ON SPECTRA OF VARIANTS OF THE CORONA OF TWO GRAPHS AND SOME NEW EQUIENERGETIC GRAPHS

Chandrashekar Adiga and B.R. Rakshith

Department of Studies in Mathematics
University of Mysore, Manasagangothri
Mysore – 570 006, India

Abstract

Let $G$ and $H$ be two graphs. The join $G \vee H$ is the graph obtained by joining every vertex of $G$ with every vertex of $H$. The corona $G \circ H$ is the graph obtained by taking one copy of $G$ and $|V(G)|$ copies of $H$ and joining the $i$-th vertex of $G$ to every vertex in the $i$-th copy of $H$. The neighborhood corona $G \star H$ is the graph obtained by taking one copy of $G$ and $|V(G)|$ copies of $H$ and joining the neighbors of the $i$-th vertex of $G$ to every vertex in the $i$-th copy of $H$. The edge corona $G \diamond H$ is the graph obtained by taking one copy of $G$ and $|E(G)|$ copies of $H$ and joining each terminal vertex of $i$-th edge of $G$ to every vertex in the $i$-th copy of $H$. Let $G_1$, $G_2$, $G_3$ and $G_4$ be regular graphs with disjoint vertex sets. In this paper we compute the spectrum of $(G_1 \vee G_2) \cup (G_1 \star G_3)$, $(G_1 \vee G_2) \cup (G_2 \star G_3) \cup (G_1 \star G_4)$, $(G_1 \vee G_2) \cup (G_1 \circ G_3)$, $(G_1 \vee G_2) \cup (G_2 \circ G_3) \cup (G_1 \circ G_4)$, $(G_1 \vee G_2) \cup (G_1 \circ G_3)$, $(G_1 \vee G_2) \cup (G_2 \circ G_3) \cup (G_1 \circ G_4)$, $(G_1 \vee G_2) \cup (G_2 \circ G_3) \cup (G_1 \circ G_4)$ and $(G_1 \vee G_2) \cup (G_2 \star G_3) \cup (G_1 \star G_4)$. As an application, we show that there exist some new pairs of equienergetic graphs on $n$ vertices for all $n \geq 11$.

Keywords: spectrum, corona, neighbourhood corona, edge corona, energy of a graph, equienergetic graphs.

2010 Mathematics Subject Classification: 05C50.

1. Introduction

Throughout this paper we consider only undirected simple graphs (i.e., graphs with no loops and multiple edges). Let $G$ be a graph on $n$ vertices. The eigenvalues of the adjacency matrix of $G$, denoted by $\lambda_i(G)$, $i = 1, 2, \ldots, n$, are
the eigenvalues of the graph $G$ and $\sigma(G) = (\lambda_1(G), \lambda_2(G), \ldots, \lambda_n(G))$, where $\lambda_1(G) \geq \lambda_2(G) \geq \cdots \geq \lambda_n(G)$ is the adjacency spectrum of $G$ [8]. The energy $E(G)$ is the sum of all absolute values of eigenvalues of $G$. The concept of energy of a graph was introduced by Gutman [12] with an application to chemistry (Huckel molecular orbital approximation for the total $\pi$-electron energy [14]). The energy and various forms of energy of a graph $G$ has been extensively studied by many mathematicians and some of their works can be found in [1, 2, 3, 5, 13, 15, 19, 21, 28, 27] and references therein. Two graphs $G_1$ and $G_2$ of the same order are said to be equienergetic if $E(G_1) = E(G_2)$. Graphs of the same order are equienergetic if they have the same spectrum. Thus, two equienergetic graphs are obviously equienergetic. For connected graphs, there are no equienergetic graphs of order $n \leq 5$. In [18] Indulal and Vijayakumar have constructed a pair of equienergetic graphs on $n$ vertices for $n = 6, 14, 18$ and for all $n \geq 20$. Later Liu et al. [22] and Ramane, Walikar [26] have independently proved that there exists a pair of equienergetic graphs on $n$ vertices for all $n \geq 9$. Studies on equienergetic graphs can be found in [6, 11, 18, 20, 22, 25, 26, 29] and references therein.

The corona of two graphs was first introduced by Frucht and Harary in [10]. Barik et al. [4] provided a complete description of the spectrum of corona $G_1 \circ G_2$ using the spectrum of $G_1$ and $G_2$. More about the spectrum of corona can be found in [4, 7, 10, 24]. The neighborhood corona and edge corona was introduced in [17] and in [16], respectively. Complete description of the spectrum of neighborhood corona and edge corona of two graphs are given in [17, 23] and [16], respectively.

