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THE PRODUCT CONNECTIVITY BANHATTI INDEX OF A GRAPH

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Abstract

Let $G = (V, E)$ be a connected graph with vertex set $V(G)$ and edge set $E(G)$. The product connectivity Banhatti index of a graph G is defined as $PB(G) = \sum_{ue} \frac{1}{\sqrt{d_G(u)d_G(e)}}$, where ue means that the vertex u and edge e are incident in G. In this paper, we determine $PB(G)$ of some standard classes of graphs. We also provide some relationship between $PB(G)$ in terms of order, size, minimum / maximum degrees and minimal non-pendant vertex degree. In addition, we obtain some bounds on $PB(G)$ in terms of Randić, Zagreb and other degree based topological indices of G.

Keywords: Randić index, Zagreb indices, Banhatti indices, product connectivity Banhatti index.

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1. Introduction

All graphs considered in this paper are finite, connected, undirected without loops and multiple edges. Let $G = (V, E)$ be a connected graph with n vertices and m edges. The degree $d_G(v)$ of a vertex v is the number of vertices adjacent to v. The degree of an edge $e = uv$ in G is defined by $d_G(e) = d_G(u) + d_G(v) - 2$. We refer to [5] for undefined term and notation.

A molecular graph is a graph such that its vertices correspond to the atoms and edges to the bonds. Chemical graph theory is a branch of Mathematical chemistry which has an important effect on the development of the chemical sciences. A single number that can be used to characterize some property of the graph of a molecule is called a topological index for that graph. There are numerous molecular descriptors, which are also referred to as topological indices, see [3] that have found some applications in theoretical chemistry, especially in QSPR/QSAR research.

One of the best known and widely used topological index is the product-connectivity index (or Randić index, connectivity index) by Randić [11], who has shown this index to reflect molecular branching. The product connectivity index of a graph G is defined as $P(G) = \sum_{uv \in E(G)} \frac{1}{\sqrt{d_G(u) d_G(v)}}$. Motivated by Randić definition of the product connectivity index, the sum connectivity index was initiated by Zhou and Trinajstic [14] and [15], which is defined by $S(G) = \sum_{uv \in E(G)} \frac{1}{\sqrt{d_G(u) + d_G(v)}}$. For more details on these type of connectivity indices, we refer to [1, 2] and [10].

The first and second K Banhatti indices of a graph G are defined as $B_1(G) = \sum_{ue} [d_G(u) + d_G(e)]$ and $B_2(G) = \sum_{ue} [d_G(u) d_G(e)]$, where ue means that the vertex u and edge e are incident in G. The K Banhatti indices were introduced by Kulli in [6]. The K Banhatti indices are closely related to Zagreb - types indices. For more details on these two types of indices refer to Gutman *et al.*, [4]. Also, recently many other indices were studied, for example, in [7] and [8].

Here, we initiate the study on this new topological index of a graph as follows. The product connectivity Banhatti index of a graph G is defined as $PB(G) = \sum_{ue} \frac{1}{\sqrt{d_G(u) d_G(e)}}$, where ue means that the vertex u and edge e are incident in G.

2. Some Standard Classes of Graphs

Proposition 1. *Let C_n be a cycle with $n \geq 3$ vertices. Then*

$$PB(C_n) = n.$$

Proof. Let $G = C_n$ be a cycle with $n \geq 3$ vertices. Every vertex of a cycle C_n is incident with exactly two edges and the number of edges in C_n is n. Consider

$$\begin{aligned} PB(G) &= \sum_{ue} \frac{1}{\sqrt{d_G(u)d_G(e)}} = \sum_{e=uv\in E(C_n)} \left[\frac{1}{\sqrt{d_G(u)d_G(e)}} + \frac{1}{\sqrt{d_G(v)d_G(e)}}\right] \\ &= n\left[\frac{1}{\sqrt{2\times 2}} + \frac{1}{\sqrt{2\times 2}}\right] = n. \end{aligned}$$

