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2-DISTANCE COLORINGS OF INTEGER DISTANCE GRAPHS

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Abstract

A 2-distance k-coloring of a graph G is a mapping from $V(G)$ to the set of colors $\{1, \ldots, k\}$ such that every two vertices at distance at most 2 receive distinct colors. The 2-distance chromatic number $\chi_2(G)$ of G is then the smallest k for which G admits a 2-distance k-coloring. For any finite set of positive integers $D = \{d_1, \ldots, d_\ell\}$, the integer distance graph $G = G(D)$ is the infinite graph defined by $V(G) = \mathbb{Z}$ and $uv \in E(G)$ if and only if $|v - u| \in D$. We study the 2-distance chromatic number of integer distance graphs for several types of sets D. In each case, we provide exact values or upper bounds on this parameter and characterize those graphs $G(D)$ with $\chi_2(G(D)) = \Delta(G(D)) + 1$.

Keywords: 2-distance coloring, integer distance graph.

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1. Introduction

All the graphs we consider in this paper are simple and loopless undirected graphs. We denote by $V(G)$ and $E(G)$ the set of vertices and the set of edges of a graph G, respectively. For any two vertices u and v of G, we denote by $d_G(u,v)$ (or simply $d(u,v)$ whenever the graph G is clear from the context) the *distance* between u and v, that is the length of a shortest path joining u and v. We denote by $\Delta(G)$ the maximum degree of G.

A (proper) *k-coloring* of a graph G is a mapping from $V(G)$ to the set of colors $\{1,\dots,k\}$ such that every two adjacent vertices receive distinct colors. The smallest k for which G admits a k-coloring is the *chromatic number* of G, denoted $\chi(G)$. A *2-distance k-coloring* of a graph G is a mapping from $V(G)$ to the set of colors $\{1,\dots,k\}$ such that every two vertices at distance at most 2 receive distinct colors. 2-distance colorings are sometimes called *$L(1,1)$-labelings* (see [5] for a survey on $L(h,k)$-labelings) or *square colorings* in the literature. The smallest k for which G admits a 2-distance k-coloring is the *2-distance chromatic number* of G, denoted $\chi_2(G)$.

The *square* G^2 of a graph G is the graph defined by $V(G^2) = V(G)$ and $uv \in E(G^2)$ if and only if $d_G(u,v) \leq 2$. Clearly, a 2-distance coloring of a graph G is nothing but a proper coloring of G^2 and, therefore, $\chi_2(G) = \chi(G^2)$ for every graph G.

The study of 2-distance colorings was initiated by Kramer and Kramer [8] (see also their survey on general distance colorings in [9]). The case of planar graphs has attracted a lot of attention in the literature (see e.g. [1–4, 6, 10, 14]), due to the conjecture of Wegner that suggests an upper bound on the 2-distance chromatic number of planar graphs depending on their maximum degree (see [15] for more details).

In this paper, we study 2-distance colorings of distance graphs. Although several coloring problems have been considered for distance graphs (see [11] for a survey), it seems that 2-distance colorings have not been considered yet. We present in Section 2 a few basic results on the chromatic number and the 2-distance chromatic number of distance graphs. We then consider specific sets D, namely $D = \{1,a\}$, $a \geq 3$ (in Section 3), $D = \{1,a,a+1\}$, $a \geq 3$ (in Section 4), and $D = \{1,\dots,m,a\}$, $2 \leq m < a$ (in Section 5). We finally propose some open problems in Section 6.

2. Preliminaries

Let $D = \{d_1,\dots,d_\ell\}$ be a finite set of positive integers. The *integer distance graph* (simply called *distance graph* in the following) $G = G(D)$ is the infinite graph defined by $V(G) = \mathbb{Z}$ and $uv \in E(G)$ if and only if $|v-u| \in D$. The following proposition follows immediately.

Proposition 1. *For every positive integers $d_1,\dots,d_\ell$ with $\gcd(\{d_1,\dots,d_\ell\}) = p > 1$, the distance graph $G(D)$ has p connected components, each of them being isomorphic to the distance graph $G(D')$ with $D' = \{d_1/p,\dots,d_\ell/p\}$.*

In this situation, we then have $\chi_2(G(D)) = \chi_2(G(D'))$ so that we can always assume $\gcd(D) = 1$ in the following.

It is easy to observe that the square of the distance graph $G(D)$ is also a distance graph, namely the distance graph $G(D^2)$ where

$$D^2 = D \cup \left\{d + d' : d, d' \in D\right\} \cup \left\{d - d' : d, d' \in D,\ d > d'\right\}.$$

For instance, for $D = \{1, 2, 5\}$, we get $D^2 = \{1, 2, 3, 4, 5, 6, 7, 10\}$. Note that if D has cardinality ℓ, then D^2 has cardinality at most $\ell(\ell + 1)$.