Motivated by the above works, in this paper we compute the spectrum of $(G_1 \vee G_2) \cup (G_1 \ast G_3), (G_1 \vee G_2) \cup (G_2 \ast G_3) \cup (G_1 \ast G_4), (G_1 \vee G_2) \cup (G_1 \circ G_3), (G_1 \vee G_2) \cup (G_2 \circ G_3) \cup (G_1 \circ G_4), (G_1 \vee G_2) \cup (G_1 \circ G_3), (G_1 \vee G_2) \cup (G_2 \circ G_3) \cup (G_1 \circ G_4), (G_1 \vee G_2) \cup (G_2 \circ G_3) \cup (G_1 \ast G_3), (G_1 \vee G_2) \cup (G_2 \circ G_3) \cup (G_1 \ast G_4)$ and $(G_1 \vee G_2) \cup (G_2 \ast G_3) \cup (G_1 \circ G_4)$, when $G_1$, $G_2$, $G_3$ and $G_4$ are regular graphs. Here the graphs $G_1$, $G_2$, $G_3$ and $G_4$ have disjoint vertex sets. As an application of our results we construct some new pairs of equienergetic graphs on $n$ vertices for all $n \geq 11$. Our method of construction and proofs are entirely different from the methods given in [18, 22, 26].

2. Preliminaries

In this section, we give some definitions and lemmas which are useful to prove our main results.

Definition [10]. Let $G_1$ and $G_2$ be two graphs on $n$ and $m$ vertices, respectively. The corona $G_1 \circ G_2$ of $G_1$ and $G_2$ is defined as the graph obtained by taking one
copy of $G_1$ and $n$ copies of $G_2$, and then joining the $i$-th vertex of $G_1$ to every vertex in the $i$-th copy of $G_2$.

**Definition** [16]. Let $G_1$ and $G_2$ be two graphs on $n_1$ and $n_2$ vertices, $m_1$ and $m_2$ edges, respectively. The edge corona $G_1 \circ G_2$ of $G_1$ and $G_2$ is defined as the graph obtained by taking one copy of $G_1$ and $m_1$ copies of $G_2$, and then joining two end vertices of the $i$-th edge of $G_1$ to every vertex in the $i$-th copy of $G_2$.

**Definition** [17]. Let $G_1$ and $G_2$ be two graphs on $n$ and $m$ vertices, respectively. The neighborhood corona $G_1 \star G_2$ of $G_1$ and $G_2$ is defined as the graph obtained by taking one copy of $G_1$ and $n$ copies of $G_2$, and then joining each neighbor of $i$-th vertex of $G_1$ to every vertex in the $i$-th copy of $G_2$.

**Definition** [8]. Let $A = (a_{ij})$ be an $n \times m$ matrix, $B = (b_{ij})$ be a $p \times q$ matrix. Then the Kronecker product $A \otimes B$ of $A$ and $B$ is the $np \times mq$ matrix obtained by replacing each entry $a_{ij}$ of $A$ by $a_{ij}B$.

**Lemma 1** [8]. If $M$, $N$, $P$, $Q$ are matrices with $M$ being a non-singular matrix, then

$$\begin{vmatrix} M & N \\ P & Q \end{vmatrix} = |M||Q - PM^{-1}N|.$$  

**Lemma 2** [26]. Let $N_1$ and $N_2$ be two graphs as shown in Figure 1. Then the line graph $L(N_1)$ of $N_1$ and the line graph $L(N_2)$ of $N_2$ are non cospectral and equienergetic.

---

**Lemma 3** [8]. The following cubic regular graphs with ten vertices are equienergetic.

---

**Figure 1**
3. Spectra of \((G_1 \lor G_2) \cup (G_1 \ast G_3)\) and \((G_1 \lor G_2) \cup (G_2 \ast G_3) \cup (G_1 \ast G_4)\)

In this section, we compute the spectrum of \((G_1 \lor G_2) \cup (G_1 \ast G_3)\) and \((G_1 \lor G_2) \cup (G_2 \ast G_3) \cup (G_1 \ast G_4)\), where \(G_1, G_2, G_3\) and \(G_4\) are regular graphs on \(n, m, l\) and \(p\) vertices, respectively.

