Discussiones Mathematicae
Graph Theory 36 (2016) 409–418
doi:10.7151/dmgt.1864

A NEIGHBORHOOD CONDITION FOR FRACTIONAL ID-$[A, B]$-FACTOR-CRITICAL GRAPHS

Sizhong Zhou, Fan Yang

School of Mathematics and Physics
Jiangsu University of Science and Technology
Mengxi Road 2, Zhenjiang, Jiangsu 212003, P.R. China

e-mail: zsz_cumt@163.com
fanyang_just@163.com

and

Zhiren Sun

School of Mathematical Sciences
Nanjing Normal University Nanjing
Jiangsu 210046, P.R. China

e-mail: 05119@njnu.edu.cn

Abstract

Let G be a graph of order n, and let a and b be two integers with $1 \leq a \leq b$. Let $h : E(G) \to [0, 1]$ be a function. If $a \leq \sum_{e \ni x} h(e) \leq b$ holds for any $x \in V(G)$, then we call $G[F_h]$ a fractional $[a, b]$-factor of G with indicator function h, where $F_h = \{e \in E(G) : h(e) > 0\}$. A graph G is fractional independent-set-deletable $[a, b]$-factor-critical (in short, fractional ID-$[a, b]$-factor-critical) if $G - I$ has a fractional $[a, b]$-factor for every independent set I of G. In this paper, it is proved that if $n \geq \frac{(a+2b)(2a+2b-3)+1}{b}$, $\delta(G) \geq \frac{bn}{a+2b} + a$ and $|N_G(x) \cup N_G(y)| \geq \frac{(a+b)n}{a+2b}$ for any two nonadjacent vertices $x, y \in V(G)$, then G is fractional ID-$[a, b]$-factor-critical. Furthermore, it is shown that this result is best possible in some sense.

Keywords: graph, minimum degree, neighborhood, fractional $[a, b]$-factor, fractional ID-$[a, b]$-factor-critical graph.

2010 Mathematics Subject Classification: 05C70, 05C72, 05C35.

1. Introduction

The graphs considered here will be finite, undirected and without loops or multiple edges. Let G be a graph. We denote by $V(G)$ and $E(G)$ the set of vertices and the set of edges of G, respectively. For any $x \in V(G)$, we denote the degree of x in G by $d_G(x)$. We write $N_G(x)$ for the set of vertices adjacent to x in G, and $N_G[x]$ for $N_G(x) \cup \{x\}$. For $S \subseteq V(G)$, we use $G[S]$ to denote the subgraph of G induced by S, and $G - S = G[V(G) \setminus S]$. Let S and T be two disjoint vertex subsets of G; we denote the number of edges from S to T by $e_G(S,T)$. We denote by $\delta(G)$ the minimum degree of G. For any nonempty subset S of $V(G)$, let

$$N_G(S) = \bigcup_{x \in S} N_G(x).$$

If G and H are vertex-disjoint graphs, then their join and union are denoted by $G \vee H$ and $G \cup H$, respectively.

A factor of a graph G is a spanning subgraph of G. Let a and b be two positive integers with $1 \le a \le b$. Then a factor F is an $[a,b]$-factor if $a \le d_F(x) \le b$ for each $x \in V(G)$. Let $h : E(G) \to [0,1]$ be a function. If $a \le \sum_{e \ni x} h(e) \le b$ holds for any $x \in V(G)$, then we call $G[F_h]$ a fractional $[a,b]$-factor of G with indicator function h, where $F_h = \{e \in E(G) : h(e) > 0\}$. If $G - I$ admits a fractional $[a,b]$-factor for every independent set I of G, then we say that G is fractional ID-$[a,b]$-factor-critical [1]. A fractional ID-$[k,k]$-factor-critical graph is simply called a fractional ID-k-factor-critical graph.

