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THE CROSSING NUMBER OF THE HEXAGONAL GRAPH $H_{3,n}$

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Abstract

In [C. Thomassen, *Tilings of the torus and the Klein bottle and vertex-transitive graphs on a fixed surface*, Trans. Amer. Math. Soc. 323 (1991) 605–635], Thomassen described completely all (except finitely many) regular tilings of the torus S_1 and the Klein bottle N_2 into (3,6)-tilings, (4,4)-tilings and (6,3)-tilings. Many authors made great efforts to investigate the crossing number (in the plane) of the Cartesian product of an m-cycle and an n-cycle, which is a special (4,4)-tiling. For other tilings, there are quite rare results concerning on their crossing numbers. This motivates us in the paper to determine the crossing number of a hexagonal graph $H_{3,n}$, which is a special kind of (3,6)-tilings.

Keywords: hexagonal graph, Cartesian product, crossing number, drawing.

2010 Mathematics Subject Classification: 05C10, 05C62.

1. Introduction

In [13], Thomassen described completely all (except finitely many) regular tilings of the torus S_1 and the Klein bottle N_2 into hexagons, quadrilaterals and triangles in which the vertices have degree 3, 4 and 6, respectively. To be more specific, let G be a connected d-regular graph ($d \geq 3$) and φ a collection of m-cycles in G, assume that each edge of G is contained in precisely two cycles in φ and that, for each vertex v in G, the edges incident with v can be labelled $e_1, e_2, \ldots, e_d$ such that for each $i = 1, 2, \ldots, d$, there is a cycle in φ containing e_i and e_{i+1} (where $e_{d+1} = e_1$). Then a surface S can be obtained by letting the cycles of φ be disjoint convex polygons in the Euclidean plane pasted together by the graph G, and G is said to be a (d, m)-tiling of S. Using Euler's formula, Thomassen observed that a regular tiling of the torus or the Klein bottle fit into three categories: (3,6)-tilings, (4,4)-tilings and (6,3)-tilings.

Note that the Cartesian product of an m-cycle and an n-cycle, denoted by $C_m \square C_n$, is a special kind of (4,4)-tilings. It is well known that $C_m \square C_n$ can be embedded in the torus whose genus is 1, but cannot be embedded in the plane. Therefore, many authors made great efforts to determine the crossing number of $C_m \square C_n$ in the plane. However, determining the crossing number of graphs is a tedious problem [6], and only very few families of graphs whose crossing number are known [3, 4, 8, 9, 14]. According to its difficulty, it is not surprising that there are very few exact results concerning on the crossing number of $C_m \square C_n$ [1, 2, 7, 10, 12].

For other regular tilings, to the best of our knowledge, there are quite rare results focus on determining their crossing numbers in the plane. Therefore, this arises our intensive interest in studying the problem, and this contribution is devoted to determine the crossing number of $H_{3,n}$, which is a special kind of (3,6)-tilings.

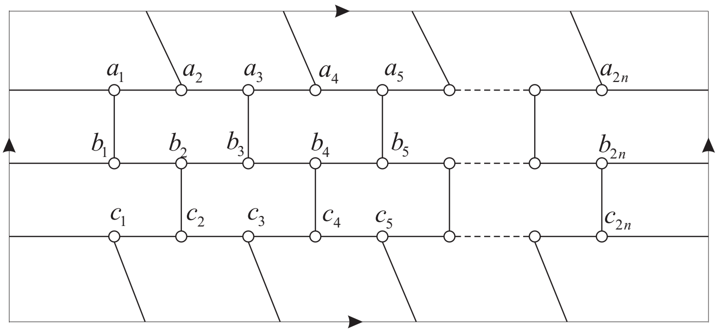

Figure 1. The embedding of the hexagonal graph $H_{3,n}$ of breadth three and length n ($n \geq 3$) in the torus.

2. DEFINITIONS

We shall introduce some basic definitions in this section.

All graphs considered here are finite, simple and connected. Let G be a graph with vertex set V and edge set E. The *crossing number* $cr(G)$ of a graph G is the minimum number of pairwise intersections of edges in a drawing of G in the plane. It is well known that the crossing number of a graph is attained only in *good drawings* of the graph, which are the drawings where no edge crosses itself, no adjacent edges cross each other, no two edges intersect more than once, and no three edges have a common point. Let D be a good drawing of the graph G, we denote the number of crossings in D by $cr_D(G)$. If D is a good drawing of G satisfying $cr_D(G) = cr(G)$, then D is an *optimal drawing* of G. In a drawing D, if an edge is not crossed by any other edge, we say that it is *clean* in D, otherwise, we say it is *crossed*. For definitions not explained here, readers are referred to [5].