**Theorem 4.** Let \(G_i\) be \(r_i\)-regular graphs \((i = 1, 2, 3)\). Suppose \(\sigma(G_1) = (\lambda_1 = r_1, \lambda_2, \ldots, \lambda_n), \sigma(G_2) = (\mu_1 = r_2, \mu_2, \ldots, \mu_m)\) and \(\sigma(G_3) = (\gamma_1 = r_3, \gamma_2, \ldots, \gamma_l)\) are the adjacency spectrum of \(G_1, G_2\) and \(G_3\), respectively. Then the adjacency spectrum of \(G = (G_1 \lor G_2) \cup (G_1 \ast G_3)\) is

\[
\sigma(G) = \begin{pmatrix}
\gamma_i & \mu_j & \left(\frac{\lambda_k + r_3 \pm \sqrt{4l\lambda_k^2 + (\lambda_k - r_3)^2}}{2}\right) & x_t \\
\lambda_k & \mu_j & 1 & 1 \\
\end{pmatrix},
\]

where \(i = 2\) to \(l\), \(j = 2\) to \(m\), \(k = 2\) to \(n\), \(t = 1, 2, 3\). Also, the entries in the first row are the eigenvalues with multiplicity written below, and \(x_t's\) are the roots of the polynomial \((x - r_2) \left((x - r_1)(x - r_3) - lr_1^2\right) - nm (x - r_3)\).

**Proof.** With suitable labelling of the vertices of \(G\), the adjacency matrix \(A(G)\) can be formulated as follows:

\[
A(G) = \begin{pmatrix}
I_n \otimes A(G_3) & 0 & A(G_1) \otimes e \\
0 & A(G_2) & J \\
A(G_1) \otimes e^T & J^T & A(G_1)
\end{pmatrix},
\]

where \(e^T = (1, 1, \ldots, 1)\), \(I_n\) is the identity matrix of order \(n\) and \(J\) is the \(m \times n\) matrix with all its entries are 1.
Since $A(G_3)$ is a real symmetric matrix, $A(G_3)$ is orthogonally diagonalizable. Let $A(G_3) = PDP^T$, where $PP^T = I_l$ and $D = \text{diag}(\gamma_1, \ldots, \gamma_l)$. Then

$$A(G) = \begin{pmatrix} I_n \otimes PDP^T & 0 & A(G_1) \otimes e \\ 0 & A(G_2) & J \\ A(G_1) \otimes e^T & J^T & A(G_1) \end{pmatrix}$$

$$= \begin{pmatrix} I_n \otimes P & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} I_n \otimes D & 0 & A(G_1) \otimes P^Te \\ 0 & A(G_2) & J \\ A(G_1) \otimes e^TP & J^T & A(G_1) \end{pmatrix} \begin{pmatrix} I_n \otimes P^T & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} I_n \otimes P & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} I_n \otimes D & 0 & A(G_1) \otimes \sqrt{\gamma_1}e \\ 0 & A(G_2) & J \\ A(G_1) \otimes \sqrt{\gamma_1}e^T & J^T & A(G_1) \end{pmatrix} \begin{pmatrix} I_n \otimes P^T & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

where $e^T_1 = (1, 0, \ldots, 0)$.

Let $B = \begin{pmatrix} I_n \otimes D & 0 & A(G_1) \otimes \sqrt{\gamma_1}e \\ 0 & A(G_2) & J \\ A(G_1) \otimes \sqrt{\gamma_1}e^T & J^T & A(G_1) \end{pmatrix}$.

Then by the above equation we have

$$|xI - A(G)| = |xI - B|.$$ 

Expanding $|xI - B|$ by Laplace’s method [9] along $(li + 2), (li + 3), \ldots, (li + l)^{th}$ columns, for $i = 0, 1, \ldots, n - 1$, we see that the only non zero $(l - 1)n \times (l - 1)n$ minor is

$$M = |I_n \otimes \text{diag}(x - \gamma_2, \ldots, x - \gamma_l)|.$$ 

The complementary minor of $M$ is

$$M_1 = \begin{vmatrix} (x - r_3)I_m & 0 & -\sqrt{\gamma_1}A(G_1) \\ 0 & xI_m - A(G_2) & -J \\ -\sqrt{\gamma_1}A(G_1) & -J^T & xI_n - A(G_1) \end{vmatrix}. $$
Corollary 7. Due to Indulal and Vijayakumar [18], there exists a pair of equienergetic graphs on \( n \) vertices for all \( n \).

Theorem 8. Now we construct some new pairs of equienergetic graphs using Corollary 7.

Remark 6. Corollary 5 is a generalization of Theorem 1 in [18]. In fact putting \( r_1 = r, n = p, r_2 = 0, m = k, r_3 = 0, l = 1 \) in Corollary 5, we obtain Theorem 1 due to Indulal and Vijayakumar [18].

Corollary 7. Let \( G_i \) (\( i = 1, 2 \)) be equienergetic regular graphs of the same degree and \( H_i \) (\( i = 1, 2 \)) be equienergetic regular graphs of the same degree. Then

\[
E(G_1 \lor H_1 \cup G_1 \star lK_1) = E(G_2 \lor H_2 \cup G_2 \star lK_1).
\]

Now we construct some new pairs of equienergetic graphs using Corollary 7.