Many authors have investigated factors and fractional factors in graphs; see, for instance, [2, 3, 4, 5, 6, 7, 8, 9]. Chang, Liu and Zhu [10] showed a minimum degree condition for a graph to be a fractional ID-k-factor-critical graph. Zhou, Bian and Wu [11] gave a degree condition for the existence of fractional ID-k-factor-critical graphs. Zhou [12] obtained a binding number condition for graphs to be fractional ID-k-factor-critical graphs. Zhou, Sun and Liu [1] obtained a minimum degree condition for a graph to be a fractional ID-$[a,b]$-factor-critical graph. In this paper, we proceed to study fractional ID-$[a,b]$-factor-critical graphs, and obtain a neighborhood condition for a graph to be fractional ID-$[a,b]$-factor-critical. The main result is the following theorem.

Theorem 1. *Let $1 \le a \le b$ be two integers, and let G be a graph of order n with $n \ge \frac{(a+2b)(2a+2b-3)+1}{b}$, and $\delta(G) \ge \frac{bn}{a+2b} + a$. If*

$$|N_G(x) \cup N_G(y)| \ge \frac{(a+b)n}{a+2b}$$

for any two nonadjacent vertices $x, y \in V(G)$, then G is fractional ID-$[a,b]$-factor-critical.

If $a = b = k$ in Theorem 1, then we obtain the following result.

Theorem 2. *Let $k \geq 1$ be an integer, and let G be a graph of order n with $n \geq 12k - 8$, and $\delta(G) \geq \frac{n}{3} + k$. If*

$$|N_G(x) \cup N_G(y)| \geq \frac{2n}{3}$$

for any two nonadjacent vertices $x, y \in V(G)$, then G is fractional ID-k-factor-critical.

If $k = 1$ in Theorem 2, then we get the following result.

Theorem 3. *Let G be a graph of order n with $n \geq 4$, and $\delta(G) \geq \frac{n}{3} + 1$. If*

$$|N_G(x) \cup N_G(y)| \geq \frac{2n}{3}$$

for any two nonadjacent vertices $x, y \in V(G)$, then G is fractional ID-factor-critical.

2. The Proof of Theorem 1

In order to prove Theorem 1, we rely heavily on the following lemma.

Lemma 4 [13]**.** *Let G be a graph. Then G has a fractional $[a, b]$-factor if and only if for every subset S of $V(G)$,*

$$\delta_G(S, T) = b|S| + d_{G-S}(T) - a|T| \geq 0,$$

where $T = \{x : x \in V(G) \setminus S, d_{G-S}(x) \leq a\}$ and $d_{G-S}(T) = \sum_{x \in T} d_{G-S}(x)$.

***Proof of Theorem* 1.** Let X be an independent set of G and $H = G - X$. In order to complete the proof of Theorem 1, we need only to prove that H has a fractional $[a, b]$-factor. By contradiction, suppose that H has no fractional $[a, b]$-factor. Then by Lemma 4, there exists some subset $S \subseteq V(H)$ such that

$$\delta_H(S, T) = b|S| + d_{H-S}(T) - a|T| \leq -1, \tag{1}$$

where $T = \{x : x \in V(H) \setminus S, d_{H-S}(x) \leq a\}$. We first prove the following claims.

Claim 1. $|X| \leq \frac{bn}{a+2b}$.

Proof. Since $n \geq \frac{(a+2b)(2a+2b-3)+1}{b}$, the inequality holds for $|X| = 1$. In the following we may assume $|X| \geq 2$. In terms of the condition of Theorem 1, there exist $x, y \in X$ such that $|N_G(x) \cup N_G(y)| \geq \frac{(a+b)n}{a+2b}$. Since X is independent, we obtain $X \cap (N_G(x) \cup N_G(y)) = \emptyset$. Thus, we have

$$|X| + \frac{(a+b)n}{a+2b} \leq |X| + |N_G(x) \cup N_G(y)| \leq n,$$

which implies

$$|X| \leq n - \frac{(a+b)n}{a+2b} = \frac{bn}{a+2b}.$$

□

Claim 2. $\delta(H) \geq a$.

Proof. Note that $H = G - X$. Combining this with Claim 1, we obtain

$$\delta(H) \geq \delta(G) - |X| \geq \left(\frac{bn}{a+2b} + a\right) - \frac{bn}{a+2b} = a.$$

□

Claim 3. $|T| \geq b + 1$.