Figure 1 shows the embedding of the hexagonal graph $H_{3,n}$ of breadth three and length n ($n \geq 2$) in the torus, it is seen that the number of 6-cycles in the meridional (respectively, longitudinal) direction is three (respectively, n). To be more specific, $H_{3,n}$ is the graph with vertex set $V(H_{3,n}) = \{a_i, b_i, c_i \,:\, i = 1, 2, \ldots, 2n\}$, and edge set $E(H_{3,n}) = \{a_i a_{i+1}, b_i b_{i+1}, c_i c_{i+1} \,:\, i = 1, 2, \ldots, 2n\} \cup \{a_{2i-1} b_{2i-1}, b_{2i} c_{2i}, c_{2i-1} a_{2i} \,:\, i = 1, 2, \ldots, n\}$. The indices are expressed modulo $2n$. See Figure 1. Clearly, the hexagonal graph $H_{3,n}$ is 3-regular, and can be viewed as by pasting together with 6-cycles. Thus, $H_{3,n}$ is a special (3,6)-tiling.

Deleting all the edges $c_{2i-1} a_{2i}$ ($i = 1, 2, \ldots, n$) from Figure 1, the resulted graph is the *hexagonal cylinder* of breadth 2 and length n. In the hexagonal cylinder, two cycles $a_1 a_2 \cdots a_{2n} a_1$ and $c_1 c_2 \cdots c_{2n} c_1$ are called *peripheral cycles*.

The hexagonal graph $H_{m,n}$ of breadth m and length n can be defined as: let $a_1 a_2 \cdots a_{2n} a_1$ and $c_1 c_2 \cdots c_{2n} c_1$ be two peripheral cycles of the hexagonal cylinder of breadth $m-1$ and length n, $H_{m,n}$ is obtained from the hexagonal cylinder of breadth $m-1$ and length n by adding all the edges $a_{2i} c_{2i-1}$ ($i = 1, 2, \ldots, n$) when m is odd, and by adding all the edges $a_{2i} c_{2i}$ ($i = 1, 2, \ldots, n$) when m is even. Figure 2 is an embedding of $H_{m,n}$ in the torus when m is even and $m \geq 4$.

It is easy to see that $H_{2,n}$ is planar, therefore, we begin to investigate the crossing number of $H_{m,n}$ for $m = 3$, and get the main result.

Theorem 1. *For $n \geq 2$, $cr(H_{3,n}) = n$.*

3. THE PROOF OF THEOREM 1

We shall proceed our proof of Theorem 1 by induction on n. The base case is $n = 2$, which needs to be discussed firstly.

Lemma 2. $cr(H_{3,2}) = 2$.

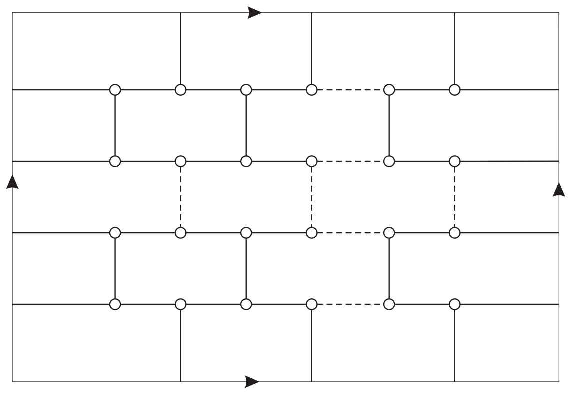

Figure 2. The embedding of $H_{m,n}$ in the torus for m is even and $m \geq 4$.

Proof. Figure 3 shows a good drawing of $H_{3,2}$ in the plane, which indicates that $cr(H_{3,2}) \leq 2$. We prove the reverse inequality by assuming to the contrary that there is a good drawing D of $H_{3,2}$ with fewer than 2 crossings, then $cr_D(H_{3,2}) = 1$ since $H_{3,2}$ contains a subdivision of $K_{3,3}$ whose crossing number is 1 [11], see Figure 4. Thus, a planar graph can be obtained from D by removing one of the crossed edge. Nevertheless, one can testify that, for any $e \in E(H_{3,2})$, $H_{3,2} - e$ contains a subdivision of $K_{3,3}$. This contradiction completes the proof. ■

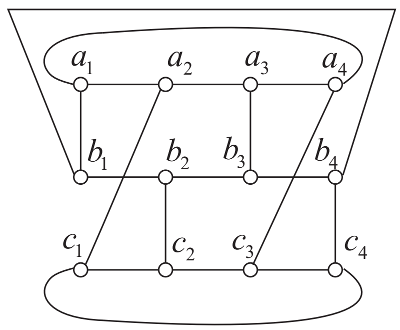

Figure 3. A good drawing of $H_{3,2}$ in the plane.