As observed in the previous section, $\chi_2(G) = \chi(G^2)$ for every graph G. Therefore, since $(G(D))^2 = G(D^2)$, determining the 2-distance chromatic number of the distance graph $G(D)$ reduces to determining the chromatic number of the distance graph $G(D^2)$. The problem of determining the chromatic number of distance graphs has been extensively studied in the literature. When $|D| \leq 2$, this question is easily solved, thanks to the following general upper bounds.

Proposition 2. *For every finite set of positive integers $D = \{d_1, \ldots, d_\ell\}$ and every positive integer p such that $d_i \not\equiv 0 \pmod{p}$ for every i, $1 \leq i \leq \ell$, $\chi(G(D)) \leq p$.*

Proof. Let $\lambda : V(G(D)) \longrightarrow \{1, \ldots, p\}$ be the mapping defined by

$$\lambda(x) = 1 + (x \mod p),$$

for every integer $x \in \mathbb{Z}$. Since $d_i \not\equiv 0 \pmod{p}$ for every i, $1 \leq i \leq \ell$, the mapping λ is clearly a proper coloring of $G(D)$. ∎

Theorem 3 (Walther [13]). *For every finite set of positive integers D,*

$$\chi(G(D)) \leq |D| + 1.$$

Proof. A $(|D| + 1)$-coloring of $G(D)$ can be easily produced using the First-Fit greedy algorithm, starting from vertex 0, from left to right and then from right to left, since every vertex has exactly $|D|$ neighbors on its left and $|D|$ neighbors on its right. ∎

Therefore, when $|D| \leq 2$, $\chi(G(D)) = 2$ if $|D| = 1$ or all elements in D are odd (since $G(D)$ is then bipartite), and $\chi(G(D)) = 3$ otherwise (since $G(D)$ then contains cycles of odd length). The case $|D| = 3$ has been settled by Zhu [16]. Whenever $|D| \geq 4$, only partial results have been obtained, namely for sets D having specific properties.

Another useful result is the following.

Theorem 4 (Voigt [12], cited in [7]). *For every finite set of positive integers* $D = \{d_1, \ldots, d_\ell\}$,

$$\chi(G(D)) \leq \min_{n \in \mathbb{N}} n(|D_n| + 1),$$

where $D_n = \{d_i : n | d_i,\ 1 \leq i \leq \ell\}$.

A coloring λ of a distance graph $G(D)$ is *p-periodic*, for some integer $p \geq 1$, if $\lambda(x + p) = \lambda(x)$ for every $x \in \mathbb{Z}$. Walther also proved the following.

Theorem 5 (Walther [13]). *For every finite set of positive integers* D*, if* $\chi(G(D)) \leq k$*, then* $G(D)$ *admits a p-periodic k-coloring for some p.*

The sequence $\lambda(x) \cdots \lambda(x+p-1)$ of such a p-periodic coloring λ is called the *pattern* of λ. In particular, the coloring defined in the proof of Proposition 2 was p-periodic with pattern $12 \cdots p$. In the following, we will describe such patterns using standard notation of Combinatorics on words. For instance, the pattern 121212345 will be denoted $(12)^3345$.

Finally, note that in any 2-distance coloring of a graph G, all vertices in the closed neighborhood of any vertex must be assigned distinct colors. Therefore, we have the following.

Observation 6. *For every graph* G, $\chi_2(G) \geq \Delta(G) + 1$.

In particular, this bound is attained by the distance graph $G(D)$ with $D = \{1, \ldots, k\}$, $k \geq 2$.

Proposition 7. *For every* $k \geq 2$,

$$\chi_2(G(\{1, \ldots, k\})) = 2k + 1 = \Delta(G(\{1, \ldots, k\})) + 1.$$

Proof. This directly follows from Theorem 3, since $|\{1, \ldots, k\}^2| = 2k$. ∎

3. The Case $D = \{1, a\}$, $a \geq 3$

We study in this section the 2-distance chromatic number of distance graphs $G(D)$ with $D = \{1, a\}$, $a \geq 3$ (note that the case $a = 2$ is already solved by Proposition 7).

When $D = \{1, a\}$, $a \geq 3$, we have $\Delta(G(D)) = 4$ and

$$D^2 = \{1, 2, a-1, a, a+1, 2a\}.$$

The following theorem gives the 2-distance chromatic number of any such graph.

Theorem 8. *For every integer $a \geq 3$,*

$$\chi_2(G(\{1,a\})) = \begin{cases} 5 & \text{if } a \equiv 2 \pmod 5, \text{ or } a \equiv 3 \pmod 5, \\ 6 & \text{otherwise.} \end{cases}$$

Proof. Since $\{1,a\}^2 = \{1,2,a-1,a,a+1,2a\}$, we get $d \not\equiv 0 \pmod 5$ for every $d \in \{1,a\}^2$ if and only if $a \equiv 2 \pmod 5$ or $a \equiv 3 \pmod 5$ and thus, by Proposition 2 and Observation 6, $\chi_2(G(\{1,a\})) = 5$.