■

Proposition 2. *Let K_n be a complete graph with $n \geq 3$ vertices. Then*

$$PB(K_n) = \frac{n\sqrt{n-1}}{\sqrt{2}\sqrt{n-2}}.$$

Proof. Let $G = K_n$ be a complete graph with $n \geq 3$ vertices. Every vertex of K_n is incident with exactly $(n-1)$ edges. Consider

$$\begin{aligned} PB(G) &= \sum_{ue} \frac{1}{\sqrt{d_G(u)d_G(e)}} = \sum_{e=uv\in E(K_n)} \left[\frac{1}{\sqrt{d_G(u)d_G(e)}} + \frac{1}{\sqrt{d_G(v)d_G(e)}}\right] \\ &= \frac{n(n-1)}{2}\left[\frac{1}{\sqrt{(n-1)\times(2n-4)}} + \frac{1}{\sqrt{(n-1)\times(2n-4)}}\right] \\ &= \frac{n\sqrt{n-1}}{\sqrt{2}\sqrt{n-2}}. \end{aligned}$$

■

Proposition 3. *Let $K_{r,s}$ be a complete bipartite graph with $1 \leq r \leq s$ and $s \geq 2$ vertices. Then*

$$PB(K_{r,s}) = \frac{r\sqrt{s} + s\sqrt{r}}{\sqrt{r+s-2}}.$$

Proof. Let $G = K_{r,s}$ be a complete bipartite graph with $r + s$ vertices and rs edges such that $|V_1| = r$, $|V_2| = s$, $V(K_{r,s}) = V_1 \cup V_2$. Every vertex of V_1 is incident with s edges and every vertex of V_2 is incident with r edges. Let $V_1 = \{u_1, u_2, \ldots, u_r\}$ and $V_2 = \{v_1, v_2, \ldots, v_s\}$. Consider

$$\begin{aligned} PB(K_{r,s}) &= \sum_{ue} \frac{1}{\sqrt{d_G(u)d_G(e)}} = \sum_{e=uv\in E(K_{r,s})} \left[\frac{1}{\sqrt{d_G(u)d_G(e)}} + \frac{1}{\sqrt{d_G(v)d_G(e)}}\right] \\ &= rs\left[\frac{1}{\sqrt{s\times(r+s-2)}} + \frac{1}{\sqrt{r\times(r+s-2)}}\right] = \frac{r\sqrt{s} + s\sqrt{r}}{\sqrt{r+s-2}}. \end{aligned}$$

■

Corollary 4. *Let $K_{r,r}$ be a complete bipartite graph with $r \geq 2$ vertices. Then*

$$PB(K_{r,r}) = \frac{\sqrt{2} \times r\sqrt{r}}{\sqrt{r-1}}.$$

Corollary 5. *Let $K_{1,s}$ be a star with $s \geq 2$ vertices. Then*

$$PB(K_{1,s}) = \frac{\sqrt{s}+s}{\sqrt{s-1}}.$$

Proposition 6. *If G is an r-regular graph with $n \geq 3$ vertices, then*

$$PB(G) = \frac{n\sqrt{r}}{\sqrt{2}\sqrt{r-1}}.$$

Proof. Let G be an r-regular graph with $n \geq 3$ vertices and $\frac{nr}{2}$ edges. Every vertex of G is incident with r edges. Consider

$$\begin{aligned} PB(G) &= \sum_{ue} \frac{1}{\sqrt{d_G(u)d_G(e)}} = \sum_{e=uv\in E(G)} \left[\frac{1}{\sqrt{d_G(u)d_G(e)}} + \frac{1}{\sqrt{d_G(v)d_G(e)}}\right] \\ &= \frac{nr}{2}\left[\frac{1}{\sqrt{r\times(2r-2)}} + \frac{1}{\sqrt{r\times(2r-2)}}\right] = \frac{n\sqrt{r}}{\sqrt{2}\sqrt{r-1}}. \end{aligned}$$

■

3. Bounds on Product Connectivity Banhatti Index

First, we start with couple of bounds in $PB(G)$ in terms of the Randić (or, product connectivity) index $P(G)$ of a graph G.