Theorem 8. There exists a pair of equienergetic graphs on \( n \) vertices for all \( n \geq 18 \).
**Proof.** From Lemma 2 we have the line graphs $L(N_1)$ and $L(N_2)$ are equienergetic. Now by Corollary 7 it is clear that the graphs $(L(N_1) \lor K_m) \cup (L(N_1) \star K_1)$ and $(L(N_2) \lor K_m) \cup (L(N_2) \star K_1)$, both of order $18 + m$ ($m = 0, 1, \ldots$), are equienergetic. This completes the proof of the theorem.

**Theorem 9.** There exists a pair of equienergetic graphs on $n$ vertices for all $n \geq 20$.

**Proof.** From Lemma 3 and Corollary 7 it is clear that the graphs $(T_1 \lor K_m) \cup (T_1 \star K_1)$ and $(T_2 \lor K_m) \cup (T_2 \star K_1)$, both of order $20 + m$ ($m = 0, 1, \ldots$), are equienergetic.

**Theorem 10.** There exists a pair of equienergetic graphs on $n$ vertices for all $n \geq 13$.

**Proof.** Case 1. $n = 9 + 2m$ ($m = 2, 3, \ldots$). For $n = 9 + 2m$ ($m = 2, 3, \ldots$), the graphs $(K_m \lor L(N_1)) \cup (K_m \star K_1)$ and $(K_m \lor L(N_2)) \cup (K_m \star K_1)$ both are of order $9 + 2m$ ($m = 2, 3, \ldots$). Now, Corollary 7 implies that these two graphs are equienergetic.

Case 2. $n = 10 + 2m$ ($m = 2, 3, \ldots$). For $n = 10 + 2m$ ($m = 2, 3, \ldots$), the graphs $(K_m \lor T_1) \cup (K_m \star K_1)$ and $(K_m \lor T_2) \cup (K_m \star K_1)$ both are of order $10 + 2m$ ($m = 2, 3, \ldots$). Now, Corollary 7 implies that these two graphs are equienergetic.

As the proof of the following theorem is similar to that of Theorem 4, we omit the details.

**Theorem 11.** Let $G_i$ be $r_i$-regular graphs ($i = 1, 2, 3, 4$). Suppose $\sigma(G_1) = (\lambda_1 = r_1, \lambda_2, \ldots, \lambda_n)$, $\sigma(G_2) = (\mu_1 = r_2, \mu_2, \ldots, \mu_m)$, $\sigma(G_3) = (\gamma_1 = r_3, \gamma_2, \ldots, \gamma_l)$ and $\sigma(G_4) = (\eta_1 = r_4, \eta_2, \ldots, \eta_p)$ are the adjacency spectrum of $G_1$, $G_2$, $G_3$ and $G_4$, respectively. Then the adjacency spectrum of $G = (G_1 \lor G_2) \cup (G_2 \star G_3) \cup (G_1 \star G_4)$ is

$$\sigma(G) = \begin{pmatrix}
\gamma_i & \eta_j \\
n & m
\end{pmatrix} \begin{pmatrix}
\lambda_k + r_4 \pm \sqrt{4p\lambda_k^2 + (\lambda_k - r_4)^2} / 2 \\
1
\end{pmatrix},$$

$$\begin{pmatrix}
\mu_s + r_3 \pm \sqrt{4l\mu_s^2 + (\mu_s - r_3)^2} / 2 \\
1
\end{pmatrix} x_t,$$

where $i = 2$ to $l$, $j = 2$ to $p$, $k = 2$ to $n$, $s = 2$ to $m$, $t = 1, 2, 3, 4$. Also, the entries in the first row are the eigenvalues with multiplicity written below, and $x_i$’s are the roots of the polynomial

$$(x - r_1)(x - r_4) - pr_1^2) ((x - r_2)(x - r_3) - lr_2^2) - nm(x - r_3)(x - r_4).$$
Corollary 12. Let $G_i$ be $r_i$-regular graphs ($i = 1, 2$). Then
\[
E(G_1 \lor G_2 \cup G_2 \star lK_1 \cup G_1 \star pK_1) = \sqrt{4p + 1}E(G_1) + \sqrt{4l + 1}E(G_2) - r_2(\sqrt{4l + 1} - 1)
\]
where $x_0$ and $x_1$ are the negative roots of the polynomial
\[
x^4 - (r_1 + r_2)x^3 + (-r_1^2p - lr_2^2 + r_1r_2 - mn)x^2 + (r_1^2r_2p + r_1r_2^2l)x + r_1^2r_2^2lp.
\]

