

ON UNIQUELY HAMILTONIAN CLAW-FREE AND TRIANGLE-FREE GRAPHS

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Abstract

A graph is uniquely Hamiltonian if it contains exactly one Hamiltonian cycle. In this note, we prove that claw-free graphs with minimum degree at least 3 are not uniquely Hamiltonian. We also show that this is best possible by exhibiting uniquely Hamiltonian claw-free graphs with minimum degree 2 and arbitrary maximum degree. Finally, we show that a construction due to Entringer and Swart can be modified to construct triangle-free uniquely Hamiltonian graphs with minimum degree 3.

Keywords: Hamiltonian cycle, uniquely Hamiltonian graphs, claw-free graphs, triangle-free graphs.

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1. INTRODUCTION

All graphs in this note will be finite, undirected, and simple. A Hamiltonian cycle of a graph G is a cycle whose vertex set is precisely $V(G)$. A graph is *uniquely Hamiltonian* if it contains exactly one Hamiltonian cycle. A classic result of Smith [12] states in any 3-regular graph there are an even number of Hamiltonian cycles through any given edge; it follows that no 3-regular graph is uniquely Hamiltonian. Thomason [9] showed that Smith's Theorem can be extended to any graph whose vertices all have odd degree, by way of a clever "lollipop argument".

Smith's Theorem inspired the following conjecture:

Conjecture 1.1 (Sheehan [8]). *There is no uniquely Hamiltonian 4-regular graph.*

Recall that Petersen's 2-Factor Theorem states that every $(2r)$ -regular graph can be decomposed into r edge-disjoint 2-factors [7]. Hence, if Conjecture 1.1 is true, then there is no uniquely Hamiltonian d -regular graph when $d \geq 3$. Thomassen [11] used the Local Lemma to show that there is no uniquely Hamiltonian d -regular graph for $d > 72$. The result was subsequently improved by Haxell, the author, and Verstraete to $d > 22$ and $d > 14$ if G has sufficiently high girth [6].

One natural strengthening of the problem is to consider uniquely Hamiltonian graphs having fixed minimum degree. Entringer and Swart [4] gave infinitely many examples of uniquely Hamiltonian graphs with minimum degree 3. Very recently, Fleischner [5] gave an infinite family of uniquely Hamiltonian graphs having minimum degree 4. The cases of higher constant minimum degree remain open. However, it was been shown by Bondy and Jackson [3] that there exists an absolute constant c for which no n -vertex graph with $\delta > c \log_2 n$ is uniquely Hamiltonian. The best result to date on the matter is a bound of $\delta > c \log_2 n + 2$ where $c \approx 1.71$ [1].

Thomassen [10] posed the following related conjecture:

Conjecture 1.2 (Thomassen [10]). *Every Hamiltonian graph G with $\delta(G) \geq 3$ has an edge $e \in E(G)$ such that both $G - e$ and G/e are Hamiltonian.*

If G is a graph containing two Hamiltonian cycles, then any edge contained in one cycle but not the other satisfies the statement of the conjecture. It was also shown by Bielak [2] that Conjecture 1.2 holds if G is also assumed to be claw-free (that is, G does not contain $K_{1,3}$ as an induced subgraph). Little else is known about Conjecture 1.2.

In this paper, we extend Bielak's result to show that claw-free graphs with $\delta \geq 3$ are not uniquely Hamiltonian. We also exhibit claw-free uniquely Hamiltonian graphs with $\delta = 2$ and Δ arbitrarily large. Finally, we show there exist triangle-free uniquely Hamiltonian graphs with $\delta = 3$.

2. CLAW-FREE GRAPHS

Bielak's proof of Conjecture 1.2 for claw-free graphs can be easily modified to prove the following:

Theorem 2.1. *If G is a claw-free graph with $\delta(G) \geq 3$, then G is not uniquely Hamiltonian.*

Proof. If G has no Hamiltonian cycle, then we are done. Let C be a Hamiltonian cycle, and consider the vertices of C in some order as they appear along C . For a vertex $x \in V(G)$, denote by x^- and x^+ the vertices preceding and following x ,

respectively. We also define x^{-n} and x^{+n} to be the vertices at distance n from x along C with respect to the ordering of $V(C)$.