Note that for every $x \in \mathbb{Z}$, the set of vertices

$$C(x) = \{x-a, x-1, x, x+1, x+a\}$$

induces a 5-clique in $G(\{1,a\}^2)$ (see Figure 1). We now claim that every 2-distance 5-coloring λ of $G(\{1,a\})$ is necessarily 5-periodic, that is $\lambda(x+5) = \lambda(x)$ for every $x \in \mathbb{Z}$. To show that, it suffices to prove that any five consecutive vertices $x, \ldots, x+4$ must be assigned distinct colors. Assume to the contrary that this is not the case and, without loss of generality, let $x = 0$. Since vertices 0, 1 and 2 necessarily get distinct colors, we only have to consider two cases.

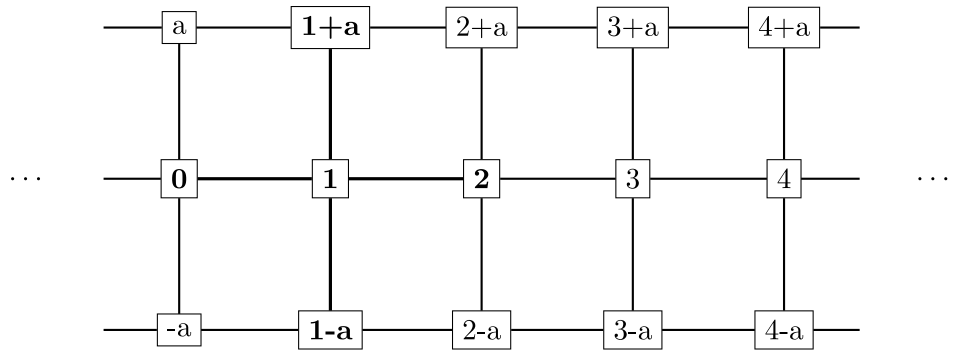

Figure 1. Subgraph of the distance graph $G(\{1,a\})$, $a \geq 3$.

Case 1. $\lambda(0) = \lambda(3) = 1$, $\lambda(1) = 2$, $\lambda(2) = 3$. Since $C(1)$ induces a 5-clique in $G(\{1,a\}^2)$ (depicted in bold in Figure 1), we have

$$\{\lambda(1-a), \lambda(1+a)\} = \{4,5\},$$

which implies

$$\{\lambda(2-a), \lambda(2+a)\} = \{4,5\}.$$

(More precisely, $\lambda(2-a) = 9 - \lambda(1-a)$ and $\lambda(2+a) = 9 - \lambda(1+a)$). This implies $\lambda(3-a) = \lambda(3+a) = 2$, a contradiction since $d(3-a, 3+a) = 2$.

Case 2. $\lambda(0) = \lambda(4) = 1$, $\lambda(1) = 2$, $\lambda(2) = 3$, $\lambda(3) = 4$. As in the previous case we have

$$\{\lambda(1-a), \lambda(1+a)\} = \{4, 5\},$$

which implies

$$\{\lambda(2-a), \lambda(2+a)\} = \{1, 5\}.$$

We then get $\lambda(3-a) = \lambda(3+a) = 2$, again a contradiction.

Therefore, $\chi_2(G(\{1,a\})) = 5$ if and only if 5 does not divide any element of $\{1,a\}^2 = \{1, 2, a-1, a, a+1, 2a\}$. This is clearly the case if and only if $a \equiv 2 \pmod 5$ or $a \equiv 3 \pmod 5$.

We finally prove that there exists a 2-distance 6-coloring of $G(\{1,a\})$ for any value of a. We consider three cases, according to the value of a (mod 3).

Case 1. $a = 3k$, $k \geq 1$. Let λ be the $(2a-1)$-periodic mapping defined by the pattern

$$(123)^k(456)^{k-1}45.$$

If $\lambda(x) = \lambda(y) = c$, $1 \leq c \leq 5$, then

$$|x-y| \in \{3q,\ 0 \leq q \leq k-1\} \ \cup\ \{(2a-1)p + 3q,\ p \geq 1,\ 1-k \leq q \leq k-1\}.$$

If $\lambda(x) = \lambda(y) = 6$ (which occurs if and only if $k \geq 2$), then

$$|x-y| \in \{3q,\ 0 \leq q \leq k-2\} \ \cup\ \{(2a-1)p + 3q,\ p \geq 1,\ 2-k \leq q \leq k-2\}.$$

Therefore, in both cases, $|x-y| \notin \{1, 2, a-1, a, a+1, 2a\}$, and thus λ is a 2-distance 6-coloring of $G(\{1,a\})$.

Case 2. $a = 3k+1$, $k \geq 1$. In that case, the result follows from Theorem 4 (taking $n = 3$), since the only element divisible by 3 in $\{1, 2, a-1, a, a+1, 2a\}$ is $a-1$.

Case 3. $a = 3k+2$, $k \geq 1$. Again, the result follows from Theorem 4 (taking $n = 3$), since the only element divisible by 3 in $\{1, 2, a-1, a, a+1, 2a\}$ is $a+1$.

