A NEIGHBORHOOD CONDITION FOR FRACTIONAL ID-\([A, B]\)-FACTOR-CRITICAL GRAPHS

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Abstract

Let \( G \) be a graph of order \( n \), and let \( a \) and \( b \) be two integers with \( 1 \leq a \leq b \). Let \( h : E(G) \rightarrow [0, 1] \) be a function. If \( a \leq \sum_{e \ni x} h(e) \leq b \) holds for any \( x \in V(G) \), then we call \( G[F_h] \) a fractional \([a, b]\)-factor of \( G \) with indicator function \( h \), where \( F_h = \{ e \in E(G) : h(e) > 0 \} \). A graph \( G \) is fractional independent-set-deletable \([a, b]\)-factor-critical (in short, fractional ID-[\(a, b]\)-factor-critical) if \( G - I \) has a fractional \([a, b]\)-factor for every independent set \( I \) of \( G \). In this paper, it is proved that if \( n \geq \frac{(a+2b)(2a+2b-3)+1}{b}, \delta(G) \geq \frac{bn}{a+b} + a \) and \( |N_G(x) \cup N_G(y)| \geq \frac{(a+b)n}{a+b} \), for any two nonadjacent vertices \( x, y \in V(G) \), then \( G \) is fractional ID-[\(a, b]\)-factor-critical. Furthermore, it is shown that this result is best possible in some sense.

Keywords: graph, minimum degree, neighborhood, fractional \([a, b]\)-factor, fractional ID-[\(a, b]\]-factor-critical graph.

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1. Introduction

The graphs considered here will be finite, undirected and without loops or multiple edges. Let $G$ be a graph. We denote by $V(G)$ and $E(G)$ the set of vertices and the set of edges of $G$, respectively. For any $x \in V(G)$, we denote the degree of $x$ in $G$ by $d_G(x)$. We write $N_G(x)$ for the set of vertices adjacent to $x$ in $G$, and $N_G(x) \cup \{x\}$ for $N_G(x) \cup \{x\}$. For $S \subseteq V(G)$, we use $G[S]$ to denote the subgraph of $G$ induced by $S$, and $G - S = G[V(G) \setminus S]$. Let $S$ and $T$ be two disjoint vertex subsets of $G$: we denote the number of edges from $S$ to $T$ by $e_G(S,T)$.

Let $S$ and $T$ be two disjoint vertex subsets of $G$; we denote the number of edges from $S$ to $T$ by $e_G(S,T)$. We denote by $\delta(G)$ the minimum degree of $G$. For any nonempty subset $S$ of $V(G)$, let $N_G(S) = \bigcup_{x \in S} N_G(x)$. If $G$ and $H$ are vertex-disjoint graphs, then their join and union are denoted by $G \lor H$ and $G \cup H$, respectively.

A factor of a graph $G$ is a spanning subgraph of $G$. Let $a$ and $b$ be two positive integers with $1 \leq a \leq b$. Then a factor $F$ is an $[a,b]$-factor if $a \leq d_F(x) \leq b$ for each $x \in V(G)$. Let $h : E(G) \to [0,1]$ be a function. If $a \leq \sum_{e \ni x} h(e) \leq b$ holds for any $x \in V(G)$, then we call $G[F_h]$ a fractional $[a,b]$-factor of $G$ with indicator function $h$, where $F_h = \{ e \in E(G) : h(e) > 0 \}$. If $G - I$ admits a fractional $[a,b]$-factor for every independent set $I$ of $G$, then we say that $G$ is fractional ID-$[a,b]$-factor-critical [1]. A fractional ID-$[k,k]$-factor-critical graph is simply called a fractional ID-$k$-factor-critical graph.

Many authors have investigated factors and fractional factors in graphs; see, for instance, [2, 3, 4, 5, 6, 7, 8, 9]. Chang, Liu and Zhu [10] showed a minimum degree condition for a graph to be a fractional ID-$k$-factor-critical graph. Zhou, Bian and Wu [11] gave a degree condition for the existence of fractional ID-$k$-factor-critical graphs. Zhou [12] obtained a binding number condition for graphs to be fractional ID-$k$-factor-critical graphs. Zhou, Sun and Liu [1] obtained a minimum degree condition for a graph to be a fractional ID-$[a,b]$-factor-critical graph. In this paper, we proceed to study fractional ID-$[a,b]$-factor-critical graphs, and obtain a neighborhood condition for a graph to be fractional ID-$[a,b]$-factor-critical. The main result is the following theorem.

