A NOTE ON PM-COMPACT BIPARTITE GRAPHS

JINFENG LIU

Center for Combinatorics and LPMC-TJKLC
Nankai University, Tianjin 300071, China

AND

XIUMEI WANG

Department of Mathematics, Zhengzhou University
Zhengzhou 450001, China

e-mail: wangxiumei@zzu.edu.cn

Abstract

A graph is called perfect matching compact (briefly, PM-compact), if its perfect matching graph is complete. Matching-covered PM-compact bipartite graphs have been characterized. In this paper, we show that any PM-compact bipartite graph $G$ with $\delta(G) \geq 2$ has an ear decomposition such that each graph in the decomposition sequence is also PM-compact, which implies that $G$ is matching-covered.

Keywords: perfect matching, PM-compact graph, matching-covered graph.

2010 Mathematics Subject Classification: 05C70.

1. Introduction

In this paper, graphs under consideration are loopless, undirected, finite and connected. Let $G$ be a graph with vertex set $V(G)$ and edge set $E(G)$. A subset $M$ of $E(G)$ is called a perfect matching of $G$ if no two edges in $M$ are adjacent and $M$ covers all vertices of $G$. The perfect matching graph of $G$, denoted by $PM(G)$, is the graph in which each perfect matching of $G$ is a vertex and two vertices $M_1$ and $M_2$ are adjacent in $PM(G)$ if and only if the symmetric difference of $M_1$ and

---

1This work is supported by NSFC (grant no. 11101383, 11201121, and 11201432).
$M_2$ is an alternating cycle. The perfect matching polytope of $G$ is the convex hull of the incidence vectors of all perfect matchings of $G$. Chvátal [4] shows that two vertices of the perfect matching polytope are adjacent if and only if the symmetric difference of the two perfect matchings is a cycle. This implies that $PM(G)$ is the 1-skeleton graph of the perfect matching polytope of $G$. Naddef and Pulleyblank [5] show that if $PM(G)$ is bipartite then $PM(G)$ is a hypercube and otherwise $PM(G)$ is Hamilton-connected. Bian and Zhang [1] give a sharp upper bound of the number of edges for the graphs whose perfect matching graphs are bipartite.

Let $G$ be a graph which has perfect matchings. If $PM(G)$ is a complete graph, i.e., the diameter of the 1-skeleton graph of the perfect matching polytope of $G$ is 1, we call $G$ perfect matching compact, or $PM$-compact for short. Clearly, $K_4$ and $K_6$ are $PM$-compact. Let $v$ be a vertex of degree 2 of $G$ which has two distinct neighbors. The bicontraction of $v$ is the graph obtained from $G$ by contracting both edges incident with $v$. The retract of $G$ is the graph obtained from $G$ by successively bicontracting vertices of degree 2 until either there are no vertices of degree 2 or at most two vertices remain. A graph with two vertices and at least two parallel edges is denoted by $K^*_2$. A graph is matching-covered if every edge of it appears in a perfect matching. Let $\delta(G)$ denote the minimum degree of $G$. For bipartite graphs, the following result is obtained in [7].

**Theorem 1.** (i) Let $G$ be a matching-covered bipartite graph. Then $G$ is $PM$-compact if and only if the retract of $G$ is $K_{3,3}$ or $K^*_2$.

(ii) The graph $K_{3,3}$ is the only simple matching-covered $PM$-compact bipartite graph $G$ with $\delta(G) \geq 3$.

Let $H$ be a subgraph of a graph $G$. An ear of $G$ with respect to $H$ is a path of odd length in $G$ which has both ends, but no edges or interior vertices, in $H$. We call an ear trivial if it is an edge. An ear decomposition of a bipartite graph $G$ is a sequence of subgraphs $(G_0, G_1, \ldots, G_r)$, where $G_0 = K_2$, $G_r = G$, and for $1 \leq i \leq r$, $G_i$ is the union of $G_{i-1}$ and an ear $P_i$ of $G_i$ with respect to $G_{i-1}$. Clearly, $G_1$ is an even cycle and $G = K_2 + P_1 + \cdots + P_r$. In [3] Theorem 4.1.1 and Theorem 4.1.6 imply the following.

**Theorem 2.** A bipartite graph $G$ is matching-covered if and only if $G$ has an ear decomposition.

This theorem implies that for an ear decomposition of a matching-covered bipartite graph, each member of the sequence is matching-covered. If $G$ is a matching-covered graph, then $G$ is 2-connected, and so has minimum degree at least 2. In this paper, we show that a $PM$-compact bipartite graph $G$ with $\delta(G) \geq 2$ has an
ear decomposition such that each member of the decomposition sequence is PM-compact, which implies that \( G \) is matching-covered. Thus the characterization of PM-compact bipartite graphs is complete. (Note that each pendant edge (of which one end has degree 1) of a graph is contained in all perfect matchings. Using the obtained results, it is easy to characterize PM-compact bipartite graphs with minimum degree one.)

2. Main Result

A vertex \( v \) of a graph \( G \) is said to be pendant if its degree is 1 in \( G \). A bipartite graph \( G \) with bipartition \((X, Y)\) is denoted by \( G[X, Y] \). The following lemma is an immediate consequence of Exercise 16.1.13 in [2].

**Lemma 3.** Let \( G[X, Y] \) be a bipartite graph. Then \( G \) has a unique perfect matching if and only if

(i) each of \( X \) and \( Y \) contains a pendant vertex, and

(ii) when the pendant vertices and their neighbors are deleted, the resulting graph (if nonempty) has a unique perfect matching.