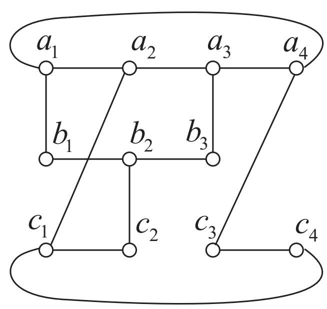

Figure 4. A subdivision of $K_{3,3}$.

For $1 \leq i \leq n$, let $F_i = \{a_{2i-2}a_{2i-1}, a_{2i-1}a_{2i}, b_{2i-2}b_{2i-1}, b_{2i-1}b_{2i}, c_{2i-2}c_{2i-1}, c_{2i-1}c_{2i}, a_{2i-1}b_{2i-1}, b_{2i}c_{2i}, c_{2i-1}a_{2i}\}$, the indices are read modulo $2n$. Then $F_1, F_2, \ldots, F_n$ is a partition of $E(H_{3,n})$, that is to say, $E(H_{3,n}) = \bigcup_{i=1}^{n} F_i$, and $F_i \cap F_j = \emptyset$ for $i \neq j$.

Let D be a good drawing of $H_{3,n}$, we define $f_D(F_i)$ $(1 \leq i \leq n)$ to be the function counting the number of crossings related to F_i in D as follows:

$$f_D(F_i) = cr_D(F_i, F_i) + \frac{1}{2} \sum_{1 \le j \le n,\, j \ne i} cr_D(F_i, F_j).$$

By counting the number of crossings in D, we can get

Lemma 3. $cr_D(H_{3,n}) = \sum_{i=1}^{n} f_D(F_i)$.

Lemma 4. $cr(H_{3,n}) \ge n$ *for* $n \ge 2$.

Proof. We prove the lemma by induction on n. Lemma 2 enforces the inequality holds for $n = 2$. Suppose that $cr(H_{3,k}) \ge k$ for $k < n$, and that there exists a good drawing D of $H_{3,n}$ satisfying $cr_D(H_{3,n}) < n$. Together with our assumption, it has $cr_D(H_{3,n}) = n - 1$ since $H_{3,n}$ contains a subdivision of $H_{3,n-1}$.

Let $E_0 = \{a_{2i-1}b_{2i-1}, b_{2i}c_{2i}, c_{2i-1}a_{2i} \,:\, i = 1, 2, \ldots, n\}$. For any $e \in E_0$, it is not difficult to see that $H_{3,n} - e$ contains a subgraph homeomorphic to $H_{3,n-1}$, therefore, e must be clean in D, otherwise, a good drawing of $H_{3,n-1}$ with less than $n - 1$ crossings can be constructed from D by removing e.

By combining Lemma 3 with the fact that $cr_D(H_{3,n}) = n - 1$, there must exist an i $(1 \le i \le n)$ such that $f_D(F_i) < 1$. Without loss of generality, let $f_D(F_2) < 1$.

The following two cases are considered.

Case 1. $f_D(F_2) = 0$. That is to say, all the edges of F_2 are clean in D. Note that the subgraph induced on six edges, $\{a_3a_4, a_4c_3, c_3c_4, c_4b_4, b_4b_3, b_3a_3\}$, of F_2 is a 6-cycle, thus, the subdrawing of the 6-cycle partite the plane into two faces.

We conclude that vertices b_2 and c_2 must lie in the same face since $b_2c_2 \in E_0$. Without loss of generality, assume that both b_2 and c_2 lie in the interior face $Int\,C$. Moreover, the vertex a_2 should also lie in $Int\,C$, otherwise, the path $a_2c_1c_2$ will cross the boundary of the 6-cycle.

Consider now the edge b_2c_2, it is clean in D since $b_2c_2 \in E_0$. Therefore, the subdrawing of $F_2 \cup \{b_2c_2\}$ must be as shown in Figure 1. The face $Int\,C$ has been divided into two regions, with vertices a_2 and c_4 do not lie on the boundary of the same region. Hence, the path $a_2c_1c_{2n}c_{2n-1}\cdots c_5c_4$ will cross F_2 at least once, which is contradicts with $f_D(F_2) = 0$.

Case 2. $f_D(F_2) > 0$. From the definition of f_D, it has $f_D(F_2) = \frac{1}{2}$ and $cr_D(F_2, F_2) = 0$, which means that exactly one edge of F_2 is crossed in D, and that F_2 does not have internal crossing in D.