This concludes the proof. ■

4. The Case $D = \{1, a, a+1\}$, $a \geq 3$

We study in this section the 2-distance chromatic number of distance graphs $G(D)$ with $D = \{1, a, a+1\}$, $a \geq 3$ (note that the case $a = 2$ is already solved by Proposition 7).

When $D = \{1, a, a+1\}$, $a \geq 3$, we have $\Delta(G(D)) = 6$ and

$$D^2 = \{1, 2, a-1, a, a+1, a+2, 2a, 2a+1, 2a+2\}.$$

First we prove the following.

Theorem 9. *For every integer a, $a \geq 3$,*

$$\chi_2(G(\{1, a, a+1\})) = 7 = \Delta(G(\{1, a, a+1\})) + 1$$

if and only if $a \equiv 2 \pmod 7$ *or* $a \equiv 4 \pmod 7$.

Proof. Since $\{1, a, a+1\}^2 = \{1, 2, a-1, a, a+1, a+2, 2a, 2a+1, 2a+2\}$, we get $d \not\equiv 0 \pmod 7$ for every $d \in \{1, a, a+1\}^2$ if and only if $a \equiv 2 \pmod 7$ or $a \equiv 4 \pmod 7$ and thus, by Proposition 2 and Observation 6, $\chi_2(G(\{1, a, a+1\})) = 7$.

Note that for every $x \in \mathbb{Z}$, the set of vertices

$$C(x) = \{x-a-1, x-a, x-1, x, x+1, x+a, x+a+1\}$$

induces a 7-clique in $G(\{1, a, a+1\}^2)$. We now claim that every 2-distance 7-coloring λ of $G(\{1, a, a+1\})$ is necessarily 7-periodic, that is $\lambda(x+7) = \lambda(x)$ for every $x \in \mathbb{Z}$. To show that, it suffices to prove that any 7 consecutive vertices $x, \ldots, x+6$ must be assigned distinct colors. Assume to the contrary that this is not the case and, without loss of generality, let $x = 0$. Since vertices 0, 1 and 2 necessarily get distinct colors, we only have to consider four cases (see Figure 2).

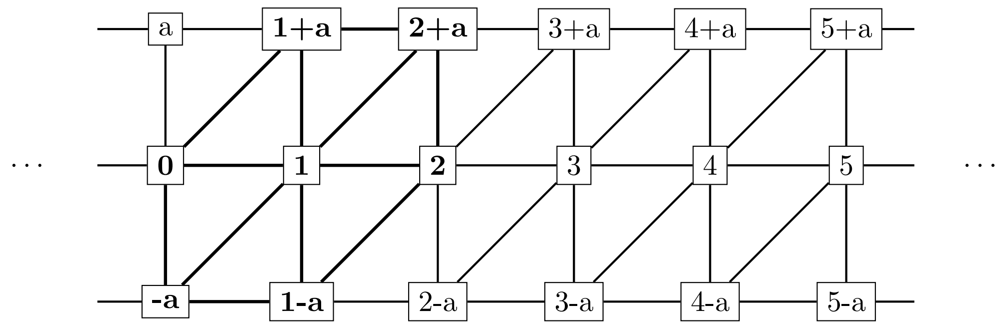

Figure 2. Subgraph of the distance graph $G(\{1, a, a+1\})$, $a \geq 3$.

Case 1. Vertices $0, 1, 2, 3$ are colored with the colors $1, 2, 3$ and 1, respectively. We consider two subcases.

Subcase (a) $\lambda(4) = 2$. Since $C(1)$ induces a 7-clique in $G(\{1, a, a+1\}^2)$ (depicted in bold in Figure 2), we have

$$\{\lambda(-a), \lambda(1-a), \lambda(1+a), \lambda(2+a)\} = \{4, 5, 6, 7\}.$$

Since $C(3)$ is also a 7-clique, we also have

$$\{\lambda(2-a), \lambda(3-a), \lambda(3+a), \lambda(4+a)\} = \{4, 5, 6, 7\}.$$

This implies $\lambda(-a) = \lambda(4-a)$ or $\lambda(1+a) = \lambda(5+a)$. Each of these cases thus corresponds to Case 2 below.

Subcase (b) $\lambda(4) \neq 2$. Note that we necessarily have $\lambda(4) \neq 3$ and $\lambda(4) \neq 1$, since vertex 4 is at distance 2 and 1 from vertices 2 and 3, respectively. We can thus assume $\lambda(4) = 4$, without loss of generality. Since $d(5,4) = 1$ and $d(5,3) = 2$, we have $\lambda(5) \notin \{1,4\}$. Moreover, if $\lambda(5) = 2$, we get $\lambda(2) = \lambda(5)$, which corresponds to Case 2 below. We can thus suppose either $\lambda(5) = 3$ or $\lambda(5) > 4$, say $\lambda(5) = 5$ without loss of generality. We consider these two cases separately.

