TETRAVALENT ARC-TRANSITIVE GRAPHS OF ORDER $3p^2$

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Abstract

Let $s$ be a positive integer. A graph is $s$-transitive if its automorphism group is transitive on $s$-arcs but not on $(s+1)$-arcs. Let $p$ be a prime. In this article a complete classification of tetravalent $s$-transitive graphs of order $3p^2$ is given.

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1. Introduction

In this paper we consider undirected finite connected graphs without loops or multiple edges. For a graph $X$ we use $V(X)$, $E(X)$ and $\text{Aut}(X)$ to denote its vertex set, edge set and its full automorphism group, respectively. For $u, v \in V(X)$, $\{u, v\}$ is the edge incident to $u$ and $v$ in $X$, and $N(u)$ is the neighborhood of $u$ in $X$, that is, the set of vertices adjacent to $u$ in $X$. A graph $X$ is locally primitive if for any vertex $v \in V(X)$, the stabilizer $\text{Aut}(X)_v$ of $v$ in $\text{Aut}(X)$ is primitive on $N(v)$. An $s$-arc in a graph is an ordered $(s+1)$-tuple $(v_0, v_1, \ldots, v_{s-1}, v_s)$ of vertices of the graph such that $v_{i-1}$ is adjacent to $v_i$ for $1 \leq i \leq s$, and $v_{i-1} \neq v_{i+1}$ for $1 \leq i \leq s-1$. For a subgroup $G \leq \text{Aut}(X)$, a graph $X$ is said to be $(G, s)$-arc-transitive or $(G, s)$-regular if $G$ acts transitively or regularly on the set of $s$-arcs of $X$, respectively. A $(G, s)$-arc-transitive graph is said to be $(G, s)$-transitive if it is not $(G, s+1)$-arc-transitive. In particular, an $(\text{Aut}(X), s)$-arc-transitive, $(\text{Aut}(X), s)$-regular or $(\text{Aut}(X), s)$-transitive graph is simply called an $s$-arc-transitive, $s$-regular or $s$-transitive graph, respectively. Note that 0-arc-transitive means vertex-transitive, and 1-arc-transitive means arc-transitive or symmetric. A graph is edge-transitive if $\text{Aut}(X)$ is transitive on $E(X)$. 
Edge-transitive graphs or s-transitive graphs of small valencies have received considerable attention in the literature. For instance, Tutte [29] initiated the investigation of cubic s-transitive graphs by proving that there exist no cubic s-transitive graphs for \( s \geq 6 \), and later much subsequent work was done along this line (see [7, 8, 9, 10, 11, 12, 13, 14, 24]). Gardiner and Praeger [15, 16] generally explored the tetravalent symmetric graphs by considering their automorphism groups. Recently, Li et al. [22] classified all vertex-primitive symmetric graphs of valency 3 or 4. Moreover, Weiss [31] proved that if \( X \) is s-transitive, then \( s \in \{1, 2, 3, 4, 5, 7\} \). Let \( p \) be a prime. Conder [6] showed that for a fixed integer \( n \) and any integer \( s > 1 \), there are only finitely many cubic s-transitive graphs of order \( np \). Li [20] generalized this result to connected symmetric graphs of any valency, and he also posed the following problem: for small values \( n \) and \( k \), classify vertex-transitive locally primitive graphs of order \( np \) and valency \( k \).

In this paper we classify all symmetric graphs of order \( np \) and valency \( k \) for certain values of \( n \) and \( k \). The classification of s-transitive graphs of order \( np \) and of valency 3 or 4 can be obtained from [4, 5, 30], where \( 1 \leq n \leq 3 \). Feng et al. [10, 12, 13] classified cubic s-transitive graphs of order \( np \) with \( n = 4, 6, 8 \) or 10. Recently, Zhou and Feng [35, 36] classified tetravalent s-transitive graphs of order \( 4p \) or \( 2p^2 \). Also Ghasemi and Zhou [18] classified tetravalent s-transitive graphs of order \( 4p^2 \). In this paper, we prove that there are no tetravalent s-transitive graphs of order \( 3p^2 \), for \( s > 1 \).