Theorem 7. *For any connected graph G with $n \geq 3$ vertices,*
(i) *$PB(G) \neq P(G)$,*
(ii) *$PB(G) > P(G)$.*

Proof. (i) For if $PB(G) = P(G)$, we have

$$\begin{aligned} \sum_{e=uv\in E(G)} \frac{\sqrt{d_G(u)}+\sqrt{d_G(v)}}{\sqrt{d_G(e)}\sqrt{d_G(u)d_G(v)}} &= \sum_{e=uv\in E(G)} \frac{1}{\sqrt{d_G(u)d_G(v)}} \\ \Leftrightarrow \sqrt{d_G(u)}+\sqrt{d_G(v)} &= \sqrt{d_G(u)+d_G(v)-2} \\ \Leftrightarrow d_G(u)+d_G(v)+2\sqrt{d_G(u)d_G(v)} &= d_G(u)+d_G(v)-2 \\ \Leftrightarrow \sqrt{d_G(u)d_G(v)} &= -1, \end{aligned}$$

which is a contradiction as $\sqrt{d_G(u)d_G(v)} > 0$. Hence the result follows.
(ii) We have

$$\begin{aligned} PB(G) &= \sum_{ue} \frac{1}{\sqrt{d_G(u)d_G(e)}} = \sum_{e=uv\in E(G)} \left[\frac{1}{\sqrt{d_G(u)d_G(e)}} + \frac{1}{\sqrt{d_G(v)d_G(e)}}\right] \\ &= \sum_{e=uv\in E(G)} \frac{1}{\sqrt{d_G(u)+d_G(v)-2}} \left[\frac{\sqrt{d_G(u)}+\sqrt{d_G(v)}}{\sqrt{d_G(u)d_G(v)}}\right]. \end{aligned}$$

Clearly, $\sqrt{d_G(u)} + \sqrt{d_G(v)} > \sqrt{d_G(u)+d_G(v)-2}$ for any connected graph G with $n \geq 3$ vertices. This implies that $\frac{\sqrt{d_G(u)}+\sqrt{d_G(v)}}{\sqrt{d_G(u)+d_G(v)-2}} > 1$. Therefore $PB(G) > \sum_{e=uv\in E(G)} \frac{1}{\sqrt{d_G(u)d_G(v)}}$. Hence $PB(G) > P(G)$ follows. ■

In order to prove our next result, we make use of the following definition of the modified second Zagreb index, $M_2^*(G)$, to obtain the lower and upper bound of $PB(G)$ in terms of degrees, Randić index $P(G)$ and $M_2^*(G)$.

The modified second Zagreb index [13] of a graph G is defined as

$$M_2^*(G) = \sum_{uv\in E(G)} \frac{1}{d_G(u)\, d_G(v)}.$$

Theorem 8. *For any connected graph G with $n \geq 3$ vertices and no pendant vertices,*

(i) $\sqrt{\frac{2\delta(G)}{\Delta(G)-1}}P(G) \leq PB(G) \leq \sqrt{\frac{2\Delta(G)}{\delta(G)-1}}P(G)$,

(ii) $\sqrt{\frac{2\delta(G)}{\Delta(G)-1}}\,\delta(G)\, M_2^*(G) \leq PB(G) \leq \sqrt{\frac{2\Delta(G)}{\delta(G)-1}}\,\Delta(G)\, M_2^*(G)$.

Proof. Let G be a connected graph with $n \geq 3$ vertices and no pendant vertices. Clearly,

$$\begin{gathered} 2(\delta(G)-1) \;\leq\; d_G(u)+d_G(v)-2 \;\leq\; 2(\Delta(G)-1) \\ \sqrt{2}(\sqrt{\delta(G)-1}) \;\leq\; \sqrt{d_G(u)+d_G(v)-2} \;\leq\; \sqrt{2}(\sqrt{\Delta(G)-1}) \\ \frac{1}{\sqrt{2}(\sqrt{\Delta(G)-1})} \;\leq\; \frac{1}{\sqrt{d_G(u)+d_G(v)-2}} \;\leq\; \frac{1}{\sqrt{2}(\sqrt{\delta(G)-1})}. \end{gathered}$$