Corollary 13. Let $G_1$, $G_2$ be equienergetic regular graphs of the same degree and $H_1$, $H_2$ be equienergetic regular graphs of the same degree. Then
\[
E(G_1 \lor H_1 \cup H_1 \star lK_1 \cup G_1 \star pK_1) = E(G_2 \lor H_2 \cup H_2 \star lK_1 \cup G_2 \star pK_1).
\]

4. Spectra of $(G_1 \lor G_2) \cup (G_1 \circ G_3)$ and $(G_1 \lor G_2) \cup (G_2 \circ G_3) \cup (G_1 \circ G_4)$

In this section, we simply state some theorems (as the proofs are quite analogous to the proof of Theorem 4) which gives the spectrum of $(G_1 \lor G_2) \cup (G_1 \circ G_3)$ and $(G_1 \lor G_2) \cup (G_2 \circ G_3) \cup (G_1 \circ G_4)$, where $G_1$, $G_2$, $G_3$ and $G_4$ are regular graphs on $n$, $m$, $l$ and $p$ vertices, respectively.

Theorem 14. Let $G_i$ be $r_i$-regular graphs ($i = 1, 2, 3$). Suppose $\sigma(G_1) = (\lambda_1 = r_1, \lambda_2, \ldots, \lambda_n)$, $\sigma(G_2) = (\mu_1 = r_2, \mu_2, \ldots, \mu_m)$ and $\sigma(G_3) = (\gamma_1 = r_3, \gamma_2, \ldots, \gamma_t)$ are the adjacency spectrum of $G_1$, $G_2$ and $G_3$, respectively. Then the adjacency spectrum of $G = (G_1 \lor G_2) \cup (G_1 \circ G_3)$ is
\[
\sigma(G) = \left(\begin{array}{c}
\gamma_i \\
\mu_j \\
\lambda_k + r_3 \pm \frac{\sqrt{4l + 1} - r_3^2}}{2} \\
1
\end{array}\right),
\]
where $i = 2$ to $l$, $j = 2$ to $m$, $k = 2$ to $n$, $t = 1, 2, 3$. Also, the entries in the first row are the eigenvalues with multiplicity written below, and $x_i$’s are the roots of the polynomial $(x - r_2)((x - r_1)(x - r_3) - l) - nm(x - r_3)$.

Theorem 15. Let $G$ be an $r$-regular graph of order $m$. Then
\[
E(K_n \lor G \cup K_n \circ lK_1) = E(G) + (n - 1)\sqrt{4l + 1} - 2x_0 + n - 1,
\]
where $x_0$ is the negative root of the polynomial $(x - r)((x - (n - 1)) - l) - nmx$.

Corollary 16. Let $G$ and $H$ be equienergetic regular graphs of the same degree. Then
\[
E(K_n \lor G \cup K_n \circ lK_1) = E(K_n \lor H \cup K_n \circ lK_1).
\]
Theorem 17. Let $G$ be an $r$-regular graph of order $m$. Then

$$E(nK_1 \lor G \cup nK_1 \circ lK_1) = E(G) + (n - 1)\sqrt{4l} - 2x_0,$$

where $x_0$ is the negative root of the polynomial $(x - r) (x^2 - l) - nmx$.

Corollary 18. Let $G$ and $H$ be equienergetic regular graphs of the same degree. Then

$$E(nK_1 \lor G \cup nK_1 \circ lK_1) = E(nK_1 \lor H \cup nK_1 \circ lK_1).$$

Now we construct some new pairs of equienergetic graphs using Corollary 16.

Theorem 19. There exists a pair of equienergetic graphs on $n$ vertices for all $n \geq 11$.

Proof. Case 1. $n = 9 + 2m$ ($m = 1, 2, \ldots$). For $n = 9 + 2m$ ($m = 1, 2, \ldots$), the graphs $(K_m \lor L(N_1)) \cup (K_m \circ K_1)$ and $(K_m \lor L(N_2)) \cup (K_m \circ K_1)$ both are of order $9 + 2m$ ($m = 1, 2, \ldots$). Now, Corollary 16 implies that these two graphs are equienergetic.

Case 2. $n = 10 + 2m$ ($m = 1, 2, \ldots$). For $n = 10 + 2m$ ($m = 1, 2, \ldots$), the graphs $(K_m \lor T_1) \cup (K_m \circ K_1)$ and $(K_m \lor T_2) \cup (K_m \circ K_1)$ both are of order $10 + 2m$ ($m = 1, 2, \ldots$). Now, Corollary 16 implies that these two graphs are equienergetic.

