subset Y \), then \( f \) is weakly \( \delta \)-continuous.

**Proof.** Let \( V \subset Y \). Since \( f^{-1}(Cl_\theta(V)) \) is \( \delta \)-semiclosed in \( X \), then \( sCl_\delta(f^{-1}(V)) \subset Cl_\delta(f^{-1}(Cl_\theta(V))) = f^{-1}(Cl_\theta(V)) \). This implies from Theorem 3 that \( f \) is weakly \( \delta \)-continuous. \[\square\]
Theorem 5. Let \( f : X \to Y \) be a function. If \( f \) is weakly \( \delta s \)-continuous, then \( f^{-1}(V) \) is \( \delta \)-semiclosed in \( X \) for every \( \theta \)-closed subset \( V \subset Y \).

Proof. It follows from Theorem 3.

Corollary 2. Let \( f : X \to Y \) be a function. If \( f \) is weakly \( \delta s \)-continuous, then \( f^{-1}(V) \) is \( \delta \)-semiopen in \( X \) for every \( \theta \)-open subset \( V \subset Y \).

Definition 5. A function \( f : X \to Y \) is said to be

1. (\( \delta, s \))-open if \( f(A) \) is semiopen for every \( \delta \)-semiopen subset \( A \subset X \).
2. neatly weak \( \delta s \)-continuous if for each \( x \in X \) and each open set \( V \) of \( X \) containing \( f(x) \), there exists a \( \delta \)-semiopen set \( U \) containing \( x \) such that \( \text{Int}(f(U)) \subset \text{Cl}(V) \).

Theorem 6. If a function \( f : X \to Y \) is neatly weak \( \delta s \)-continuous and (\( \delta, s \))-open, then \( f \) is weakly \( \delta s \)-continuous.

Proof. Let \( x \in X \) and \( V \) be an open subset of \( Y \) containing \( f(x) \). Since \( f \) is neatly weak \( \delta s \)-continuous, there exists a \( \delta \)-semiopen set \( U \) of \( X \) containing \( x \) such that \( \text{Int}(f(U)) \subset \text{Cl}(V) \). Since \( f \) is (\( \delta, s \))-open, then \( f(U) \) is semiopen in \( Y \). Then \( f(U) \subset \text{Cl}(\text{Int}(f(U))) \subset \text{Cl}(V) \). Thus, \( f \) is weakly \( \delta s \)-continuous.

Theorem 7. If \( f : X \to Y \) is weakly \( \delta s \)-continuous and \( Y \) is Hausdorff, then for each \( (x, y) \notin G(f) \), there exist a \( \delta \)-semiopen set \( V \subset X \) and an open set \( U \subset Y \) containing \( x \) and \( y \) respectively, such that \( f(V) \cap \text{Int}(\text{Cl}(U)) = \emptyset \).

Proof. Let \( (x, y) \notin G(f) \). We have \( y \neq f(x) \). Since \( Y \) is Hausdorff, there exist disjoint open sets \( U \) and \( V \) containing \( y \) and \( f(x) \), respectively. We have \( \text{Int}(\text{Cl}(U)) \cap \text{Cl}(V) = \emptyset \). Since \( f \) is weakly \( \delta s \)-continuous, there exists an \( \delta \)-semiopen set \( G \) containing \( x \) such that \( f(G) \subset \text{Cl}(V) \). Hence, \( f(G) \cap \text{Int}(\text{Cl}(U)) = \emptyset \).

Definition 6. A function \( f : X \to Y \) is said to be faintly \( \delta s \)-continuous if for each \( x \in X \) and each \( \theta \)-open set \( V \) of \( Y \) containing \( f(x) \), there exists a \( \delta \)-semiopen set \( U \) containing \( x \) such that \( f(U) \subset V \).

Theorem 8. Let \( f : X \to Y \) be a function. The following are equivalent:

1. \( f \) is faintly \( \delta s \)-continuous,
2. \( f^{-1}(V) \) is \( \delta \)-semiopen in \( X \) for every \( \theta \)-open subset \( V \subset Y \),
3. \( f^{-1}(V) \) is \( \delta \)-semiclosed in \( X \) for every \( \theta \)-closed subset \( V \subset Y \).

Proof. Obvious.
Theorem 9. Let \( f : X \to Y \) be a function, where \( Y \) is regular. The following are equivalent:

1. \( f \) is \( \delta \)-semicontinuous,
2. \( f^{-1}(\text{Cl}_\theta(V)) \) is \( \delta \)-semiclosed in \( X \) for every subset \( V \subset Y \),
3. \( f \) is weakly \( \delta s \)-continuous,
4. \( f \) is faintly \( \delta s \)-continuous.

Proof. (1) \( \Rightarrow \) (2): Let \( V \subset Y \). Since \( \text{Cl}_\theta(V) \) is closed, then \( f^{-1}(\text{Cl}_\theta(V)) \) is \( \delta \)-semiclosed in \( X \).

(2) \( \Rightarrow \) (3): It follows from Theorem 4.