**Theorem 1.** Let $1 \leq a \leq b$ be two integers, and let $G$ be a graph of order $n$ with $n \geq \frac{(a+2b)(2a+2b-3)+1}{b}$, and $\delta(G) \geq \frac{bn}{a+2b} + a$. If

$$|N_G(x) \cup N_G(y)| \geq \frac{(a+b)n}{a+2b}$$

for any two nonadjacent vertices $x, y \in V(G)$, then $G$ is fractional ID-$[a,b]$-factor-critical.
If $a = b = k$ in Theorem 1, then we obtain the following result.

**Theorem 2.** Let $k \geq 1$ be an integer, and let $G$ be a graph of order $n$ with $n \geq 12k - 8$, and $\delta(G) \geq \frac{n}{3} + k$. If

$$|N_G(x) \cup N_G(y)| \geq \frac{2n}{3}$$

for any two nonadjacent vertices $x, y \in V(G)$, then $G$ is fractional ID-$k$-factor-critical.

If $k = 1$ in Theorem 2, then we get the following result.

**Theorem 3.** Let $G$ be a graph of order $n$ with $n \geq 4$, and $\delta(G) \geq \frac{n}{3} + 1$. If

$$|N_G(x) \cup N_G(y)| \geq \frac{2n}{3}$$

for any two nonadjacent vertices $x, y \in V(G)$, then $G$ is fractional ID-factor-critical.

## 2. The Proof of Theorem 1

In order to prove Theorem 1, we rely heavily on the following lemma.

**Lemma 4 [13].** Let $G$ be a graph. Then $G$ has a fractional $[a, b]$-factor if and only if for every subset $S$ of $V(G)$,

$$\delta_G(S, T) = b|S| + d_{G-S}(T) - a|T| \geq 0,$$

where $T = \{x : x \in V(G) \setminus S, d_{G-S}(x) \leq a\}$ and $d_{G-S}(T) = \sum_{x \in T} d_{G-S}(x)$.

**Proof of Theorem 1.** Let $X$ be an independent set of $G$ and $H = G - X$. In order to complete the proof of Theorem 1, we need only to prove that $H$ has a fractional $[a, b]$-factor. By contradiction, suppose that $H$ has no fractional $[a, b]$-factor. Then by Lemma 4, there exists some subset $S \subseteq V(H)$ such that

$$\delta_H(S, T) = b|S| + d_{H-S}(T) - a|T| \leq -1,$$

where $T = \{x : x \in V(H) \setminus S, d_{H-S}(x) \leq a\}$. We first prove the following claims.

**Claim 1.** $|X| \leq \frac{bn}{a + 2b}$. 

**Proof.** Since $n \geq \frac{(a+2b)(2a+2b-3)+1}{b}$, the inequality holds for $|X| = 1$. In the following we may assume $|X| \geq 2$. In terms of the condition of Theorem 1, there
exist \ x, y \in X \text{ such that } |N_G(x) \cup N_G(y)| \geq \frac{(a+b)n}{a+2b}. \text{ Since } X \text{ is independent, we obtain } X \cap (N_G(x) \cup N_G(y)) = \emptyset. \text{ Thus, we have }

\begin{align*}
|X| + \frac{(a+b)n}{a+2b} &\leq |X| + |N_G(x) \cup N_G(y)| \leq n,
\end{align*}

which implies

\begin{align*}
|X| &\leq n - \frac{(a+b)n}{a+2b} = \frac{bn}{a+2b}.
\end{align*}

\begin{claim}
\delta(H) \geq a.
\end{claim}

\begin{proof}
Note that \( H = G - X \). Combining this with Claim 1, we obtain

\begin{align*}
\delta(H) &\geq \delta(G) - |X| \geq \left( \frac{bn}{a+2b} + a \right) - \frac{bn}{a+2b} = a.
\end{align*}