(i) $\lambda(5) = 3$. In that case, we necessarily have

$$\{\lambda(-a), \lambda(1+a)\} \subseteq \{4,5,6,7\},\ \{\lambda(1-a), \lambda(2+a)\} \subseteq \{4,5,6,7\},$$
$$\{\lambda(2-a), \lambda(3+a)\} \subseteq \{5,6,7\},\ \{\lambda(3-a), \lambda(4+a)\} \subseteq \{2,5,6,7\},$$
$$\{\lambda(4-a), \lambda(5+a)\} \subseteq \{5,6,7\}.$$

By setting $\{x,y,z\} = \{5,6,7\}$, we get

$$\{\lambda(-a), \lambda(1+a)\} = \{x,y\},\ \{\lambda(1-a), \lambda(2+a)\} = \{4,z\},$$
$$\{\lambda(2-a), \lambda(3+a)\} = \{x,y\},\ \{\lambda(3-a), \lambda(4+a)\} = \{2,z\},$$
$$\{\lambda(4-a), \lambda(5+a)\} = \{x,y\}.$$

Since $\lambda(-a), \lambda(2-a), \lambda(4-a) \in \{x,y\}$ and $\lambda(-a) \neq \lambda(2-a)$, $\lambda(2-a) \neq \lambda(4-a)$, it follows that $\lambda(-a) = \lambda(4-a)$. That case corresponds to Case 2 below.

(ii) $\lambda(5) = 5$. In that case, we necessarily have

$$\{\lambda(-a), \lambda(1+a)\} \subseteq \{4,5,6,7\},\ \{\lambda(1-a), \lambda(2+a)\} \subseteq \{4,5,6,7\},$$
$$\{\lambda(2-a), \lambda(3+a)\} \subseteq \{5,6,7\},\ \{\lambda(3-a), \lambda(4+a)\} \subseteq \{2,6,7\},$$
$$\{\lambda(4-a), \lambda(5+a)\} \subseteq \{3,6,7\}.$$

By setting $\{x,y\} = \{6,7\}$, we get

$$\{\lambda(-a), \lambda(1+a)\} = \{5,x\},\ \{\lambda(1-a), \lambda(2+a)\} = \{4,y\},$$
$$\{\lambda(2-a), \lambda(3+a)\} = \{5,x\},\ \{\lambda(3-a), \lambda(4+a)\} = \{2,y\},$$
$$\{\lambda(4-a), \lambda(5+a)\} = \{3,x\}.$$

We then necessarily have either $\lambda(1+a) = \lambda(5+a)$ or $\lambda(-a) = \lambda(4-a)$ and, in both cases, we are in the situation of Case 2 below.

Case 2. Vertices $0,1,2,3,4$ are colored with the colors $1,2,3,4$ and 1, respectively. Again considering the 7-cliques $C(1)$, $C(2)$ and $C(3)$ in $G(\{1,a,a+1\}^2)$, we get

$$\{\lambda(1-a), \lambda(2+a)\} \subseteq \{5,6,7\},$$

and

$$\{\lambda(2-a), \lambda(3+a)\} \subseteq \{5,6,7\},$$

a contradiction, since vertices $1-a$, $2-a$, $a+2$ and $a+3$ induce a 4-clique in $G(\{1,a,a+1\}^2)$.

Case 3. Vertices $0,1,2,3,4,5$ are colored with the colors $1,2,3,4,5$ and 1, respectively. Considering the 7-cliques $C(1)$, $C(2)$ and $C(3)$ in $G(\{1,a,a+1\}^2)$, we get

$$\{\lambda(-a),\lambda(1-a),\lambda(1+a),\lambda(2+a)\}=\{4,5,6,7\},$$
$$\{\lambda(2-a),\lambda(3+a)\}\subseteq\{1,\lambda(-a),\lambda(1+a)\}\setminus\{4,5\},$$
$$\{\lambda(3-a),\lambda(4+a)\}\subseteq\{2,\lambda(1-a),\lambda(2+a)\}\setminus\{4,5\},$$

and thus

$$\{\lambda(2-a),\lambda(3+a)\}\subseteq\{1,6,7\}\quad\text{and}\quad\{\lambda(3-a),\lambda(4+a)\}\subseteq\{2,6,7\}.$$

Assuming that none of Cases 1 or 2 occurs, we have to consider two subcases.

Subcase (a) $\lambda(6)=2$. Considering the 7-clique $C(4)$ in $G(\{1,a,a+1\}^2)$, we get

$$\{\lambda(4-a),\lambda(5+a)\}\subseteq\{3,\lambda(2-a),\,\lambda(3+a)\}\setminus\{1,2\}=\{3,6,7\}.$$

If $\{\lambda(4-a),\lambda(5+a)\}=\{3,6\}$, then

$$\{\lambda(3-a),\lambda(4+a)\}=\{2,7\},$$
$$\{\lambda(2-a),\lambda(3+a)\}=\{1,6\},$$
$$\{\lambda(1-a),\lambda(2+a)\}=\{5,7\}$$

and

$$\{\lambda(-a),\lambda(1+a)\}=\{4,6\}.$$

If $\lambda(-a)=6$, then $\lambda(2-a)=1$ and thus $\lambda(4-a)=\lambda(-a)=6$ which corresponds to Case 2. If $\lambda(1+a)=6$, then $\lambda(3+a)=1$ and thus $\lambda(5+a)=\lambda(1+a)=6$ which again corresponds to Case 2.