(3) \( \Rightarrow \) (4): Let \( V \) be a \( \theta \)-closed subset of \( Y \). By Theorem 3, we have \( s\text{Cl}_\delta(f^{-1}(V)) \subset f^{-1}(\text{Cl}_\theta(V)) = f^{-1}(V) \). This shows that \( f^{-1}(V) \) is \( \delta \)-semiclosed and hence \( f \) is faintly \( \delta s \)-continuous.

(4) \( \Rightarrow \) (1): Let \( V \) be an open subset of \( Y \). Since \( Y \) is regular, \( V \) is \( \theta \)-open in \( Y \). Since \( f \) is faintly \( \delta s \)-continuous, then \( f^{-1}(V) \) is \( \delta \)-semiopen in \( X \). Thus, \( f \) is \( \delta \)-semicontinuous. \( \blacksquare \)

Definition 7. A space \((X, \tau)\) is said to be \( \delta \)-semi \( T_2 \) (see [1]) if for each pair of distinct points \( x \) and \( y \) in \( X \), there exist \( U \in \delta \text{SO}(X, x) \) and \( V \in \delta \text{SO}(X, y) \) such that \( U \cap V = \emptyset \).

Theorem 10. Let \( f : (X, \tau) \to (Y, \sigma) \) be a weakly \( \delta s \)-continuous injective function. If \( (Y, \sigma) \) is Urysohn, then \( (X, \tau) \) is \( \delta \)-semi \( T_2 \).

Proof. Let \( x_1 \) and \( x_2 \) be any two distinct points of \( X \). Since \( f \) is injective, \( f(x_1) \neq f(x_2) \). Since \( (Y, \sigma) \) is Urysohn, there exist disjoint \( V_1, V_2 \in \sigma \) such that \( f(x_1) \in V_1, f(x_2) \in V_2 \) and \( \text{Cl}(V_1) \cap \text{Cl}(V_2) = \emptyset \). Since \( f \) is weakly \( \delta s \)-continuous at \( x_i \), then there exists \( \delta \)-semiopen sets \( U_i \) for \( i = 1, 2 \) containing \( x_i \) such that \( f(U_i) \subset \text{Cl}(V_i) \). This indicates that \( (X, \tau) \) is \( \delta \)-semi \( T_2 \). \( \blacksquare \)

Theorem 11. If \( f : X \to Y \) is weakly \( \delta s \)-continuous and \( g : Y \to Z \) is continuous, then the composition \( \text{gof} : X \to Z \) is weakly \( \delta s \)-continuous.

Proof. Let \( x \in X \) and \( A \) be an open set of \( Z \) containing \( g(f(x)) \). We have \( g^{-1}(A) \) is an open set of \( Y \) containing \( f(x) \). Then there exists a \( \delta \)-semiopen set \( B \) containing \( x \) such that \( f(B) \subset \text{Cl}(g^{-1}(A)) \). Since \( g \) is continuous, then \( (\text{gof})(B) \subset g(\text{Cl}(g^{-1}(A))) \subset \text{Cl}(A) \). Thus, \( \text{gof} \) is weakly \( \delta s \)-continuous. \( \blacksquare \)

Definition 8. We say that the product space \( X = X_1 \times \ldots \times X_n \) has property \( P_{\delta s} \) [3] if \( A_i \) is a \( \delta \)-semiopen set in a topological space \( X_i \), for \( i = 1, 2, \ldots, n \), then \( A_1 \times \ldots \times A_n \) is also \( \delta \)-semiopen in the product space \( X = X_1 \times \ldots \times X_n \).
Theorem 12. If \( f_i : X_i \rightarrow Y_i \) is weakly \( \delta s \)-continuous for each \( i \in I = \{1, 2, 3, \ldots, n\} \) and \( \prod X_i \) has property \( P_{\delta s} \), then the function \( f : \prod X_i \rightarrow \prod Y_i \) which is defined by \( f((x_i)) = (f_i(x_i)) \) is weakly \( \delta s \)-continuous.

Proof. Let \( x = (x_i) \in \prod X_i \) and \( V \) be an open set containing \( f(x) \). There exists an open set \( \prod U_i \) such that \( f(x) \in \prod_{i=1}^{n} U_i \times \prod_{i\neq j} Y_j \subset V \), where \( U_i \) is open in \( Y_i \). Since \( f_i \) is weakly \( \delta s \)-continuous, there exists \( \delta \)-semiopen sets \( G_i \) in \( X_i \) containing \( x_i \) such that \( f_i(G_i) \subset Cl(U_i) \) for each \( i = 1, 2, \ldots, n \). Take \( G = \prod_{i=1}^{n} G_i \times \prod_{i\neq j} X_j \). Then \( G \) is \( \delta \)-semiopen in \( \prod X_i \) containing \( x \) and \( f(G) \subset \prod_{i=1}^{n} f_i(G_i) \times \prod_{i\neq j} Y_j \subset \prod_{i=1}^{n} Cl(U_i) \times \prod_{i\neq j} Y_j \subset Cl(V) \). This shows that \( f \) is weakly \( \delta s \)-continuous.