Proof. If $|T| \leq b$, then from Claim 2 and since $|S| + d_{H-S}(x) \geq d_H(x) \geq \delta(H)$ for each $x \in T$, we have

$$\begin{aligned}\delta_H(S,T) &= b|S| + d_{H-S}(T) - a|T| \geq |T||S| + d_{H-S}(T) - a|T| \\ &= \sum\nolimits_{x\in T}(|S| + d_{H-S}(x) - a) \geq \sum\nolimits_{x\in T}(\delta(H) - a) \geq 0,\end{aligned}$$

which contradicts (1). □

Claim 4. $a|T| > b|S|$.

Proof. If $a|T| \leq b|S|$, then from (1) we obtain

$$-1 \geq \delta_H(S,T) = b|S| + d_{H-S}(T) - a|T| \geq b|S| - a|T| \geq 0,$$

which is a contradiction. □

Claim 5. $|S| + |X| < \frac{(a+b)n}{a+2b}$.

Proof. According to Claim 1, Claim 4 and since $|S| + |T| + |X| \leq n$, we have

$$\begin{aligned}an &\geq a|S| + a|T| + a|X| > a|S| + b|S| + a|X| = (a+b)(|S| + |X|) - b|X| \\ &\geq (a+b)(|S| + |X|) - \frac{b^2 n}{a+2b},\end{aligned}$$

which implies

$$|S| + |X| < \frac{(a+b)n}{a+2b}.$$

□

In view of Claim 3, $T \neq \emptyset$. Define

$$h_1 = \min\{d_{H-S}(x) : x \in T\}$$

and

$$R = \{x : x \in T, d_{H-S}(x) = 0\}.$$

We write $r = |R|$ and choose $x_1 \in T$ such that $d_{H-S}(x_1) = h_1$. If $T \setminus N_T[x_1] \neq \emptyset$, let

$$h_2 = \min\{d_{H-S}(x) : x \in T \setminus N_T[x_1]\}.$$

Thus, we have $0 \leq h_1 \leq h_2 \leq a$ by the definition of T.

We shall consider various cases by the value of r and derive a contradiction in each case.

Case 1. $r \geq 2$. Obviously, there exist $x, y \in R$ such that $d_{H-S}(x) = d_{H-S}(y) = 0$ and $xy \notin E(G)$. In terms of $H = G - X$, Claim 5 and the condition of Theorem 1, we obtain

$$\begin{aligned}\frac{(a+b)n}{a+2b} &\leq |N_G(x) \cup N_G(y)| \leq |N_H(x) \cup N_H(y)| + |X| \\ &\leq d_{H-S}(x) + d_{H-S}(y) + |S| + |X| = |S| + |X| < \frac{(a+b)n}{a+2b},\end{aligned}$$

which is a contradiction.

Case 2. $r = 1$. Clearly, $h_1 = 0$ and $|N_T[x_1]| = 1$. According to Claim 3, $r = 1$ and $|N_T[x_1]| = 1$, we have $T \setminus N_T[x_1] \neq \emptyset$ and $1 \leq h_2 \leq a$. Choose $x_2 \in T \setminus N_T[x_1]$ such that $d_{H-S}(x_2) = h_2$. It is easy to see that $x_1x_2 \notin E(G)$. According to $H = G - X$ and the condition of Theorem 1, we have

$$\begin{aligned}\frac{(a+b)n}{a+2b} &\leq |N_G(x_1) \cup N_G(x_2)| \leq |N_H(x_1) \cup N_H(x_2)| + |X| \\ &\leq d_{H-S}(x_1) + d_{H-S}(x_2) + |S| + |X| = h_2 + |S| + |X|,\end{aligned}$$

which implies

$$|S| \geq \frac{(a+b)n}{a+2b} - h_2 - |X|. \tag{2}$$

Note that $|T \setminus N_T[x_1]| = |T| - 1$. Combining this with $|S| + |T| + |X| \leq n$, (2), Claim 1, $b \geq a \geq 1$, $1 \leq h_2 \leq a$ and $n \geq \frac{(a+2b)(2a+2b-3)+1}{b} > \frac{(a+2b)(2a+2b-3)}{b}$, we obtain

$$\begin{aligned}\delta_H(S,T) &= b|S| + d_{H-S}(T) - a|T| \\ &= b|S| + d_{H-S}(N_T[x_1]) + d_{H-S}(T \setminus N_T[x_1]) - a|T|\end{aligned}$$