The case $\{\lambda(4-a),\lambda(5+a)\}=\{3,7\}$ is similar and leads to the same conclusion.

Finally, if $\{\lambda(4-a),\lambda(5+a)\}=\{6,7\}$, then $\lambda(3-a)=\lambda(4+a)=2$, a contradiction since $d(3-a,4+a)=2$.

Subcase (b) $\lambda(6)=6$. Considering the 7-clique $C(4)$ in $G(\{1,a,a+1\}^2)$, we get

$$\{\lambda(4-a),\lambda(5+a)\}\subseteq\{3,\lambda(2-a),\lambda(3+a)\}\setminus\{1,6\}=\{3,7\}.$$

This implies

$$\{\lambda(3-a),\lambda(4+a)\}=\{2,6\},$$
$$\{\lambda(2-a),\lambda(3+a)\}=\{1,7\},$$
$$\{\lambda(1-a),\lambda(2+a)\}=\{5,6\}$$

and

$$\{\lambda(-a), \lambda(1+a)\} = \{4, 7\}.$$

If $\lambda(-a) = 7$, then $\lambda(2-a) = 1$ and thus $\lambda(4-a) = \lambda(-a) = 7$ which corresponds to Case 2. If $\lambda(1+a) = 7$, then $\lambda(3+a) = 1$ and thus $\lambda(5+a) = \lambda(1+a) = 7$ which again corresponds to Case 2.

Case 4. Vertices $0, 1, 2, 3, 4, 5, 6$ are colored with the colors $1, 2, 3, 4, 5, 6$ and 1, respectively. Again considering the 7-cliques $C(1)$, $C(2)$ and $C(3)$ in $G(\{1, a, a+1\}^2)$, we get

$$\{\lambda(-a), \lambda(1-a), \lambda(1+a), \lambda(2+a)\} = \{4, 5, 6, 7\},$$
$$\{\lambda(2-a), \lambda(3+a)\} \subseteq \{1, \lambda(-a), \lambda(1+a)\} \setminus \{4, 5\},$$

and thus

$$\{\lambda(3-a), \lambda(4+a)\} \subseteq \{2, \lambda(1-a), \lambda(2+a)\} \setminus \{4, 5, 6\} = \{2, 7\}.$$

This implies

$$\{\lambda(2-a), \lambda(3+a)\} = \{1, 6\},$$
$$\{\lambda(1-a), \lambda(2+a)\} = \{5, 7\}$$

and

$$\{\lambda(-a), \lambda(1+a)\} = \{4, 6\}.$$

Therefore,

$$\{\lambda(4-a), \lambda(5+a)\} \subseteq \{3, \lambda(2-a), \lambda(3+a)\} \setminus \{1, 6\} = \{3\},$$

a contradiction since $d(4-a, 5+a) = 2$.

Therefore, every 2-distance 7-coloring λ of $G(\{1, a, a+1\})$ is necessarily 7-periodic, and thus $\chi_2(G(\{1, a, a+1\})) = 7$ if and only if 7 does not divide any element of $\{1, 2, a-1, a, a+1, a+2, 2a, 2a+1, 2a+2\}$. This is clearly the case if and only if $a \equiv 2 \pmod 7$ or $a \equiv 4 \pmod 7$. ■

The following result provides an upper bound on $\chi_2(G(\{1, a, a+1\}))$ for any value of a.

Theorem 10. *For every integer a, $a \geq 3$,*

$$\chi_2(G(\{1, a, a+1\})) \leq 9 = \Delta(G(\{1, a, a+1\})) + 3.$$

Proof. First recall that $\{1, a, a+1\}^2 = \{1, 2, a-1, a, a+1, a+2, 2a, 2a+1, 2a+2\}$. We consider three cases, according to the value of a (mod 3).

Case 1. $a = 3k$, $k \geq 1$. Since the only elements divisible by 3 in $\{1, a, a+1\}^2$ are a and $2a$, the result follows by Theorem 4 (taking $n = 3$).

Case 2. $a = 3k+1$, $k \geq 1$. Let λ be the $(3a+2)$-periodic mapping defined by the pattern

$$(123)^k(456)^k7123(789)^{k-1}4568.$$

If $\lambda(x) = \lambda(y) = c$, $1 \leq c \leq 6$, then

$$\begin{aligned}|x-y| \in\ & \{3q,\ 0 \leq q \leq k-1\}\\ &\cup \{3q+2a-1,\ 1-k \leq q \leq 0\}\\ &\cup \{(3a+2)p+2a-1,\ p>0\}\\ &\cup \{(3a+2)p-2a+1,\ p>0\}\\ &\cup \{(3a+2)p+3q,\ p>0,\ 1-k \leq q < 0\}\\ &\cup \{(3a+2)p+3q+2a-1,\ p>0,\ 1-k \leq q < 0\}\\ &\cup \{(3a+2)p+3q,\ p>0,\ 0<q \leq k-1\}\\ &\cup \{(3a+2)p+3q-2a+1,\ p>0,\ 0<q \leq k-1\}.\end{aligned}$$