By the analogous arguments to those of Case 1, the subdrawing of the 6-cycle $a_3a_4c_3c_4b_4b_3a_3$ partite the plane into two faces, moreover, the vertices b_2 and c_2 should lie in the same face since the edge b_2c_2 is clean in D. Without loss of

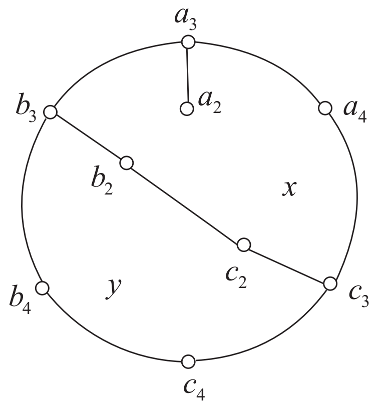

Figure 5. The subdrawing of $F_2 \cup \{b_2c_2\}$.

generality, assume that both b_2 and c_2 lie in the interior face $Int\,C$. By adding two edges b_2b_3 and c_2c_3 without any internal crossing occurred in F_2, and by adding the clean edge b_2c_2, one can see that the face $Int\,C$ has been divided into two regions, denoted as x and y. See Figure 5.

Consider the vertex a_2. If a_2 lies in the exterior face $Ext\,C$, then the two edge-disjoint paths $a_2a_1b_1b_2$ and $a_2c_1c_2$ will cross the boundary of the 6-cycle at least once respectively, contradicts with $f_D(F_2) = \frac{1}{2}$. If a_2 lies in the region y, then there will be at least one internal crossing in F_2 made by the edge a_2a_3, which is impossible. Hence, a_2 must lie in the region x.

Now consider the vertices a_5 and b_5, they must lie in the same region since $a_5b_5 \in E_0$. The following three subcases are discussed according to in which region do a_5 and b_5 lie.

Subcase 2.1. Both a_5 and b_5 lie in $Ext\,C$. Notice that vertices a_2 and a_5 do not lie on the boundary of a same region, thus the two edge-disjoint paths $a_2a_1a_{2n}a_{2n-1}\cdots a_5$ and $a_2c_1c_{2n}c_{2n-1}\cdots c_6b_6b_5a_5$ will cross at least once with the edges of F_2, respectively, which implies $f_D(F_2) \geq 1$, contradicts with $f_D(F_2) = \frac{1}{2}$.

Subcase 2.2. Both a_5 and b_5 lie in the region x. Remind that the edge b_2c_2 is clean in D, therefore, the edge b_4b_5 and the path $c_4c_5c_6b_6b_5$ will cross at least once with the edges of F_2, respectively, which is impossible.

Subcase 2.3. Both a_5 and b_5 lie in the region y. By the analogous arguments to that of Subcase 2.2, the edge a_4a_5 and the path $a_2a_1a_{2n}a_{2n-1}\cdots a_5$ will cross at least once with the edges of F_2, respectively, which is absurd.

All the above contradictions confirm that $cr(H_{3,n}) \geq n$. ∎

Lemma 5. *$cr(H_{m,n}) \leq (m-2)n$ for $n \geq 2$.*

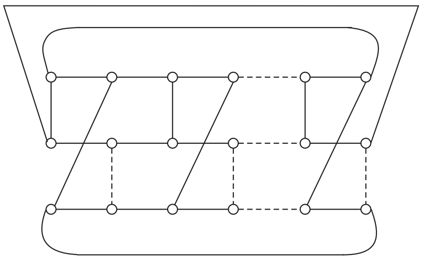

Figure 6. A good drawing of $H_{m,n}$ when m is odd.

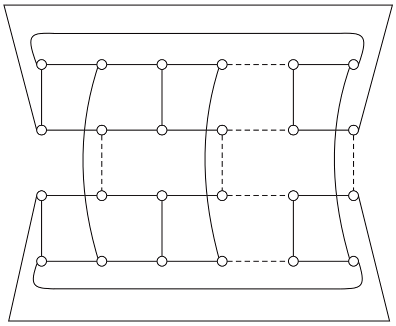

Figure 7. A good drawing of $H_{m,n}$ when m is even.

Proof. For m is odd (respectively, even), Figure 6 (respectively, Figure 7) demonstrates a good drawing of $H_{m,n}$ in the plane with exactly $(m-2)n$ crossings. Thus, $cr(H_{m,n}) \leq (m-2)n$. ■

According to Lemmas 4 and 5, Theorem 1 is easily followed.

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