2. Preliminaries

In this section, we introduce some notation and definitions as well as some preliminary results which will be used later in the paper.

For a regular graph \( X \), use \( d(X) \) to represent the valency of \( X \), and for any subset \( B \) of \( V(X) \), the subgraph of \( X \) induced by \( B \) will be denoted by \([B]\).

For a positive integer \( n \), denote by \( Z_n \) the cyclic group of order \( n \) as well as the ring of integers modulo \( n \), by \( Z_n^* \) the multiplicative group of \( Z_n \) consisting of numbers coprime to \( n \), by \( D_{2n} \) the dihedral group of order \( 2n \), and by \( C_n \) and \( K_n \) the cycle and the complete graph of order \( n \), respectively. We call \( C_n \) an \( n \)-cycle.

Let \( G \) be a permutation group on a set \( \Omega \) and \( \alpha \in \Omega \). Denote by \( G_\alpha \) the stabilizer of \( \alpha \) in \( G \), that is, the subgroup of \( G \) fixing the point \( \alpha \). We say that \( G \) is semiregular on \( \Omega \) if \( G_\alpha = 1 \) for every \( \alpha \in \Omega \) and regular if \( G \) is transitive and semiregular. For any \( g \in G \), \( g \) is said to be semiregular if \( \langle g \rangle \) is semiregular. The following proposition gives a characterization for Cayley graphs in terms of their automorphism groups.

**Proposition 2.1** (Lemma 16.3 [2]). A graph \( X \) is isomorphic to a Cayley graph on a group \( G \) if and only if its automorphism group has a subgroup isomorphic
to $G$, acting regularly on the vertex set of $X$.

Let $X$ be a connected symmetric graph and let $G \leq \text{Aut}(X)$ be arc-transitive on $X$. For a normal subgroup $N$ of $G$, the quotient graph $X_N$ of $X$ relative to the orbits of $N$ is defined as the graph with vertices the orbits of $N$ on $V(X)$ and with two orbits adjacent if there is an edge in $X$ between those two orbits. If $X_N$ and $X$ have the same valency, then $X$ is called a normal cover of $X_N$. Let $X$ be a connected tetravalent symmetric graph and $N$ an elementary abelian $p$-group. A classification of connected tetravalent symmetric graphs was obtained when $N$ has at most two orbits in [15] and a characterization of such graphs was given when $X_N$ is a cycle in [16].

The following proposition is due to Praeger et al. (refer to Theorem 1.1 [15] and [27]).

**Proposition 2.2.** Let $X$ be a connected tetravalent $(G, 1)$-arc-transitive graph. For each normal subgroup $N$ of $G$, one of the following holds.

1. $N$ is transitive on $V(X)$,
2. $X$ is bipartite and $N$ acts transitively on each part of the bipartition,
3. $N$ has $r \geq 3$ orbits on $V(X)$, the quotient graph $X_N$ is a cycle of length $r$, and $G$ induces the full automorphism group $D_{2r}$ on $X_N$,
4. $N$ has $r \geq 5$ orbits on $V(X)$, $N$ acts semiregularly on $V(X)$, the quotient graph $X_N$ is a connected tetravalent $G/N$-symmetric graph, and $X$ is a $G$-normal cover of $X_N$.

Moreover, if $X$ is also $(G, 2)$-arc-transitive, then case (3) cannot happen.

The following proposition characterizes the vertex stabilizer of the connected tetravalent $s$-transitive graphs, which can be deduced from Lemma 2.5 [23], or Proposition 2.8 [22], or Theorem 2.2 [21].

**Proposition 2.3.** Let $X$ be a connected tetravalent $(G, s)$-transitive graph. Let $G_v$ be the stabilizer of a vertex $v \in V(X)$ in $G$. Then $s = 1, 2, 3, 4$ or 7. Furthermore, either $G_v$ is a 2-group for $s = 1$, or $G_v$ is isomorphic to $A_4$ or $S_4$ for $s = 2$; $A_4 \times \mathbb{Z}_3$, $Z_3 \times S_4$, $S_3 \times S_4$ for $s = 3$; $\mathbb{Z}_3 \times \text{GL}(2, 3)$ for $s = 4$; or $[3^5] \times \text{GL}(2, 3)$ for $s = 7$, where $[3^5]$ represents an arbitrary group of order $3^5$.