If $\lambda(x) = \lambda(y) = 7$ (which occurs if and only if $k \geq 2$), then

$$\begin{aligned}|x-y| \in\ & \{3q,\ 0 \leq q \leq k-2\}\\ &\cup \{3q+4,\ 0 \leq q \leq k-2\}\\ &\cup \{(3a+2)p+3q-4,\ p>0,\ 2-k \leq q \leq 0\}\\ &\cup \{(3a+2)p+3q+4,\ p>0,\ 0 \leq q \leq k-2\}\\ &\cup \{(3a+2)p+3q,\ p>0,\ 2-k \leq q \leq k-2\}.\end{aligned}$$

If $\lambda(x) = \lambda(y) = 8$ (which occurs if and only if $k \geq 2$), then

$$\begin{aligned}|x-y| \in\ & \{3q,\ 0 \leq q \leq k-2\}\\ &\cup \{3q+a-2,\ 2-k \leq q \leq 0\}\\ &\cup \{(3a+2)p+a-2,\ p>0\}\\ &\cup \{(3a+2)p-a+2,\ p>0\}\\ &\cup \{(3a+2)p+3q,\ p>0,\ 2-k \leq q < 0\}\\ &\cup \{(3a+2)p+3q+a-2,\ p>0,\ 2-k \leq q < 0\}\\ &\cup \{(3a+2)p+3q,\ p>0,\ 0<q \leq k-2\}\\ &\cup \{(3a+2)p+3q-a+2,\ p>0,\ 0<q \leq k-2\}.\end{aligned}$$

If $\lambda(x) = \lambda(y) = 9$ (which occurs if and only if $k \geq 2$), then

$$|x-y| \in \{3q,\ 0 \leq q \leq k-2\} \ \cup\ \{(3a+2)p+3q,\ p \geq 1,\ 2-k \leq q \leq k-2\}.$$

Therefore, in all these cases, $|x-y| \notin \{1, 2, a-1, a, a+1, a+2, 2a, 2a+1, 2a+2\}$, and thus λ is a 2-distance 9-coloring of $G(\{1, a, a+1\})$.

Case 3. $a = 3k+2$, $k \geq 1$. Since the only elements divisible by 3 in $\{1, a, a+1\}^2$ are $a+1$ and $2a+2$, the result follows by Theorem 4 (taking $n = 3$).

This concludes the proof. ∎

From Theorems 9 and 10, we thus get the following.

Corollary 11. *For every integer a, $a \geq 3$, $a \not\equiv 2, 4 \pmod 7$,*

$$8 \leq \chi_2(G(\{1, a, a+1\})) \leq 9.$$

5. The Case $D = \{1, \ldots, m, a\}$, $2 \leq m < a$

We study in this section the 2-distance chromatic number of distance graphs $G(D)$ with $D = \{1, \ldots, m, a\}$, $2 \leq m < a$ (note that the case $a = m+1$ is already solved by Proposition 7).

When $D = \{1, \ldots, m, a\}$, we have $\Delta(G(D)) = 2m + 2$ and

$$D^2 = \{1, 2, \ldots, 2m\} \cup \{a - m, a - m + 1, \ldots, a + m\} \cup \{2a\}.$$

First we prove the following.

Theorem 12. *For all integers m and a, $2 \leq m < a$,*

$$\chi_2(G(\{1, \ldots, m, a\})) = 2m + 3 = \Delta(G(\{1, \ldots, m, a\})) + 1$$

if and only if $a \equiv m + 1 \pmod{2m+3}$ or $a \equiv m + 2 \pmod{2m+3}$.

Proof. Since $\{1, \ldots, m, a\}^2 = \{1, \ldots, 2m\} \cup \{a - m, a - m + 1, \ldots, a + m\} \cup \{2a\}$, we have $d \not\equiv 0 \pmod{2m+3}$ for every $d \in \{1, \ldots, m, a\}^2$ if and only if $a \equiv m+1 \pmod{2m+3}$ or $a \equiv m + 2 \pmod{2m+3}$, and thus, by Proposition 2 and Observation 6, $\chi_2(G(\{1, \ldots, m, a\})) = 2m + 3$.

We now claim that every 2-distance $(2m + 3)$-coloring λ of $G(\{1, \ldots, m, a\})$ is necessarily $(2m+3)$-periodic, that is $\lambda(x + 2m + 3) = \lambda(x)$ for every $x \in \mathbb{Z}$. To show that, it suffices to prove that any $2m+3$ consecutive vertices $x, \ldots, x+2m+2$ must be assigned distinct colors. Assume to the contrary that this is not the case and, without loss of generality, let $x = 0$. Since vertices $0, 1, \ldots, 2m$ necessarily get distinct colors, we only have to consider two cases.