\end{proof}

\begin{claim}
|T| \geq b + 1.
\end{claim}

\begin{proof}
If \( |T| \leq b \), then from Claim 2 and since \( |S| + d_{H-S}(x) \geq d_H(x) \geq \delta(H) \) for each \( x \in T \), we have

\begin{align*}
\delta_H(S,T) &= b|S| + d_{H-S}(T) - a|T| \geq |T||S| + d_{H-S}(T) - a|T| \\
&= \sum_{x \in T} (|S| + d_{H-S}(x) - a) \geq \sum_{x \in T} (\delta(H) - a) \geq 0,
\end{align*}

which contradicts (1).

\end{proof}

\begin{claim}
a|T| > b|S|.
\end{claim}

\begin{proof}
If \( a|T| \leq b|S| \), then from (1) we obtain

\begin{align*}
-1 &\geq \delta_H(S,T) = b|S| + d_{H-S}(T) - a|T| \geq b|S| - a|T| \geq 0,
\end{align*}

which is a contradiction.

\end{proof}

\begin{claim}
|S| + |X| < \frac{(a+b)n}{a+2b}.
\end{claim}

\begin{proof}
According to Claim 1, Claim 4 and since \( |S| + |T| + |X| \leq n \), we have

\begin{align*}
an &\geq a|S| + a|T| + a|X| > a|S| + b|S| + a|X| = (a+b)(|S| + |X|) - b|X| \\
&\geq (a+b)(|S| + |X|) - \frac{b^2n}{a+2b},
\end{align*}

which implies

\begin{align*}
|S| + |X| &< \frac{(a+b)n}{a+2b}.
\end{align*}

\end{proof}
In view of Claim 3, $T \neq \emptyset$. Define

$$h_1 = \min \{d_{H-S}(x) : x \in T\}$$

and

$$R = \{x : x \in T, d_{H-S}(x) = 0\}.$$  

We write $r = |R|$ and choose $x_1 \in T$ such that $d_{H-S}(x_1) = h_1$. If $T \setminus N_T[x_1] \neq \emptyset$, let

$$h_2 = \min \{d_{H-S}(x) : x \in T \setminus N_T[x_1]\}.$$  

Thus, we have $0 \leq h_1 \leq h_2 \leq a$ by the definition of $T$.

We shall consider various cases by the value of $r$ and derive a contradiction in each case.

**Case 1.** $r \geq 2$. Obviously, there exist $x,y \in R$ such that $d_{H-S}(x) = d_{H-S}(y) = 0$ and $xy \notin E(G)$. In terms of $H = G - X$, Claim 5 and the condition of Theorem 1, we obtain

$$\frac{(a + b)n}{a + 2b} \leq |N_G(x) \cup N_G(y)| \leq |N_H(x) \cup N_H(y)| + |X| \leq d_{H-S}(x) + d_{H-S}(y) + |S| + |X| = |S| + |X| < \frac{(a + b)n}{a + 2b},$$

which is a contradiction.

**Case 2.** $r = 1$. Clearly, $h_1 = 0$ and $|N_T[x_1]| = 1$. According to Claim 3, $r = 1$ and $|N_T[x_1]| = 1$, we have $T \setminus N_T[x_1] \neq \emptyset$ and $1 \leq h_2 \leq a$. Choose $x_2 \in T \setminus N_T[x_1]$ such that $d_{H-S}(x_2) = h_2$. It is easy to see that $x_1x_2 \notin E(G)$. According to $H = G - X$ and the condition of Theorem 1, we have

$$\frac{(a + b)n}{a + 2b} \leq |N_G(x_1) \cup N_G(x_2)| \leq |N_H(x_1) \cup N_H(x_2)| + |X| \leq d_{H-S}(x_1) + d_{H-S}(x_2) + |S| + |X| = h_2 + |S| + |X|,$$

which implies

$$|S| \geq \frac{(a + b)n}{a + 2b} - h_2 - |X|.$$  

(2)  