Let $X$ be a tetravalent one-regular graph of order $3p^2$. If $p \leq 13$, then $|V(X)| = 12, 27, 75, 147, 363, 507$. Now, a complete census of the tetravalent arc-transitive graphs of order at most 640 has been recently obtained by Potočnik, Spiga and Verret [25, 26]. Therefore, a quick inspection through this list (with the invaluable help of magma (see [3])) gives the number of tetravalent one-regular graphs in the case $p \leq 13$. The following Proposition can be extracted from Theorem 3.4 [17].
Proposition 2.4. Let \( p \) be a prime and \( p > 13 \). A tetravalent graph \( X \) of order \( 3p^2 \) is 1-regular if and only if one of the following holds:

(i) \( X \) is a Cayley graph over \( \langle x, y \rangle x^p = y^6 = [x, y] = 1 \), with connection set \( \{y, y^{-1}, xy, x^{-1}y^{-1}\} \).

(ii) \( X \) is a connected arc-transitive circulant graph with respect to every connection set \( S \).

(iii) \( X \) is one of the graphs described in Lemma 8.4 [16].

Proposition 2.5 (Theorem 1.2 [16]). Let \( X \) be a connected tetravalent symmetric graph of order \( 3p^2 \) where \( p > 5 \) is a prime. Let \( A = \text{Aut}(X) \) and let \( N = \mathbb{Z}_p^2 \) be a minimal normal subgroup of \( A \). Let \( K \) denote the kernel of \( G \) acting on \( N \)-orbits. If the quotient graph \( X_N \) is a 3-cycle, then \( K_v \cong \mathbb{Z}_2 \), and \( X \) is 1-regular.

Finally in the following example we introduce \( G(3p, r) \), which was first defined in [5].

Example 2.6. For each positive divisor \( r \) of \( p - 1 \) we use \( H_r \) to denote the unique subgroup of \( \text{Aut}(\mathbb{Z}_p) \) of order \( r \), which is isomorphic to \( \mathbb{Z}_r \). Define a graph \( G(3p, r) \) by \( V(G(3p, r)) = \{x_i \mid i \in \mathbb{Z}_3, x \in \mathbb{Z}_p\} \), and \( E(G(3p, r)) = \{x_iy_{i+1} \mid i \in \mathbb{Z}_3, x \in \mathbb{Z}_p, y - x \in H_r\} \). Then \( G(3p, r) \) is a connected symmetric graph of order \( 3p \) and valency \( 2r \). Also \( \text{Aut}(G(3p, p-1)) \cong S_p \times S_3 \). For \( r \neq p - 1 \), \( \text{Aut}(G(3p, r)) \) is isomorphic to \( (\mathbb{Z}_p, H_r), S_3 \) and acts regularly on the arc set, where \( X.Y \) denotes an extension of \( X \) by \( Y \).

3. Main Results

In this section, we classify tetravalent \( s \)-transitive graphs of order \( 3p^2 \) for each prime \( p \). To do so, we need the following lemmas.

Lemma 3.1. Let \( p \) be a prime and let \( n > 1 \) be an integer. Let \( X \) be a connected tetravalent graph of order \( 3p^n \). If \( G \leq \text{Aut}(X) \) is transitive on the arc set of \( X \), then every minimal normal subgroup of \( G \) is solvable.

Proof. Let \( v \in V(X) \). Since \( G \) is arc-transitive on \( X \), by Proposition 2.3, \( G_v \) either is a 2-group or has order dividing \( 2^4 \cdot 3^6 \). It follows that \( |G| = 2^4 \cdot 3^7 \cdot p^n \) or \( |G| = 2^m \cdot 3 \cdot p^n \) for some integer \( m \). Let \( N \) be a minimal normal subgroup of \( G \).