We begin with the simple observation that if $a, b \in V(G)$ are vertices that are not joined by an edge of C and $ab^-, a^+b \in E(G)$, then G has a Hamiltonian cycle distinct from C , namely $C - \{aa^+, b^-b\} + \{ab^-, a^+b\}$. We call such a structure a “cycle exchange”; we may assume from this point forward that G has no cycle exchange. In particular, if $v_i, v_{i+1}, v_{i+2}, v_{i+3}$ are such that $v_j^+ = v_{j+1}$ for each $j = 0, 1, 2$, then either $v_i v_{i+2}$ or $v_{i+1} v_{i+3}$ is a non-edge.

Let v be a vertex such that $v^- v^+ \notin E(G)$. Let u be a neighbour of v distinct from v^- and v^+ . Since $\{u, v^-, v, v^+\}$ may not induce a claw, u must be adjacent to at least one of v^- and v^+ . Without loss of generality, say $uv^+ \in E(G)$. We now consider u and its neighbours $\{v^+, u^-, u^+\}$. Since G is claw-free, one of $v^+ u^+$, $u^- u^+$ or $u^- v^+$ is an edge of G . In the first case, G would have a cycle exchange. In the second case, $C - u^- u u^+ + u^- u^+ + v u v^+ - v v^+$ is a second Hamiltonian cycle of G . We thus proceed assuming that $u^- v^+ \in E(G)$. We now consider v^+ and its neighbours $\{u^-, v, v^{+2}\}$. As before, if $v u^-$ or $v v^{+2} \in E(G)$, then G has a second Hamiltonian cycle, so we proceed assuming that $u^- v^{+2} \in E(G)$. By iteratively applying this argument, we see that $E(G)$ must contain the edges $vu, uv^+, v^+ u^-, u^- v^{+2}, v^{+2} u^{-2}, \dots$ with the sequence ending when the final edge added has endpoints which lie at distance 2 from one another along C .

Now, consider the vertex u and its neighbours $\{v, u^-, u^+\}$. Again, if $v u^-$ or $u^- u^+ \in E(G)$, then G contains a second Hamiltonian cycle. We thus assume that $v u^+ \in E(G)$. A symmetric argument to the one above shows that $E(G)$ contains the edges $vu^+, u^+ v^-, v^- u^{+2}, u^{+2} v^{-2}, \dots$ with the sequence again ending when the final edge added has endpoints which lie at distance 2 from one another along C .

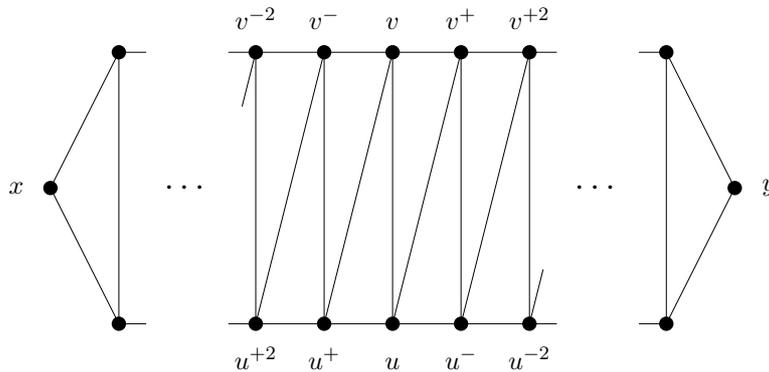


Figure 1. The structure of G .