Theorem 13. Let \( f, g : X \rightarrow Y \) be weakly \( \delta s \)-continuous functions and \( Y \) be Urysohn. If \( \delta SO(X) \) is closed under the finite intersections, then the set \( \{x \in X : f(x) = g(x)\} \) is \( \delta \)-semiclosed in \( X \).

Proof. Let \( x \in X \setminus \{x \in X : f(x) = g(x)\} \). We have \( f(x) \neq g(x) \). Since \( Y \) is Urysohn, then there exist open sets \( A \) and \( B \) of \( Y \) such that \( f(x) \in A \), \( g(x) \in B \) and \( Cl(A) \cap Cl(B) = \emptyset \). Since \( f \) is weakly \( \delta s \)-continuous, there exists \( \delta \)-semiopen set \( G \) in \( X \) containing \( x \) such that \( f(G) \subset Cl(A) \). Since \( g \) is weakly \( \delta s \)-continuous, there exists a \( \delta \)-semiopen set \( K \) of \( X \) containing \( x \) such that \( g(K) \subset Cl(B) \). Take \( W = G \cap K \). Then \( W \) is \( \delta \)-semiopen containing \( x \) and \( f(W) \cap g(W) \subset Cl(A) \cap Cl(B) = \emptyset \). This implies that \( W \cap \{x \in X : f(x) = g(x)\} = \emptyset \) and hence \( \{x \in X : f(x) = g(x)\} \) is \( \delta \)-semiclosed in \( X \).

Definition 9. A subset \( U \) of a topological space \( X \) is called \( N \)-closed if there exists a finite number of points \( x_1, x_2, \ldots, x_n \) in \( U \) such that \( U \subset \bigcup_{i=1}^{n} Int(Cl(X(x_i))) \), where the family \( \{V(x) \mid x \in U\} \) is a cover of \( U \) by open sets of \( X \).

Theorem 14. Let \( f : X \rightarrow Y \) be a function, where \( \delta SO(X) \) is semiclosed under the finite intersections. If for each \( (x, y) \notin G(f) \), there exist a \( \delta \)-semiopen set \( U \subset X \) and an open set \( V \subset Y \) containing \( x \) and \( y \), respectively, such that \( f(U) \cap Int(Cl(V)) = \emptyset \), then inverse image of each \( N \)-closed set of \( Y \) is \( \delta \)-semiclosed in \( X \).

Proof. Suppose that there exists a \( N \)-closed set \( W \subset Y \) such that \( f^{-1}(W) \) is not \( \delta \)-semiclosed in \( X \). We have a point \( x \in sCl_{\delta}(f^{-1}(W)) \setminus f^{-1}(W) \). Since \( x \notin f^{-1}(W) \), then \( (x, y) \notin G(f) \) for each \( y \in W \). There exist \( \delta \)-semiopen sets \( U_y(x) \subset X \) and an open set \( V(y) \subset Y \) containing \( x \) and \( y \), respectively, such that \( f(U_y(x)) \cap Int(Cl(V(y))) = \emptyset \). The family \( \{V(y) : y \in W \} \) is a cover of \( W \) by open sets of \( Y \). Since \( W \) is \( N \)-closed, there exist a finite number of points \( y_1, y_2, \ldots, y_n \) in \( W \) such that \( W \subset \bigcup_{i=1}^{n} Int(Cl(V(y_i))) \).
Take $U = \cap_{i=1}^{n} U_{y_{i}}(x)$. We have $f(U) \cap W = \emptyset$. Since $x \in Cl_{s}(f^{-1}(W))$, then $f(U) \cap W \neq \emptyset$. This is a contradiction. ■

For a function $f : X \rightarrow Y$, the graph function $g : X \rightarrow X \times Y$ of $f$ is defined by $g(x) = (x, f(x))$ for each $x \in X$.

**Theorem 15.** If the graph function $g$ of a function $f : X \rightarrow Y$ is weakly $\delta s$-continuous, then $f$ is weakly $\delta s$-continuous.

**Proof.** Let $g$ be weakly $\delta s$-continuous and $x \in X$ and $U$ be an open set of $X$ containing $f(x)$. Then $X \times U$ is an open set containing $g(x)$. There exists a $\delta$-semiopen set $V$ containing $x$ such that $g(V) \subset Cl(X \times U) = X \times Cl(U)$. This implies that $f(V) \subset Cl(U)$ and hence $f$ is weakly $\delta s$-continuous. ■

**References**


More on weak $\delta$-continuity . . .

[18] Noiri T., Remarks on $\delta$-semi-open sets and $\delta$-preopen sets, Demonstratio Math., 36(2003), 1007-1020.

Saeid Jafari
College of Vestsjaelland South
Herrestraede 11
4200 Slagelse, Denmark
e-mail: jafaripersia@gmail.com

Takashi Noiri
2949-1 Shiokita-cho
Hinagu, Yatsushiro-shi
Kumamoto-ken, 869-5142 Japan
e-mail: t.noiri@nifty.com

Received on 25.01.2016 and, in revised form, on 12.04.2016.