Note that $|T \setminus N_T[x_1]| = |T| - 1$. Combining this with $|S| + |T| + |X| \leq n$, (2), Claim 1, $b \geq a \geq 1$, $1 \leq h_2 \leq a$ and $n \geq \frac{(a + 2b)(2a + 2b - 3) + 1}{b} > \frac{(a + 2b)(2a + 2b - 3)}{b}$, we obtain

$$\delta_H(S, T) = b|S| + d_{H-S}(T) - a|T|$$

$$= b|S| + d_{H-S}(N_T[x_1]) + d_{H-S}(T \setminus N_T[x_1]) - a|T|$$

$$= h_2 + |S| + |X| - a|T|,$$
Obviously, Claim 6.

Proof. Suppose that $a(3) h_2 = (n - |X|) - h_2$

We have

$$1 \leq n \leq a - 1.$$ 

We now prove the following claim.

Claim 6. $T \setminus N_T[x_1] \neq \emptyset$.

Proof. Suppose that $T = N_T[x_1]$. Then from (3) we have

$$|T| = |N_T[x_1]| \leq |N_{H-S}[x_1]| = d_{H-S}(x_1) + 1 = h_1 + 1 \leq a,$$

which contradicts Claim 3.

In view of Claim 6, there exists $x_2 \in T \setminus N_T[x_1]$ such that $d_{H-S}(x_2) = h_2$. Obviously, $x_1x_2 \notin E(G)$. According to the condition of Theorem 1, we obtain

$$\frac{(a+b)n}{a+2b} \leq |N_G(x_1) \cup N_G(x_2)| \leq |N_H(x_1) \cup N_H(x_2)| + |X|$$

$$\leq d_{H-S}(x_1) + d_{H-S}(x_2) + |S| + |X| = h_1 + h_2 + |S| + |X|,$$

that is,

$$|S| \geq \frac{(a+b)n}{a+2b} - h_1 - h_2 - |X|.$$
It is easy to see that

\[(5) \quad |N_T[x_1]| \leq |N_{H-S}[x_1]| = d_{H-S}(x_1) + 1 = h_1 + 1.\]

Using \(1 \leq h_1 \leq h_2 \leq a, |S| + |T| + |X| \leq n, (4), (5)\) and Claim 1, we have

\[
\delta_H(S, T) = b|S| + d_{H-S}(T) - a|T|
\geq b|S| + h_1|N_T[x_1]| + h_2(|T| - |N_T[x_1]|) - a|T|
= b|S| - (h_2 - h_1)|N_T[x_1]| - (a - h_2)|T|
\geq b|S| - (h_2 - h_1)(h_1 + 1) - (a - h_2)(n - |S| - |X|)
= (a + b - h_2)|S| - (h_2 - h_1)(h_1 + 1) - (a - h_2)n + (a - h_2)|X|
\geq (a + b - h_2) \left( \frac{(a + b)n}{a + 2b} - h_1 - h_2 - |X| \right) - (h_2 - h_1)(h_1 + 1)
- (a - h_2)n + (a - h_2)|X|
= (a + b - h_2) \left( \frac{(a + b)n}{a + 2b} - h_1 - h_2 \right) - (h_2 - h_1)(h_1 + 1)
- (a - h_2)n - \frac{b^2n}{a + 2b}
\]
\[
= \frac{bn}{a + 2b} h_2 - (a + b - h_2)(h_1 + h_2) - (h_2 - h_1)(h_1 + 1),
\]

that is,

\[(6) \quad \delta_H(S, T) \geq \frac{bn}{a + 2b} h_2 - (a + b - h_2)(h_1 + h_2) - (h_2 - h_1)(h_1 + 1).\]

Let \(F(h_1, h_2) = \frac{bn}{a + 2b} h_2 - (a + b - h_2)(h_1 + h_2) - (h_2 - h_1)(h_1 + 1)\). Thus, by (3) we have

\[
\frac{\partial F(h_1, h_2)}{\partial h_1} = -(a + b - h_2) - (-h_1 - 1 + h_2 - h_1) = -(a + b) + 2h_1 + 1
\leq -(a + b) + 2(a - 1) + 1 \leq -1.
\]