Case 1. Vertices $0, 1, \ldots, 2m + 1$ are colored with the colors $1, 2, \ldots, 2m + 1$ and 1, respectively. Note that vertices $m - a$ and $m + a$ are both adjacent to all vertices $0, 1, \ldots, 2m$. Hence,

$$\{\lambda(m - a), \lambda(m + a)\} = \{2m + 2, 2m + 3\},$$

which implies

$$\{\lambda(m + 1 - a), \lambda(m + 1 + a)\} = \{2m + 2, 2m + 3\}$$

(more precisely, $\lambda(m+1-a) = 4m+5-\lambda(m-a)$ and $\lambda(m+1+a) = 4m+5-\lambda(m+a)$). This implies $\lambda(m+2-a) = \lambda(m+2+a) = 2$, a contradiction since $d(m+2-a, m+2+a) = 2$.

Case 2. Vertices $0, 1, \ldots, 2m+2$ are colored with the colors $1, 2, \ldots, 2m+2$ and 1, respectively. As in the previous case we have

$$\{\lambda(m-a), \lambda(m+a)\} = \{2m+2, 2m+3\},$$

which implies

$$\{\lambda(m+1-a), \lambda(m+1+a)\} = \{1, 2m+3\}.$$

We thus get $\lambda(m+2-a) = \lambda(m+2+a) = 2$, again a contradiction.

Therefore, every 2-distance $(2m+3)$-coloring λ of $G(\{1, \ldots, m, a\})$ is necessarily $(2m+3)$-periodic, and thus $\chi_2(G(\{1, \ldots, m, a\})) = 2m+3$ if and only if $2m+3$ does not divide any element of $\{1, 2, \ldots, 2m\} \cup \{a-m, a-m+1, \ldots, a+m\} \cup \{2a\}$. This is clearly the case if and only if $a \equiv m+1 \pmod{2m+3}$ or $a \equiv m+2 \pmod{2m+3}$. ■

For other values of a, we propose the following general upper bound.

Theorem 13. *For all integers m and a, $2 \leq m < a$,*

$$\chi_2(G(\{1, \ldots, m, a\})) \leq 4m+2 = 2\Delta(G(\{1, \ldots, m, a\})) - 2.$$

Proof. First note that $\{1, \ldots, m, a\}^2 = \{1, \ldots, 2m\} \cup \{a-m, \ldots, a+m\} \cup \{2a\}$. Therefore, if $2m+1$ does not divide a, then the set $\{1, \ldots, m, a\}^2$ contains only one element e divisible by $2m+1$ (with $e \in \{a-m, \ldots, a+m\}$). In that case, the result follows by Theorem 4 (taking $n = 2m+1$).

Suppose now that $a = k(2m+1)$, with $k \geq 1$. Let λ be the $(2a-m)$-periodic mapping defined by the pattern

$$[12\cdots(2m+1)]^k[(2m+2)(2m+3)\cdots(4m+2)]^{k-1}(2m+2)(2m+3)\cdots(3m+2).$$

If $\lambda(x) = \lambda(y) = c$, $1 \leq c \leq 3m+2$, then

$$\begin{aligned} |x-y| \in &\{q(2m+1),\ 0 \leq q \leq k-1\} \\ &\cup \{p(2a-m) + q(2m+1),\ p \geq 1,\ 1-k \leq q \leq k-1\}. \end{aligned}$$

If $\lambda(x) = \lambda(y) = c$, $3m+3 \leq c \leq 4m+2$ (which occurs if and only if $k \geq 2$), then

$$\begin{aligned} |x-y| \in &\{q(2m+1),\ 0 \leq q \leq k-2\} \\ &\cup \{p(2a-m) + q(2m+1),\ p \geq 1,\ 2-k \leq q \leq k-2\}. \end{aligned}$$

Therefore, in both cases, $|x-y| \notin \{1, \ldots, m, a\}^2$, and thus λ is a 2-distance $(4m+2)$-coloring of $G(\{1, \ldots, m, a\})$. This concludes the proof. ■

From Theorems 12 and 13, we thus get the following.

Corollary 14. *For all integers m and a, $2 \leq m < a$, $a \not\equiv m+1, m+2 \pmod{2m+3}$,*

$$2m + 4 \leq \chi_2(G(\{1, \ldots, m, a\})) \leq 4m + 2.$$

6. Discussion

In this paper, we studied 2-distance colorings of several types of distance graphs. In each case, we characterized those distance graphs that admit an optimal 2-distance coloring, that is distance graphs $G(D)$ with $\chi_2(G(D)) = \Delta(G(D)) + 1$. We also provided general upper bounds for the 2-distance chromatic number of the considered graphs. Note here that all our results can be extended to a larger class of integer distance graphs, thanks to Proposition 1, by multiplying all the elements of the set D by the same constant $k > 1$.

We leave as open problems the question of completely determining the 2-distance chromatic number of distance graphs $G(D)$ when $D = \{1, a, a + 1\}$, $a \geq 3$, or $D = \{1, \ldots, m, a\}$, $2 \leq m < a$.

Considering other types of sets D would certainly be also an interesting direction for future research.

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