Theorem 20. Let $G_i$ be $r_i$-regular graphs ($i = 1, 2, 3, 4$). Suppose $\sigma(G_1) = (\lambda_1 = r_1, \lambda_2, \ldots, \lambda_n)$, $\sigma(G_2) = (\mu_1 = r_2, \mu_2, \ldots, \mu_m)$, $\sigma(G_3) = (\gamma_1 = r_3, \gamma_2, \ldots, \gamma_t)$ and $\sigma(G_4) = (\eta_1 = r_4, \eta_2, \ldots, \eta_s)$ are the adjacency spectrum of $G_1$, $G_2$, $G_3$ and $G_4$, respectively. Then the adjacency spectrum of $G = (G_1 \lor G_2) \cup (G_2 \circ G_3) \cup (G_1 \circ G_4)$ is

$$\sigma(G) = \left( \begin{array}{c c c c}
\gamma_i & \eta_j & \left(\lambda_k + r_4 \pm \sqrt{4p + (\lambda_k - r_4)^2}\right)/2 & 1 \\
m & n & \left((\mu_s + r_3 \pm \sqrt{4l + (\mu_s - r_3)^2})/2\right) & x_t \\
 & & 1 & 1
\end{array} \right),$$

where $i = 2$ to $t$, $j = 2$ to $p$, $k = 2$ to $n$, $s = 2$ to $m$, $t = 1, 2, 3, 4$. Also, the entries in the first row are the eigenvalues with multiplicity written below, and $x_t$'s are the roots of the polynomial

$$(x - r_1) (x - r_3 - p) ((x - r_2) (x - r_3) - l) - nm (x - r_3) (x - r_4).$$
5. Spectra of \((G_1 \lor G_2) \cup (G_1 \diamond G_3)\) and \((G_1 \lor G_2) \cup (G_2 \diamond G_3) \cup (G_1 \diamond G_4)\)

In Theorems 21 and 25 of this section, we compute the spectrum of \((G_1 \lor G_2) \cup (G_1 \diamond G_3)\) and \((G_1 \lor G_2) \cup (G_2 \diamond G_3) \cup (G_1 \diamond G_4)\), where \(G_1, G_2, G_3\) and \(G_4\) are regular graphs on \(n, m, l\) and \(p\) vertices, respectively. Proofs of these theorems are not given as they are similar to the proof of Theorem 4.

**Theorem 21.** Let \(G_i\) be \(r_i\)-regular graphs \((i = 1, 2, 3)\) and \(r_1 \geq 2\). Suppose \(\sigma(G_1) = (\lambda_1 = r_1, \lambda_2, \ldots, \lambda_n)\), \(\sigma(G_2) = (\mu_1 = r_2, \mu_2, \ldots, \mu_m)\) and \(\sigma(G_3) = (\gamma_1 = r_3, \gamma_2, \ldots, \gamma_l)\) are the adjacency spectrum of \(G_1, G_2\) and \(G_3\), respectively. Then the adjacency spectrum of \(G = (G_1 \lor G_2) \cup (G_1 \diamond G_3)\) is

\[
\sigma(G) = \left( \begin{array}{ccc}
\gamma_i & r_3 & \mu_j \\
\frac{r_1}{n} & \frac{r_3}{n} & \frac{\mu_1}{k} \\
1 & 1 & 1
\end{array} \right)
\]

where \(i = 2\) to \(l\), \(j = 2\) to \(m\), \(k = 2\) to \(n\), \(t = 1, 2, 3\). Also, the entries in the first row are the eigenvalues with multiplicity written below, and \(x_t\)’s are the roots of the polynomial \((x - x_2)(x - x_3) - 2x_1 - mn(x - x_3)\).

**Theorem 22.** Let \(G\) be an \(r\)-regular graph of order \(m\). Then

\[
E(K_n \lor G \cup K_n \diamond lK_l) = E(G) + (n - 1)(\sqrt{4l + n - 2} + 1) - 2x_0,
\]

where \(x_0\) is the negative root of the polynomial

\[
x^3 - (n - 1 + r)x^2 + ((n - 1)r - 2(n - 1)l - mn)x + 2(n - 1)rl.
\]

**Corollary 23.** Let \(G\) and \(H\) be equienergetic regular graphs of the same degree. Then

\[
E(K_n \lor G \cup K_n \diamond lK_l) = E(K_n \lor H \cup K_n \diamond lK_l).
\]

Now we construct some new pairs of equienergetic graphs using Corollary 23.

**Theorem 24.** There exists a pair of equienergetic graphs on \(n\) vertices for all \(n \geq 15\).

**Proof.** Case 1. Let \(n = 9 + 2m\) \((m = 3, 4, \ldots)\) and \(C_m\) be the cycle of length \(m\). Then, by Corollary 23 and Lemma 2 the graphs \((C_m \lor L(N_1)) \cup (C_m \diamond K_1)\) and \((C_m \lor L(N_2)) \cup (C_m \diamond K_1)\), both of order \(9 + 2m\) \((m = 3, 4, \ldots)\), are equienergetic.