Let $F = \{\dots, v^{-2}u^{+2}, u^{+2}v^{-}, v^{-}u^{+}, u^{+}v, vu, uv^{+}, v^{+}u^{-}, u^{-}v^{+2}, v^{+2}u^{-2}, \dots\}$ be the set of edges added above. Let x, y be the two vertices not incident to any edge of F . Since $\delta(G) \geq 3$, x has at least one neighbour yet to be determined. If $xy \in E(G)$, it is easy to see that G contains a Hamiltonian cycle consisting of the edges of F together with some cycle edge incident to each of x and y . To finish the proof, we relabel the vertices of $V(G) \setminus \{x, y\}$ for ease of notation. Noting that the edges of F form a Hamiltonian path of $G - \{x, y\}$, we label the vertices, in order along this path, $a_1, b_1, a_2, b_2, \dots$ so that $a_1x \in E(C)$.

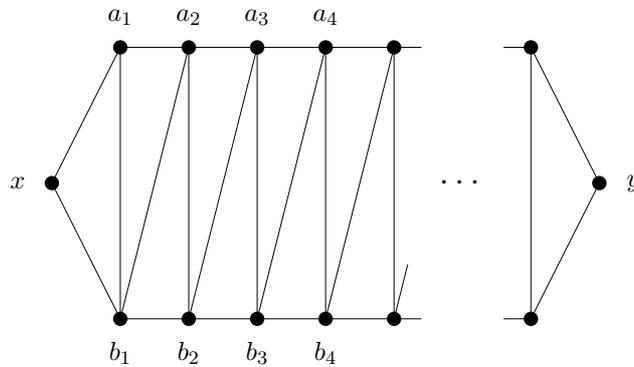


Figure 2. The structure of G , with relabelled vertices.

There are two remaining cases to consider, namely $xa_j \in E(G)$ for some j or $xb_k \in E(G)$ for some k . If $xa_j \in E(G)$, then $C - a_1a_2 \cdots a_j - xb_1b_2 \cdots b_{j-1} + xa_j + a_1b_1a_2b_2 \cdots b_{j-1}$ is a Hamiltonian cycle of G . If $xb_k \in E(G)$, then $C - a_1a_2 \cdots a_k - xb_1b_2 \cdots b_k + xb_k + a_1b_1a_2b_2 \cdots b_{k-1}a_k$ is a Hamiltonian cycle of G . Having exhausted all cases, we conclude that G is not uniquely Hamiltonian. ■

We now show that the minimum degree requirement cannot be removed from the statement of Theorem 2.1. While cycles would suffice to show this, we present a graph in Figure 3 that is claw-free, uniquely Hamiltonian, has minimum degree 2, and arbitrarily high maximum degree.

The graph in Figure 3, which we call H , is obtained from the complete graph K_k and a cycle C_{3k} for some fixed positive integer k . Let the vertices of K_k be $V(K_k) = \{v_1, \dots, v_k\}$ and the vertices of C_{3k} be $V(C_{3k}) = \{u_1, \dots, u_{3k}\}$. The graph H is defined as $V(H) = V(K_k) \cup V(C_{3k})$ and $E(H) = E(K_k) \cup E(C_{3k}) \cup \{v_iu_{3i-2}, v_iu_{3i-1} : 1 \leq i \leq k\}$. It is easy to check that H is claw-free, that $\delta(H) = 2$, and that $\Delta(H) = k + 1$. The edges in bold in Figure 3 show that H is Hamiltonian, and it is not hard to see that this is the only possible Hamiltonian cycle in H .

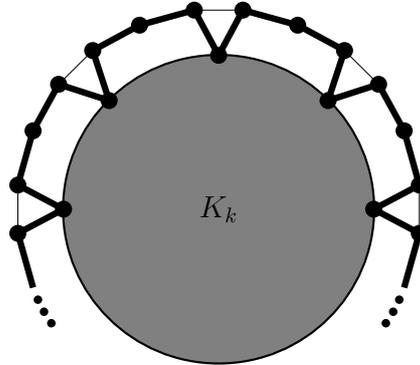


Figure 3. A uniquely Hamiltonian claw-free graph with arbitrarily large maximum degree

Theorem 2.2. *For any positive integer d , there exists a uniquely Hamiltonian claw-free graph with maximum degree at least d .*