Also, $2\sqrt{\delta(G)} \leq \sqrt{d_G(u)} + \sqrt{d_G(v)} \leq 2\sqrt{\Delta(G)}$, and

$$\frac{1}{\Delta(G)} \leq \frac{1}{\sqrt{d_G(u)d_G(v)}} \leq \frac{1}{\delta(G)}.$$

We have,

$$PB(G) = \sum_{uv\in E(G)} \frac{\sqrt{d_G(u)}+\sqrt{d_G(v)}}{\sqrt{d_G(u)+d_G(v)-2}} \left[\frac{1}{\sqrt{d_G(u)d_G(v)}}\right].$$

(i) By virtue of the above facts, we see that

$$\frac{2\sqrt{\delta(G)}}{\sqrt{2}\sqrt{\Delta(G)-1}}P(G) \leq PB(G) \leq \frac{2\sqrt{\Delta(G)}}{\sqrt{2}\sqrt{\delta(G)-1}}P(G)$$
$$\sqrt{\frac{2\delta(G)}{\Delta(G)-1}}P(G) \leq PB(G) \leq \sqrt{\frac{2\Delta(G)}{\delta(G)-1}}P(G).$$

(ii) Also $\delta(G) \leq \sqrt{d_G(u)\, d_G(v)} \leq \Delta(G)$. We have

$$PB(G) = \sum_{uv\in E(G)} \frac{\sqrt{d_G(u)}+\sqrt{d_G(v)}}{\sqrt{d_G(u)+d_G(v)-2}} \left[\frac{\sqrt{d_G(u)d_G(v)}}{d_G(u)d_G(v)}\right].$$

By virtue of the above facts, we see that

$$\frac{2\sqrt{\delta(G)}}{\sqrt{2}\sqrt{\Delta(G)-1}}\,\delta(G)\, M_2^*(G) \leq PB(G) \leq \frac{2\sqrt{\Delta(G)}}{\sqrt{2}\sqrt{\delta(G)-1}}\,\Delta(G)\, M_2^*(G)$$
$$\sqrt{\frac{2\delta(G)}{\Delta(G)-1}}\,\delta(G)\, M_2^*(G) \leq PB(G) \leq \sqrt{\frac{2\Delta(G)}{\delta(G)-1}}\,\Delta(G)\, M_2^*(G).$$

Thus the results follow. ∎

To prove our next result, we make use of the following definition of the sum connectivity Banhatti index $SB(G)$ to obtain the upper bound and characterization of $PB(G)$.

The sum connectivity Banhatti index of a graph G is defined as

$$SB(G) = \sum_{ue} \frac{1}{\sqrt{d_G(u)+d_G(e)}},$$

where ue means that the vertex u and edge e are incident in G. This connectivity based index is put forward by Kulli *et al.* [9].

Theorem 9. *For any (n, m)-connected graph G with $\delta(G) \geq 2$ and $n \geq 3$ vertices,*

$$PB(G) \leq SB(G).$$

Further, equality is attained if and only if $G \cong C_n$.

Proof. Let G be a connected graph with $\delta(G) \geq 2$ and $n \geq 3$ vertices. Then

$$\begin{aligned} d_G(u)d_G(e) &\geq d_G(u) + d_G(e) \\ \sum_{ue} \frac{1}{\sqrt{d_G(u)d_G(e)}} &\leq \sum_{ue} \frac{1}{\sqrt{d_G(u) + d_G(e)}} \\ PB(G) &\leq SB(G). \end{aligned}$$

Clearly, equality is attained

$$\begin{aligned} &\Leftrightarrow \quad d_G(u)\, d_G(e) = d_G(u) + d_G(e) \\ &\Leftrightarrow \quad d_G(u) = d_G(v) = 2 \\ &\Leftrightarrow \quad G \cong C_n. \end{aligned}$$

■

Now, we obtain the lower bound of $PB(G)$ in terms of the order n, $\delta(G)$ and $\Delta(G)$ of a graph G.