$$
\begin{aligned}
&= b|S| + d_{H-S}(T \setminus N_T[x_1]) - a|T| \geq b|S| + h_2(|T|-1) - a|T| \\
&= b|S| - (a-h_2)|T| - h_2 \geq b|S| - (a-h_2)(n - |S| - |X|) - h_2 \\
&= (a+b-h_2)|S| - (a-h_2)n + (a-h_2)|X| - h_2 \\
&\geq (a+b-h_2)\left(\frac{(a+b)n}{a+2b} - h_2 - |X|\right) - (a-h_2)n + (a-h_2)|X| - h_2 \\
&= (a+b-h_2)\left(\frac{(a+b)n}{a+2b} - h_2\right) - (a-h_2)n - b|X| - h_2 \\
&\geq (a+b-h_2)\left(\frac{(a+b)n}{a+2b} - h_2\right) - (a-h_2)n - \frac{b^2n}{a+2b} - h_2 \\
&= h_2^2 + \left(\frac{bn}{a+2b} - a - b - 1\right)h_2 \\
&> h_2^2 + \left(\frac{(a+2b)(2a+2b-3)}{a+2b} - a - b - 1\right)h_2 \\
&= h_2^2 + (a+b-4)h_2 \geq h_2^2 - 2h_2 = (h_2-1)^2 - 1 \geq -1,
\end{aligned}
$$

which contradicts (1).

Case 3. $r = 0$. If $h_1 = a$, then by (1) we obtain $-1 \geq \delta_H(S,T) = b|S| + d_{H-S}(T) - a|T| \geq b|S| + h_1|T| - a|T| = b|S| \geq 0$, which is a contradiction. Thus, we have

$$1 \leq h_1 \leq a-1. \tag{3}$$

We now prove the following claim.

Claim 6. $T \setminus N_T[x_1] \neq \emptyset$.

Proof. Suppose that $T = N_T[x_1]$. Then from (3) we have

$$|T| = |N_T[x_1]| \leq |N_{H-S}[x_1]| = d_{H-S}(x_1) + 1 = h_1 + 1 \leq a,$$

which contradicts Claim 3. □

In view of Claim 6, there exists $x_2 \in T \setminus N_T[x_1]$ such that $d_{H-S}(x_2) = h_2$. Obviously, $x_1x_2 \notin E(G)$. According to the condition of Theorem 1, we obtain

$$
\begin{aligned}
\frac{(a+b)n}{a+2b} &\leq |N_G(x_1) \cup N_G(x_2)| \leq |N_H(x_1) \cup N_H(x_2)| + |X| \\
&\leq d_{H-S}(x_1) + d_{H-S}(x_2) + |S| + |X| = h_1 + h_2 + |S| + |X|,
\end{aligned}
$$

that is,

$$|S| \geq \frac{(a+b)n}{a+2b} - h_1 - h_2 - |X|. \tag{4}$$

It is easy to see that

$$|N_T[x_1]| \le |N_{H-S}[x_1]| = d_{H-S}(x_1) + 1 = h_1 + 1. \tag{5}$$

Using $1 \le h_1 \le h_2 \le a$, $|S| + |T| + |X| \le n$, (4), (5) and Claim 1, we have

$$\begin{aligned}
\delta_H(S,T) &= b|S| + d_{H-S}(T) - a|T| \\
&= b|S| + d_{H-S}(N_T[x_1]) + d_{H-S}(T \setminus N_T[x_1]) - a|T| \\
&\ge b|S| + h_1|N_T[x_1]| + h_2(|T| - |N_T[x_1]|) - a|T| \\
&= b|S| - (h_2 - h_1)|N_T[x_1]| - (a - h_2)|T| \\
&\ge b|S| - (h_2 - h_1)(h_1 + 1) - (a - h_2)(n - |S| - |X|) \\
&= (a + b - h_2)|S| - (h_2 - h_1)(h_1 + 1) - (a - h_2)n + (a - h_2)|X| \\
&\ge (a + b - h_2)\left(\frac{(a+b)n}{a+2b} - h_1 - h_2 - |X|\right) - (h_2 - h_1)(h_1 + 1) \\
&- (a - h_2)n + (a - h_2)|X| \\
&= (a + b - h_2)\left(\frac{(a+b)n}{a+2b} - h_1 - h_2\right) - (h_2 - h_1)(h_1 + 1) \\
&- (a - h_2)n - b|X| \\
&\ge (a + b - h_2)\left(\frac{(a+b)n}{a+2b} - h_1 - h_2\right) - (h_2 - h_1)(h_1 + 1) \\
&- (a - h_2)n - \frac{b^2 n}{a+2b} \\
&= \frac{bn}{a+2b}h_2 - (a + b - h_2)(h_1 + h_2) - (h_2 - h_1)(h_1 + 1),
\end{aligned}$$