3. TRIANGLE-FREE GRAPHS

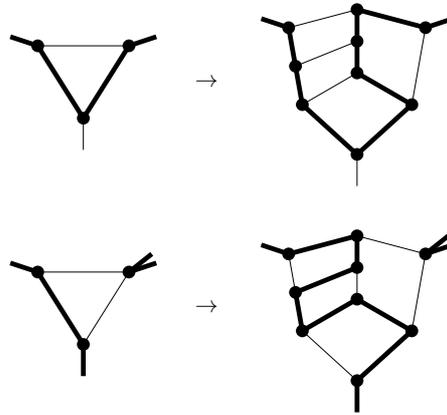
In this section, we show that there exist triangle-free, uniquely Hamiltonian graphs having $\delta = 3$, and so any minimum degree result analogous to Theorem 2.1 for triangle-free graphs would be for minimum degree at least 4. Since the net (a triangle with a pendant edge on each vertex) and the bull (a triangle with a pendant edge on each of two vertices) both rely on the presence of a triangle, a similar statement holds for net-free and bull-free graphs as well.

We will make use of the following constructive result:

Theorem 3.1 (Entringer, Swart [4]). *For every $n = 2k$, $k \geq 11$, there exists a uniquely Hamiltonian graph on n vertices with two vertices having degree 4 and all others having degree 3.*

Let G be a graph having Hamiltonian cycle C , and let uvw be a triangle in G . We define a C -blowup of a triangle uvw in G to be the graph obtained by replacing the triangle uvw with the graph given in Figure 4. Note that the replacement operation does depend on which edges of the triangle are in C . We call the graph that replaces the triangle X .

The edges of C are indicated in bold in the figures on the left of Figure 4. The bold edges in the corresponding copy of X show how C extends to a Hamiltonian cycle in the graph containing the C -blowup. Note that, in a general graph containing a triangle uvw , it would be possible for a Hamiltonian cycle to use

Figure 4. C -blowups of a triangle uvw .

none of the edges uv , vw , or uw . However, since we are only considering triangles which contain at least one vertex of degree 3, this situation will not arise; the only possibilities left are the two pictured in Figure 4.

Lemma 3.2. *Let G be a graph with $\delta(G) = 3$ such that every triangle contains a vertex of degree 3. Suppose that G contains a unique Hamiltonian cycle, C . If G' is the graph obtained by applying a C -blowup to each triangle of G , then G' is uniquely Hamiltonian. Furthermore, $\delta(G') = \delta(G) = 3$ and $\Delta(G') = \Delta(G)$.*

Proof. It is easy to see that, in C -blowup of a triangle uvw , the vertices u, v, w maintain the same degree and all other vertices added to the graph have degree 3. Hence, we have $\delta(G') = \delta(G) = 3$ and $\Delta(G') = \Delta(G)$.

By construction, G' is Hamiltonian. Let C' be a Hamiltonian cycle of G' , and let u, v, w be vertices as in Figure 4. Clearly, C' must either:

1. contain a path that covers every vertex of X and has its ends in $\{u, v, w\}$;
or
2. contain a path that covers every vertex of X except for one of $\{u, v, w\}$ and has its ends in $\{u, v, w\}$.

In either case, C' gives rise to a Hamiltonian cycle in G by reversing the C -blowup. Since G is uniquely Hamiltonian, this cycle must be precisely C . We can easily check that, for each C -blowup shown in Figure 4, the edges of the Hamiltonian cycle in the graph containing the blowup constitute the only way to extend C to a Hamiltonian cycle. Hence G' must be uniquely Hamiltonian. ■

By applying Lemma 3.2, we obtain the following corollary to Theorem 3.1.

Corollary 3.3. *There exist triangle-free uniquely Hamiltonian graphs with two vertices having degree 4 and all others having degree 3.*

Thus, if one wishes to find uniquely Hamiltonian H -free graphs where H is any graph containing a triangle (say, for instance, the bull or the net), then one must begin with graphs having minimum degree at least 4.

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