Theorem 10. *For any (n, m)-connected graph G with $n \geq 3$ vertices,*

$$PB(G) \geq \frac{n\,\delta(G)}{\sqrt{2\Delta(G)(\Delta(G) - 1)}}\,.$$

Further, equality is attained if and only if G is regular.

Proof. We have

$$\begin{aligned} PB(G) &= \sum_{e=uv \in E(G)} \left[\frac{1}{\sqrt{d_G(u)d_G(e)}} + \frac{1}{\sqrt{d_G(v)d_G(e)}} \right] \\ &= \frac{1}{2} \sum_{u \in V(G)} \sum_{v \in N(u)} \left[\frac{1}{\sqrt{d_G(u)d_G(uv)}} + \frac{1}{\sqrt{d_G(v)d_G(uv)}} \right], \end{aligned}$$

where $N(u) = \{v \in V(G) : uv \in E(G)\}$,

$$\begin{aligned} &= \frac{1}{2} \sum_{u \in V(G)} \sum_{v \in N(u)} \frac{1}{\sqrt{d_G(e)}} \left[\frac{1}{\sqrt{d_G(u)}} + \frac{1}{\sqrt{d_G(v)}} \right] \\ &\geq \frac{1}{2} \sum_{u \in V(G)} \frac{d_G(u)}{\sqrt{d_G(u) + \Delta(G) - 2}} \left[\frac{1}{\sqrt{d_G(u)}} + \frac{1}{\sqrt{\Delta(G)}} \right] \\ &\geq \frac{n}{2} \times \frac{\delta(G)}{\sqrt{2}\sqrt{\Delta(G) - 1}} \times \frac{2}{\sqrt{\Delta(G)}} \geq \frac{n\,\delta(G)}{\sqrt{2\Delta(G)(\Delta(G) - 1)}}\,. \end{aligned}$$

Further, equality is attained if and only if $d_G(u) + d_G(v) = 2\delta(G) = 2\Delta(G)$, for each $uv \in E(G)$, which implies that G is a regular graph. ■

Now, we obtain lower and upper bounds of $PB(G)$ in terms of order n.

Let a_{ij} be the number of edges of a connected graph G, which joins the vertices of degree i and j, where $1 \leq i \leq j \leq \Delta(G)$ and n_i denotes the number of vertices with degree i in G for $i = 1, 2, \ldots, \Delta(G)$. Further, let $\overline{xy} = \frac{1}{\sqrt{x(x+y-2)}}$.

Lemma 11. *If $x \leq y$ with $x, y > 0$, then $\overline{xi} \geq \overline{yi}$, where $\overline{xi} = \frac{1}{\sqrt{x(x+i-2)}}$.*

Proof. Since $x \leq y$, we have $\sqrt{x(x+i-2)} \leq \sqrt{y(y+i-2)}$, $\frac{1}{\sqrt{x(x+i-2)}} \geq \frac{1}{\sqrt{y(y+i-2)}}$. Hence the result follows. ■

For instance, we have

$$\begin{aligned} \overline{1i} &= \tfrac{1}{\sqrt{1(1+i-2)}} = \frac{1}{\sqrt{i-1}}\,, \\ \overline{2i} &= \tfrac{1}{\sqrt{2(2+i-2)}} = \frac{1}{\sqrt{2i}} \end{aligned}$$

and so on.

Theorem 12. *For any connected graph G with $n \geq 3$ vertices and no pendant vertices,*

$$\frac{n\sqrt{2}}{\sqrt{(n-1)(n-2)}} \leq PB(G) \leq n.$$

Further, equality holds in lower bound if and only if $G \cong C_3$ and an equality holds in upper bound if and only if $G \cong C_n$, $n \geq 3$.