Suppose that \( N \) is non-solvable. Then \( p > 3 \) because a \( \{2, 3\}\)-group is solvable by a theorem of Burnside Theorem 8.5.3 [28]. Since \( N \) is minimal, it is a product of isomorphic non-abelian simple groups. Since \( |N| = 2^4 \cdot 3^7 \cdot p^n \), or \( |N| = 2^m \cdot 3 \cdot p^n \) by [19], pp.12–14, each direct factor of \( N \) is one of the following: \( A_5, A_6, \) PSL(2, 7), PSL(2, 8), PSL(2, 17), PSL(3, 3), PSU(3, 3) or PSU(4, 2).

An inspection of the orders of such groups gives \( n = 2 \) and \( |N| = 2^4 \cdot 3^7 \cdot p^n \). It follows that \( X \) is \( (G, 2) \)-arc transitive and we have \( N \cong A_5 \times A_5 \). Then \( p = 5 \) and
\[|X| = 75.\] However, from [32] we know that all tetravalent arc-transitive graphs of order 75 are 1-transitive, a contradiction. \[\mathbf{\Box}\]

**Lemma 3.2.** Let \(X\) be a connected tetravalent \(G\)-arc-transitive graph of order \(3p^2\), where \(p > 13\). Assume that \(G\) has a normal subgroup \(N\) of prime order. If \(N\) has at least three orbits on \(V(X)\), then either \(|X_N|\) is of valency 4 or \(G\) is regular on the arcs of \(X\).

**Proof.** By our assumption \(N\) has at least three orbits on \(V(X)\). If \(N\) has \(r \geq 5\) orbits on \(V(X)\), then by Proposition 2.2, \(X_N\) has valency 4 and \(X\) is a normal cover of \(X_N\). Thus we may suppose that \(N\) has \(r \geq 3\) orbits. Thus \(d(X_N) = 2\) and \(|X_N| = 3p\) or \(|X_N| = p^2\).

First suppose that \(|X_N| = 3p\). Thus \(X_N \cong C_{3p}\) and hence \(G/K \cong \text{Aut}(C_{3p}) \cong D_{6p}\). Let \(\Delta\) and \(\Delta'\) be two adjacent orbits of \(N\) in \(V(X)\). Then the subgraph \([\Delta \cup \Delta']\) of \(X\) induced by \(\Delta \cup \Delta'\) has valency 2. Since \(p > 13\), one has \(|\Delta \cup \Delta'| \cong C_{2p}\). The subgroup \(K^*\) of \(K\) fixing \(\Delta\) pointwise also fixes \(\Delta'\) pointwise. The connectivity of \(X\) and the transitivity of \(G/K\) on \(V(X_N)\) imply that \(K^* = 1\), and consequently, \(K \leq \text{Aut}([\Delta \cup \Delta']) \cong D_{4p}\). Since \(K\) fixes \(\Delta\), one has \(|K| \leq 2p\). It follows that \(|G| = |G/K||K| \leq 12p^2\), and hence \(G\) is regular on the arcs of \(X\).

Now suppose that \(|X_N| = p^2\). Thus \(X_N \cong C_{p^2}\). It follows that \(G/K \cong D_{2p^2}\). Let \(\Delta\) and \(\Delta'\) be two adjacent orbits of \(N\) in \(V(X)\). Then the subgraph \([\Delta \cup \Delta']\) of \(X\) induced by \(\Delta \cup \Delta'\) has valency 2. Clearly, we have \(|\Delta \cup \Delta'| \cong C_{p}\). The subgroup \(K^*\) of \(K\) fixing \(\Delta\) pointwise also fixes \(\Delta'\) pointwise. The connectivity of \(X\) and the transitivity of \(G/K\) on \(V(X_N)\) imply that \(K^* = 1\), and consequently, \(K \leq \text{Aut}([\Delta \cup \Delta']) \cong D_{12}\). Since \(K\) fixes \(\Delta\), one has \(|K| \leq 6\). It follows that \(|G| = |G/K||K| \leq 12p^2\), and hence \(G\) is regular on the arcs of \(X\). Now the proof is complete. \[\mathbf{\Box}\]

**Theorem 3.3.** Let \(p\) be a prime and let \(X\) be a connected tetravalent graph of order \(3p^2\). Then \(X\) is \(s\)-transitive for some positive integer \(s\) if and only if it is isomorphic to one of the graphs in Proposition 2.4.