Combining this with \(1 \leq h_1 \leq h_2 \leq a, \) we obtain

\[(7) \quad F(h_1, h_2) \geq F(h_2, h_2).\]
In terms of (6), (7), \(1 \leq h_2 \leq a\) and \(n \geq \frac{(a+2b)(2a+2b-3)+1}{a+2b} > \frac{(a+2b)(2a+2b-3)}{a+2b}\), we have

\[
\delta_H(S, T) \geq F(h_1, h_2) \geq F(h_2, h_2) = \frac{bn}{a+2b} h_2 - 2(a+b-h_2)h_2 > \frac{(a+2b)(2a+2b-3)}{a+2b} h_2 - 2(a+b-h_2)h_2 \]

\[
= h_2(2h_2-3) \geq -1,
\]

which contradicts (1).

In all the cases above we obtained contradictions. Hence, \(H\) has a fractional \([a, b]\)-factor, that is, \(G\) is fractional ID-[a, b]-factor-critical. The proof of Theorem 1 is complete.

3. Remarks

**Remark 5.** In Theorem 1, the bound in the condition

\[
|N_G(x) \cup N_G(y)| \geq \frac{(a+b)n}{a+2b}
\]

is sharp. We can show this by constructing a graph \(G = (nt)K_1 \lor (bt)K_1 \lor (bt+1)K_1\), where \(t\) is a sufficiently large positive integer. It is easy to see that \(|V(G)| = n = (a+2b)t+1\) and

\[
\frac{(a+b)n}{a+2b} > \frac{(a+b)n}{a+2b} = (a+b)t = (a+b) \cdot \frac{n-1}{a+2b}
\]

\[
= \frac{(a+b)n}{a+2b} - \frac{a+b}{a+2b} > \frac{(a+b)n}{a+2b} - 1
\]

for each pair of nonadjacent vertices \(x, y\) of \((bt+1)K_1 \subset G\). Set \(X = (bt)K_1\). Clearly, \(X\) is an independent set of \(G\). Put \(H = G - X = (at)K_1 \lor (bt+1)K_1\), \(S = (at)K_1\) and \(T = (bt+1)K_1\). Then \(|S| = at\), \(|T| = bt+1\) and \(d_H - S(T) = 0\). Thus, we have

\[
\delta_H(S, T) = b|S| + d_{H-S}(T) - a|T| = abt - a(bt+1) = -a < 0.
\]

In terms of Lemma 4, \(H\) has no fractional \([a, b]\)-factor. Hence, \(G\) is not fractional ID-[a, b]-factor-critical.

**Remark 6.** We show that the bound on minimum degree \(\delta(G) \geq \frac{bn}{a+2b} + a\) in Theorem 1 is also best possible. Consider a graph \(G\) constructed from \(btK_1\), \((at-1)K_1\), \(\frac{b-1}{2}K_2\) and \(K_1\) as follows: let \(\{x_1, x_2, \ldots, x_{a-1}\} \subset (at-1)K_1\) and
$K_1 = \{u\}$, where $t$ is a sufficiently large positive integer and $bt$ is even. Set $V(G) = V(btK_1 \cup (at-1)K_1 \cup \frac{bt}{2}K_2 \cup \{u\})$ and $E(G) = E(btK_1 \cup (at-1)K_1 \cup \frac{bt}{2}K_2)$ 
∪ $E(btK_1 \cup \{u\})$ ∪ $\{ux_i : i = 1, 2, \ldots, a-1\}$. It is easily seen that $|N_G(x) \cup N_G(y)| \geq \frac{(a+b)n}{a+2b}$ for each pair of nonadjacent vertices $x, y$ of $G$, $n = (a+2b)t$ and $\delta(G) = \frac{bn}{a+2b} + a - 1$. Let $X = btK_1$. It is easy to see that $X$ is an independent set of $G$. Set $H = G - X$. Then $\delta(H) = d_H(u) = a - 1$. Clearly, $H$ has no fractional $[a, b]$-factor, that is, $G$ is not fractional ID-$[a, b]$-factor-critical.

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References


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