Proof. Let G be a connected graph with $n \geq 3$ vertices and $\Delta(G) \leq n-1$. If G has no pendant vertices, we have $\delta(G) \geq 2$. Since $\frac{1}{\sqrt{\Delta(G)-1}} \geq \frac{1}{\sqrt{n-1}}$ and by Theorem 10, the desired lower bound of $PB(G)$ follows. For obtaining the upper bound of $PB(G)$, we consider

$$\begin{aligned} PB(G) &= \sum_{uv \in E(G)} \frac{1}{\sqrt{d_G(u) d_G(e)}} \\ &= 2 \sum_{2 \leq i \leq \Delta(G)} x_{2i} \overline{2i} + 2 \sum_{3 \leq i \leq j \leq \Delta(G)} x_{ij} \overline{ij}. \end{aligned}$$

From the above lemma, for $x \leq y$, $\overline{xi} \geq \overline{yi}$, we have $\overline{2i} \geq \overline{3i} \geq \cdots$. Therefore

$$\begin{aligned} PB(G) &\leq 2 \left(\sum_{2 \leq i \leq \Delta(G)} x_{2i} \right) \overline{2i} + 2 \left(\sum_{3 \leq i \leq j \leq \Delta(G)} x_{ij} \right) \overline{2i} \\ &\leq 2 \left(n_2 + n_3 + \cdots + n_{\Delta(G)} \right) \overline{2i}, \quad \text{since} \quad n_1 = 0. \end{aligned}$$

But as $i \geq 2$, $\overline{2i} \leq \frac{1}{2}$. Therefore $PB(G) \leq 2n\left(\frac{1}{2}\right) = n$. The equality case attains directly from Proposition 1. ■

Next, we obtain lower and upper bounds of $PB(G)$ in terms of the size m, minimum degree $\delta(G)$ and the maximum degree $\Delta(G)$ of G.

Theorem 13. *For any connected graph G with $n \geq 3$ vertices and no pendant vertices,*

$$\frac{m}{\Delta(G)}\sqrt{\frac{2\delta(G)}{\Delta(G)-1}} \leq PB(G) \leq \frac{m}{\delta(G)}\sqrt{\frac{2\Delta(G)}{\delta(G)-1}}.$$

Further, equality in both lower and upper bounds is attained if and only if G is regular.

Proof. Let G be a connected graph with $n \geq 3$ vertices and no pendant vertices. Clearly,

$$\begin{aligned} 2(\delta(G)-1) \leq&\ d_G(u)+d_G(v)-2 \leq 2(\Delta(G)-1) \\ \sqrt{2}(\sqrt{\delta(G)-1}) \leq&\ \sqrt{d_G(u)+d_G(v)-2} \leq \sqrt{2}(\sqrt{\Delta(G)-1}) \\ \frac{1}{\sqrt{2}(\sqrt{\Delta(G)-1})} \leq&\ \frac{1}{\sqrt{d_G(u)+d_G(v)-2}} \leq \frac{1}{\sqrt{2}(\sqrt{\delta(G)-1})} \end{aligned}$$

Also, $2\sqrt{\delta(G)} \leq \sqrt{d_G(u)}+\sqrt{d_G(v)} \leq 2\sqrt{\Delta(G)}$, and

$$\frac{1}{\Delta(G)} \leq \frac{1}{\sqrt{d_G(u)d_G(v)}} \leq \frac{1}{\delta(G)}.$$

By virtue of the above facts, we have

$$\begin{aligned} PB(G) &= \sum_{uv\in E(G)} \frac{1}{\sqrt{d_G(u)+d_G(v)-2}}\left[\frac{\sqrt{d_G(u)}+\sqrt{d_G(v)}}{\sqrt{d_G(u)d_G(v)}}\right] \\ &\leq \frac{1}{\sqrt{2}\sqrt{\delta(G)-1}}\frac{2m\sqrt{\Delta(G)}}{\delta(G)} \leq \frac{m}{\delta(G)}\sqrt{\frac{2\Delta(G)}{\delta(G)-1}}. \end{aligned}$$

Thus the upper bound follows. Similarly, $PB(G) \geq \frac{m}{\Delta(G)}\sqrt{\frac{2\delta(G)}{\Delta(G)-1}}$ follows. Second part is obvious as for a regular graph G with $\delta(G) = \Delta(G)$. ■

Now, we obtain lower and upper bounds of $PB(G)$ in terms of the minimum and maximum degrees, the number of pendant vertices and minimal non-pendant vertices of G.