Case 2. \(n = 10 + 2m\) \((m = 3, 4, \ldots)\). For \(n = 10 + 2m\) \((m = 3, 4, \ldots)\), the graphs \((C_m \lor T_1) \cup (C_m \diamond K_1)\) and \((C_m \lor T_2) \cup (C_m \diamond K_1)\) both are of order \(9 + 2m\) \((m = 3, 4, \ldots)\). Now, Corollary 23 and Lemma 3 implies that these two graphs are equienergetic. 

\[\square\]
Theorem 25. Let $G_i$ be $r_i$-regular graphs ($i = 1, 2, 3, 4$), $r_1 \geq 2$ and $r_2 \geq 2$.
Suppose $\sigma(G_1) = (\lambda_1 = r_1, \lambda_2, \ldots, \lambda_n)$, $\sigma(G_2) = (\mu_1 = r_2, \mu_2, \ldots, \mu_m)$, $\sigma(G_3) = (\gamma_1 = r_3, \gamma_2, \ldots, \gamma_l)$ and $\sigma(G_4) = (\eta_1 = r_4, \eta_2, \ldots, \eta_p)$ are the adjacency spectrum of $G_1$, $G_2$, $G_3$ and $G_4$, respectively. Then the adjacency spectrum of $G = (G_1 \vee G_2) \cup (G_2 \circ G_3) \cup (G_1 \circ G_4)$ is

$$\sigma(G) = \begin{pmatrix} \gamma_i & \eta_j & r_4 \\ r_2 n/2 & (r_2 - 2)n/2 & r_1 n/2 & (r_1 - 2)n/2 \\ \lambda_k + r_4 \pm \sqrt{4p(\lambda_k + r_1) + (\lambda_k - r_4)^2} / 2 & (\mu_s + r_3 \pm \sqrt{4l(\mu_s + r_2) + (\mu_s - r_3)^2} / 2 & x_t \\ 1 & 1 \\ \end{pmatrix},$$

where $i = 2$ to $l$, $j = 2$ to $p$, $k = 2$ to $n$, $s = 2$ to $m$, $t = 1, 2, 3, 4$. Also, the entries in the first row are the eigenvalues with multiplicity written below, and $x_t$'s are the roots of the polynomial

$$(x - r_1)(x - r_4) - pr_1 \left((x - r_2)(x - r_3) - 2r_2l\right) - nm(x - r_3)(x - r_4).$$

6. Spectra of $(G_1 \vee G_2) \cup (G_2 \circ G_3) \cup (G_1 \circ G_3)$, $(G_1 \vee G_2) \cup (G_2 \circ G_3) \cup (G_1 \circ G_4)$ and $(G_1 \vee G_2) \cup (G_2 \circ G_3) \cup (G_1 \circ G_4)$

In this section we just give the spectrum of $(G_1 \vee G_2) \cup (G_2 \circ G_3) \cup (G_1 \circ G_3)$, $(G_1 \vee G_2) \cup (G_2 \circ G_3) \cup (G_1 \circ G_4)$ and $(G_1 \vee G_2) \cup (G_2 \circ G_3) \cup (G_1 \circ G_4)$, where $G_1$, $G_2$, $G_3$ and $G_4$ are regular graphs on $n$, $m$, $l$ and $p$ vertices, respectively. Proofs of Theorems 26–28 are similar to the proof of Theorem 4.

Theorem 26. Let $G_i$ be $r_i$-regular graphs ($i = 1, 2, 3, 4$). Suppose $\sigma(G_1) = (\lambda_1 = r_1, \lambda_2, \ldots, \lambda_n)$, $\sigma(G_2) = (\mu_1 = r_2, \mu_2, \ldots, \mu_m)$, $\sigma(G_3) = (\gamma_1 = r_3, \gamma_2, \ldots, \gamma_l)$ and $\sigma(G_4) = (\eta_1 = r_4, \eta_2, \ldots, \eta_p)$ are the adjacency spectrum of $G_1$, $G_2$, $G_3$ and $G_4$, respectively. Then the adjacency spectrum of $G = (G_1 \vee G_2) \cup (G_2 \circ G_3) \cup (G_1 \circ G_4)$ is

$$\sigma(G) = \begin{pmatrix} \gamma_i & \eta_j & r_4 \\ n & m & \lambda_k + r_4 \pm \sqrt{4p(\lambda_k + r_1) + (\lambda_k - r_4)^2} / 2 & 1 \\ \mu_s + r_3 \pm \sqrt{4l(\mu_s + r_2) + (\mu_s - r_3)^2} / 2 & x_t & 1 \\ 1 \\ \end{pmatrix},$$