that is,

$$\delta_H(S,T) \ge \frac{bn}{a+2b}h_2 - (a + b - h_2)(h_1 + h_2) - (h_2 - h_1)(h_1 + 1). \tag{6}$$

Let $F(h_1, h_2) = \frac{bn}{a+2b}h_2 - (a + b - h_2)(h_1 + h_2) - (h_2 - h_1)(h_1 + 1)$. Thus, by (3) we have

$$\begin{aligned}
\frac{\partial F(h_1,h_2)}{\partial h_1} &= -(a + b - h_2) - (-h_1 - 1 + h_2 - h_1) = -(a + b) + 2h_1 + 1 \\
&\le -(a + b) + 2(a - 1) + 1 \le -1.
\end{aligned}$$

Combining this with $1 \le h_1 \le h_2 \le a$, we obtain

$$F(h_1, h_2) \ge F(h_2, h_2). \tag{7}$$

In terms of (6), (7), $1 \leq h_2 \leq a$ and $n \geq \frac{(a+2b)(2a+2b-3)+1}{b} > \frac{(a+2b)(2a+2b-3)}{b}$, we have

$$\begin{aligned}\delta_H(S,T) &\geq F(h_1,h_2) \geq F(h_2,h_2) = \frac{bn}{a+2b}h_2 - 2(a+b-h_2)h_2 \\ &> \frac{(a+2b)(2a+2b-3)}{a+2b}h_2 - 2(a+b-h_2)h_2 \\ &= h_2(2h_2-3) \geq -1,\end{aligned}$$

which contradicts (1).

In all the cases above we obtained contradictions. Hence, H has a fractional $[a,b]$-factor, that is, G is fractional ID-$[a,b]$-factor-critical. The proof of Theorem 1 is complete. ■

3. Remarks

Remark 5. In Theorem 1, the bound in the condition

$$|N_G(x) \cup N_G(y)| \geq \frac{(a+b)n}{a+2b}$$

is sharp. We can show this by constructing a graph $G = (at)K_1 \vee (bt)K_1 \vee (bt+1)K_1$, where t is a sufficiently large positive integer. It is easy to see that $|V(G)| = n = (a+2b)t+1$ and

$$\begin{aligned}\frac{(a+b)n}{a+2b} &> |N_G(x) \cup N_G(y)| = (a+b)t = (a+b)\cdot\frac{n-1}{a+2b} \\ &= \frac{(a+b)n}{a+2b} - \frac{a+b}{a+2b} > \frac{(a+b)n}{a+2b} - 1\end{aligned}$$

for each pair of nonadjacent vertices x, y of $(bt+1)K_1 \subset G$. Set $X = (bt)K_1$. Clearly, X is an independent set of G. Put $H = G - X = (at)K_1 \vee (bt+1)K_1$, $S = (at)K_1$ and $T = (bt+1)K_1$. Then $|S| = at$, $|T| = bt+1$ and $d_{H-S}(T) = 0$. Thus, we have

$$\begin{aligned}\delta_H(S,T) &= b|S| + d_{H-S}(T) - a|T| \\ &= abt - a(bt+1) = -a < 0.\end{aligned}$$

In terms of Lemma 4, H has no fractional $[a,b]$-factor. Hence, G is not fractional ID-$[a,b]$-factor-critical.