Theorem 14. *For any (n,m)-connected graph G with η pendant vertices and minimal non-pendant vertex degree $\delta_1(G)$,*

$$\frac{\eta(1+\sqrt{\Delta(G)})+(m-\eta)\sqrt{2}}{\sqrt{\Delta(G)(\Delta(G)-1)}} \leq PB(G) \leq \frac{\eta(1+\sqrt{\delta_1(G)})+(m-\eta)\sqrt{2}}{\sqrt{\delta_1(G)(\delta_1(G)-1)}}.$$

Proof. We have

$$\begin{aligned}
PB(G) &= \sum_{e=uv\in E(G)}\left[\frac{1}{\sqrt{d_G(u)d_G(e)}}+\frac{1}{\sqrt{d_G(v)d_G(e)}}\right]\\
&= \sum_{e=uv\in E(G);d_G(u)=1}\left[\frac{1}{\sqrt{d_G(v)-1}}+\frac{1}{\sqrt{d_G(v)(d_G(v)-1)}}\right]\\
&+ \sum_{e=uv\in E(G);d_G(u),d_G(v)\neq 1}\frac{1}{\sqrt{d_G(e)}}\left[\frac{1}{\sqrt{d_G(u)}}+\frac{1}{\sqrt{d_G(v)}}\right]\\
&= \sum_{e=uv\in E(G);d_G(u)=1}\frac{\sqrt{d_G(v)}+1}{\sqrt{d_G(v)}\sqrt{d_G(v)-1}}\\
&+ \sum_{e=uv\in E(G);d_G(u),d_G(v)\neq 1}\frac{1}{\sqrt{d_G(e)}}\left[\frac{1}{\sqrt{d_G(u)}}+\frac{1}{\sqrt{d_G(v)}}\right]\\
&\leq \frac{\eta(\sqrt{\delta_1(G)}+1)}{\sqrt{\delta_1(G)}\sqrt{\delta_1(G)-1}}+\frac{(m-\eta)}{\sqrt{2}\sqrt{\delta_1(G)-1}}\times\frac{2}{\sqrt{\delta_1(G)}}.
\end{aligned}$$

Since $2(\Delta(G)-1)\geq d_G(u)+d_G(v)-2\geq 2(\delta_1(G)-1)$

$$\Rightarrow \frac{1}{\sqrt{2}\sqrt{\Delta(G)-1}} \leq \frac{1}{\sqrt{d_G(u)+d_G(v)-2}} \leq \frac{1}{\sqrt{2}\sqrt{\delta_1(G)-1}}$$

and $\frac{1}{\sqrt{\Delta(G)}} \leq \frac{1}{\sqrt{d_G(u)}} \leq \frac{1}{\sqrt{\delta_1(G)}}$. Thus the upper bound follows. Similarly,

$$\begin{aligned}
PB(G) &\geq \frac{\eta(1+\sqrt{\Delta(G)})}{\sqrt{\Delta(G)(\Delta(G)-1)}}+\frac{(m-\eta)\sqrt{2}}{\sqrt{2}\sqrt{\Delta(G)(\Delta(G)-1)}}\\
&\geq \frac{\eta(1+\sqrt{\Delta(G)})+(m-\eta)\sqrt{2}}{\sqrt{\Delta(G)(\Delta(G)-1)}}.
\end{aligned}$$

Thus the lower bound follows. ■

Remark 15. Equality is attained on both sides if and only if $d_G(u) = d_G(v) = \Delta(G) = \delta_1(G)$ for each $uv \in E(G)$ with $d_G(u), d_G(v) \neq 1$ and $d_G(v) = \Delta(G) = \delta_1(G)$ for each $uv \in E(G)$ with $d_G(u) = 1$.

In order to prove our next result (lower bound) of $PB(G)$ in terms of an inverse edge degree of G, we make use of the following definition.