**Proof.** Let \(X\) be a tetravalent \(s\)-transitive graph of order \(3p^2\) for a positive integer \(s\). By [25, 26], we may assume that \(p > 13\). If \(X\) is one-regular, then \(X\) is one of the graphs in Proposition 2.4 and so \(s = 1\). In what follows, we assume that \(p > 13\) and that \(X\) is not one-regular. Set \(A = \text{Aut}(X)\) and let \(P\) be a Sylow \(p\)-subgroup. Then \(|P| = p^2\) and by Lemma 3.1, \(A\) is solvable. First we prove a claim.

**Claim 1.** \(P\) is not normal in \(A\).

**Proof.** Suppose to, the contrary that \(P \subseteq A\). If \(P\) is a minimal normal subgroup of \(A\) then by Proposition 2.5, \(X\) is one-regular, a contradiction. Suppose that \(P\) contains a non-trivial subgroup, say \(N\), which is normal in \(A\). Consider the
quotient graph $X_N$ of $X$ relative to the orbit set of $N$, and let $K$ be the kernel of $A$ on $V(X_N)$. Since $p > 13$, one has $|X_N| = 3p$. By Lemma 3.2 either $X$ is a normal cover of $X_N$ or $d(X_N) = 2$ and $X$ is one-regular. Since $X$ is not one-regular, we may suppose that $d(X_N) = 4$. By [30], $G(3p, 2)$ is the only tetravalent symmetric graph of order $3p$, (see Example 2.6). Also $|\text{Aut}(G(3p, 2))| = 12p$ and $G(3p, 2)$ is one-regular. Thus $|A/M| = 12p$ and so $|A| = 12p^2$. Thus $X$ is one-regular, a contradiction.

Let $M$ be the maximal normal 2-subgroup of $A$ and assume $|M| > 1$. Consider the quotient graph $X_M$ of $X$ relative to the orbit set of $M$, and let $K$ be the kernel of $A$ acting on $V(X_M)$. Since $p > 13$, every orbit of $M$ has length 2 or 4, a contradiction. So $A$ has no non-trivial normal 2-subgroup.

Now we are ready to complete the proof. Let $M$ be a minimal normal subgroup of $A$. Clearly, $M$ is a 3-group or a $p$-group. First suppose that $M$ is a $p$-group. Thus $|M| = p$ or $p^2$. If $|M| = p^2$, then $M = P$ is a Sylow $p$-subgroup of $A$. By Claim 1, $P$ is not normal in $A$, a contradiction. Suppose that $|M| = p$. By Lemma 3.2 either $X$ is a normal cover of $X_M$ or $d(X_M) = 2$ and $X$ is one-regular. Since $X$ is not one-regular, we may suppose that $d(X_M) = 4$. By [30], $G(3p, 2)$ is the only tetravalent symmetric graph of order $3p$ (see Example 2.6). Also $|\text{Aut}(G(3p, 2))| = 12p$ and $G(3p, 2)$ is one-regular. Thus $|A/M| = 12p$ and so $|A| = 12p^2$. Thus $X$ is one-regular, a contradiction.

Now suppose that $M$ is a 3-group. Thus $|X_M| = p^2$. If $d(X_M) = 4$, then by Proposition 2.5, $K = M$ is semiregular on $V(X_M)$. Therefore $K = M \cong \mathbb{Z}_3$. Since $P > 13$, $PM = P \times M$ is abelian. Clearly, $PM$ is transitive on $V(X)$. Thus $PM$ is regular on $V(X)$, because $|PM| = 3p^2$. Thus $X$ is a Cayley graph on abelian group of order $3p^2$. By Theorem 1.2 [1], $X$ is normal. If $PM$ is cyclic, then by [33] $X$ is one-regular, a contradiction. Thus $PM$ is not cyclic. Now by Proposition 3.3 [34], $X$ is one-regular, a contradiction. If $d(X_M) = 2$, then $X_M \cong C_{p^2}$. By Lemma 3.2, $X$ is one-regular, a contradiction.

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