where $i = 2$ to $l$, $j = 2$ to $p$, $k = 2$ to $n$, $s = 2$ to $m$, $t = 1, 2, 3, 4$. Also, the entries in the first row are the eigenvalues with multiplicity written below, and $x_t$'s are the roots of the polynomial

$$(x - r_2)(x - r_3) - l \left((x - r_1)(x - r_4) - pr_1^2\right) - nm(x - r_3)(x - r_4).$$
Theorem 27. Let \( G_i \) be \( r_i \)-regular graphs \((i = 1, 2, 3, 4)\) and \( r_1 \geq 2 \). Suppose \( \sigma(G_1) = (\lambda_1 = r_1, \lambda_2, \ldots, \lambda_n) \), \( \sigma(G_2) = (\mu_1 = r_2, \mu_2, \ldots, \mu_m) \), \( \sigma(G_3) = (\gamma_1 = r_3, \gamma_2, \ldots, \gamma_l) \) and \( \sigma(G_4) = (\eta_1 = r_4, \eta_2, \ldots, \eta_p) \) are the adjacency spectrum of \( G_1 \), \( G_2 \), \( G_3 \) and \( G_4 \), respectively. Then the adjacency spectrum of \( G = (G_1 \vee G_2) \cup (G_2 \odot G_3) \cup (G_1 \odot G_4) \) is

\[
\sigma(G) = \begin{pmatrix}
\gamma_i & \eta_j & r_4 \\
n & m & (r_1 - 2)n/2 \\
\lambda_k + r_4 \pm \sqrt{4p(\lambda_k + r_1) + (\lambda_k - r_4)^2} / 2 & 1 \\
\mu_s + r_3 \pm \sqrt{4l + (\mu_s - r_3)^2} / 2 & x_t & 1 \\
\end{pmatrix},
\]

where \( i = 2 \) to \( l \), \( j = 2 \) to \( p \), \( k = 2 \) to \( n \), \( s = 2 \) to \( m \), \( t = 1, 2, 3, 4 \). Also, the entries in the first row are the eigenvalues with multiplicity written below, and \( x_t \)'s are the roots of the polynomial

\[
((x - r_1)(x - r_4) - 2pr_1)((x - r_2)(x - r_3) - l) - nm(x - r_3)(x - r_4).
\]

Theorem 28. Let \( G_i \) be \( r_i \)-regular graphs \((i = 1, 2, 3, 4)\) and \( r_1 \geq 2 \). Suppose \( \sigma(G_1) = (\lambda_1 = r_1, \lambda_2, \ldots, \lambda_n) \), \( \sigma(G_2) = (\mu_1 = r_2, \mu_2, \ldots, \mu_m) \), \( \sigma(G_3) = (\gamma_1 = r_3, \gamma_2, \ldots, \gamma_l) \) and \( \sigma(G_4) = (\eta_1 = r_4, \eta_2, \ldots, \eta_p) \) are the adjacency spectrum of \( G_1 \), \( G_2 \), \( G_3 \) and \( G_4 \), respectively. Then the adjacency spectrum of \( G = (G_1 \vee G_2) \cup (G_2 \odot G_3) \cup (G_1 \odot G_4) \) is

\[
\sigma(G) = \begin{pmatrix}
\gamma_i & \eta_j & r_4 \\
n & m & (r_1 - 2)n/2 \\
\lambda_k + r_4 \pm \sqrt{4p(\lambda_k + r_1) + (\lambda_k - r_4)^2} / 2 & 1 \\
\mu_s + r_3 \pm \sqrt{4l + (\mu_s - r_3)^2} / 2 & x_t & 1 \\
\end{pmatrix},
\]

where \( i = 2 \) to \( l \), \( j = 2 \) to \( p \), \( k = 2 \) to \( n \), \( s = 2 \) to \( m \), \( t = 1, 2, 3, 4 \). Also, the entries in the first row are the eigenvalues with multiplicity written below, and \( x_t \)'s are the roots of the polynomial

\[
((x - r_1)(x - r_4) - 2pr_1)((x - r_2)(x - r_3) - tr_4^2) - nm(x - r_3)(x - r_4).
\]

Acknowledgement

The authors are thankful to the referee for useful suggestions. The first author is thankful to the University Grants Commission, Government of India, for the financial support under Grant F.510/2/SAP-DRS/2011. The second author is thankful to UGC, New Delhi, for UGC-JRF, under which this work has been done.
References


Received 29 April 2015
Revised 25 May 2015
Accepted 25 May 2015