An inverse edge degree [12] of a graph G with no isolated edges is defined as

$$IED(G) = \sum_{e=uv\in E(G)} \frac{1}{d_G(e)}.$$

Theorem 16. *For any (n, m)-connected graph G with $n \geq 3$ and no pendant vertices,*

$$PB(G) \geq 2\, IED(G).$$

Further, equality is attained if and only if $G \cong C_n$.

Proof. Since $\delta(G) \geq 2$, we have $d_G(u) \leq d_G(e)$, for any $e = uv \in E(G)$. This implies that $d_G(u)\, d_G(e) \leq (d_G(e))^2$. Therefore

$$\begin{aligned} PB(G) &= \sum_{e=uv\in E(G)} [d_G(u)\, d_G(e)]^{-\frac{1}{2}} + \sum_{e=uv\in E(G)} [d_G(v)\, d_G(e)]^{-\frac{1}{2}} \\ &\geq \sum_{e=uv\in E(G)} \frac{2}{d_G(uv)} \geq 2 \sum_{e=uv\in E(G)} \frac{1}{d_G(e)}. \end{aligned}$$

Thus the lower bound follows. Equality holds if and only if $d_G(u) d_G(e) = (d_G(e))^2$

$$\begin{aligned} &\Leftrightarrow \; d_G(u) = d_G(e) \\ &\Leftrightarrow \; G \cong C_n. \end{aligned}$$

■

In order to prove our next result (lower bound) of $PB(G)$ in terms of the size m and second K Banhatti index $B_2(G)$ of a graph G, we recall the following facts.

If real valued function $f(x)$ defined on an interval has a second derivative $f''(x)$, then a necessary and sufficient condition for it to be strictly convex on that interval is that $f''(x) > 0$. For positive integer k, if $f(x)$ is strictly convex, then (by Jensen's inequality) we have $f\left(\sum_{i=1}^{k} \frac{x_i}{k}\right) \leq f(x_i)$ with equality if and only if $x_1 = x_2 = \cdots = x_k$, and if $-f(x)$ is strictly convex, then the inequality is reversed.

Theorem 17. *For any (n, m)-connected graph G with $n \geq 3$ vertices,*

$$PB(G) \geq \frac{(2m)^{\frac{3}{2}}}{\sqrt{B_2(G)}}.$$

Further, equality is attained if and only if G is a regular graph.

Proof. Let G be a connected graph with $n \geq 3$ vertices. Then

$$PB(G) = \sum_{ue} \frac{1}{\sqrt{d_G(u) d_G(e)}} = \sum_{ue} [d_G(u)\, d_G(e)]^{-\frac{1}{2}}.$$

By Jensen's inequality, $\frac{1}{\sqrt{x}}$ is a convex function for $x > 0$, we have

$$\sum_{ue} \frac{[d_G(u)\, d_G(e)]^{-\frac{1}{2}}}{2m} \geq \left[\sum_{ue} \frac{d_G(u)\, d_G(e)}{2m} \right]^{-\frac{1}{2}}.$$

Therefore

$$PB(G) \geq 2m \left[\sum_{ue} \frac{d_G(u)\, d_G(e)}{2m} \right]^{-\frac{1}{2}} \geq \frac{2\sqrt{2}m\sqrt{m}}{\sqrt{\sum_{ue} [d_G(u) d_G(e)]}}.$$

Thus the result follows. The equality case attains directly from Proposition 6. ■

4. Conclusions

Being new topological index of a graph G in terms of incident vertex-edge degrees, the product connectivity Banhatti index is an invariant, whose properties are relatively unknown. For the comparative advantages, applications and mathematical point of view, many questions are suggested by this research, among them are the following.

1. Find the extremal values and extremal graphs of the product connectivity Banhatti index.
2. Find the relationship between $PB(G)$ and other degree based topological indices.
3. Find the values of the product connectivity Banhatti index of all classes of chemical graphs and compare with other degree based topological indices, when $\Delta(G) \leq 4$. Also, explore some results towards QSPR/QSAR Model.
4. Characterize the product connectivity Banhatti index in terms of other degree based topological indices.

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