Remark 6. We show that the bound on minimum degree $\delta(G) \geq \frac{bn}{a+2b} + a$ in Theorem 1 is also best possible. Consider a graph G constructed from btK_1, $(at-1)K_1$, $\frac{bt}{2}K_2$ and K_1 as follows: let $\{x_1, x_2, \ldots, x_{a-1}\} \subset (at-1)K_1$ and

$K_1 = \{u\}$, where t is a sufficiently large positive integer and bt is even. Set $V(G) = V(btK_1 \cup (at-1)K_1 \cup \frac{bt}{2}K_2 \cup \{u\})$ and $E(G) = E(btK_1 \vee (at-1)K_1 \vee \frac{bt}{2}K_2) \cup E(btK_1 \vee \{u\}) \cup \{ux_i : i = 1, 2, \ldots, a-1\}$. It is easily seen that $|N_G(x) \cup N_G(y)\}| \geq \frac{(a+b)n}{a+2b}$ for each pair of nonadjacent vertices x, y of G, $n = (a+2b)t$ and $\delta(G) = \frac{bn}{a+2b} + a - 1$. Let $X = btK_1$. It is easy to see that X is an independent set of G. Set $H = G - X$. Then $\delta(H) = d_H(u) = a - 1$. Clearly, H has no fractional $[a, b]$-factor, that is, G is not fractional ID-$[a, b]$-factor-critical.

Acknowledgements

The authors would like to thank the anonymous referees for their comments on this paper. This work is supported by the National Natural Science Foundation of China (Grant No. 11371009, 11501256, 11326215) and the National Social Science Foundation of China (Grant No. 14AGL001).

References

[1] S. Zhou, Z. Sun and H. Liu, *A minimum degree condition for fractional ID-$[a, b]$-factor-critical graphs*, Bull. Aust. Math. Soc. **86** (2012) 177–183. doi:10.1017/S0004972711003467

[2] H. Matsuda, *A neighborhood condition for graphs to have $[a, b]$-factors*, Discrete Math. **224** (2000) 289–292. doi:10.1016/S0012-365X(00)00140-0

[3] J. Ekstein, P. Holub, T. Kaiser, L. Xiong and S. Zhang, *Star subdivisions and connected even factors in the square of a graph*, Discrete Math. **312** (2012) 2574–2578. doi:10.1016/j.disc.2011.09.004

[4] H. Lu, *Simplified existence theorems on all fractional $[a, b]$-factors*, Discrete Appl. Math. **161** (2013) 2075–2078. doi:10.1016/j.dam.2013.02.006

[5] S. Zhou, *Independence number, connectivity and (a, b, k)-critical graphs*, Discrete Math. **309** (2009) 4144–4148. doi:10.1016/j.disc.2008.12.013

[6] S. Zhou, *A sufficient condition for graphs to be fractional (k, m)-deleted graphs*, Appl. Math. Lett. **24** (2011) 1533–1538. doi:10.1016/j.aml.2011.03.041

[7] S. Zhou and H. Liu, *Neighborhood conditions and fractional k-factors*, Bull. Malays. Math. Sci. Soc. **32** (2009) 37–45.

[8] K. Kimura, *f-factors, complete-factors, and component-deleted subgraphs*, Discrete Math. **313** (2013) 1452–1463. doi:10.1016/j.disc.2013.03.009

[9] M. Kouider and Z. Lonc, *Stability number and $[a, b]$-factors in graphs*, J. Graph Theory **46** (2004) 254–264. doi:10.1002/jgt.20008

[10] R. Chang, G. Liu and Y. Zhu, *Degree conditions of fractional ID-k-factor-critical graphs*, Bull. Malays. Math. Sci. Soc. **33** (2010) 355–360.

[11] S. Zhou, Q. Bian and J. Wu, *A result on fractional ID-k-factor-critical graphs*, J. Combin. Math. Combin. Comput. **87** (2013) 229–236.

[12] S. Zhou, *Binding numbers for fractional ID-k-factor-critical graphs*, Acta Math. Sin. Engl. Ser. **30** (2014) 181–186.
doi:10.1007/s10114-013-1396-9

[13] G. Liu and L. Zhang, *Fractional (g, f)-factors of graphs*, Acta Math. Sci. Ser. B **21** (2001) 541–545.

Received 12 February 2015
Accepted 17 June 2015