**Lemma 4.** Let \( G \) be a PM-compact graph and \( H \) a subgraph of \( G \) which has a perfect matching. If either (i) \( H \) is a spanning subgraph of \( G \) or (ii) \( G - V(H) \) has a perfect matching, then \( H \) is PM-compact.

**Proof.** If (i) holds, the assertion follows directly from the definition of PM-compact graphs.

If (ii) holds, let \( M \) be a perfect matching of \( G - V(H) \). Suppose that \( M'_1 \) and \( M'_2 \) are two distinct perfect matchings of \( H \). Then \( M_1 = M'_1 \cup M \) and \( M_2 = M'_2 \cup M \) are two perfect matchings of \( G \). Since \( G \) is PM-compact, \( M_1 \triangle M_2 \) is an alternating cycle of \( G \). So \( M'_1 \triangle M'_2 = M_1 \triangle M_2 \) is an alternating cycle of \( H \), and hence \( H \) is PM-compact.

**Theorem 5.** Let \( G \) be a PM-compact bipartite graph with \( \delta(G) \geq 2 \). Then \( G \) has an ear decomposition \((G_0, G_1, \ldots, G_r)\) such that each \( G_i \), \( 1 \leq i \leq r \), is PM-compact.

**Proof.** Suppose that \( H \) is a subgraph of \( G \) such that \( G - V(H) \) has a unique perfect matching \( M^* \). If a nontrivial ear \( P \) of \( G \) with respect to \( H \) is an \( M^* \)-alternating path, then we call \( P \) a normal ear.

**Claim.** The graph \( G \) has a normal ear with respect to \( H \).

**Proof.** To show this, write \( G^* = G - V(H) \). Let \( P^* \) be a longest \( M^* \)-alternating path in \( G^* \). Let \( x \) and \( y \) be the two ends of \( P^* \). We assert that both \( x \) and \( y \)
are covered by $M^* \cap E(P^*)$ and each have a unique neighbor in $G^*$, that is, their other neighbors are all in $H$. We show this by way of contradiction. If $x$ is not covered by $M^* \cap E(P^*)$, let $y'$ be the vertex matched to $x$ under $M^*$ (clearly, $y' \in V(G^*)$); otherwise, let $y'$ be an arbitrary neighbor of $x$ in $G^* - E(P^*)$. When $y' \notin V(P^*)$, $P^* + xy'$ is an $M^*$-alternating path which is longer than $P^*$. But this contradicts the choice of $P^*$. When $y' \in V(P^*)$, let $C^*$ be the union of the edge $xy'$ and the segment of $P^*$ from $x$ to $y'$. Since $G$ is bipartite, $C^*$ is an even cycle which is an $M^*$-alternating cycle. Hence $M^* \triangle E(C^*)$ is another perfect matching of $G^*$, which contradicts the uniqueness of $M^*$. Therefore $x$ is covered by $M^* \cap E(P^*)$ and has only one neighbor in $G^*$ (namely, a member of $V(P^*)$).

By symmetry, $y$ also has these properties. The assertion follows.

Since $\delta(G) \geq 2$, by the above assertion, $x$ and $y$ have neighbors in $H$. Let $x_1, y_1 \in V(H)$ be two neighbors of $x$ and $y$, respectively. The above assertion also implies that the length of $P^*$ is odd. Since $G$ is bipartite, we have $x_1 \neq y_1$. Write $P = P^* + x_1y_1 + y_1x_1$. By the above assertion again, $P$ is an $M^*$-alternating path with odd length. So $P$ is a normal ear of $G$ with respect to $H$. The claim follows.

We now proceed inductively to get an ear decomposition of $G$. For an even cycle $C$ of $G$, if $G - V(C)$ has a perfect matching, we call $C$ a $PM$-alternating cycle.

Recall $\delta(G) \geq 2$. By Lemma 3, $G$ has at least two perfect matchings. Since each cycle in the symmetric difference of any two perfect matchings of $G$ is a $PM$-alternating cycle of $G$, $G$ has $PM$-alternating cycles. Let $C$ be a $PM$-alternating cycle of $G$, and set $H_1 = C$. If $G - V(H_1)$ has two perfect matchings $M'_1$ and $M'_2$, let $E_1$ and $E_2$ be the two disjoint perfect matchings in $H_1$. Then $M_1 = M'_1 \cup E_1$ and $M_2 = M'_2 \cup E_2$ are two perfect matchings of $G$. Since $M_1 \triangle M_2$ contains at least two alternating cycles, namely, $C$ and an alternating cycle in $M'_1 \triangle M'_2$, $M_1$ and $M_2$ are not adjacent in $PM(G)$. This contradicts the assumption that $G$ is $PM$-compact. So either $G - V(H_1)$ has a unique perfect matching, say $M'$, or $G - V(H_1)$ is null.

For the former case, by the above claim, $G$ has a normal ear $P_2$ with respect to $H_1$. Set $H_2 = H_1 + P_2$. If $H_2$ is not spanning, then $M' \setminus E(P_2)$ is the unique perfect matching of $G - V(H_2)$. So we can proceed to find a normal ear $P_3$ of $G$ with respect to $H_2$. Continue in this way until $H_k = H_{k-1} + P_k$, $k \geq 1$, is a spanning subgraph of $G$. Write $E' = E(G) \setminus E(H_k)$. Then each edge in $E'$ is a trivial ear of $G$ with respect to $H_k$. Write $r = k + |E'|$. Then we get an ear decomposition $(H_1, H_2, \ldots, H_k)$ of $G$, where $H_i = H_{i-1} + P_i$ such that $P_i$ is a normal ear of $H_i$ with respect to $H_{i-1}$ for each $2 \leq i \leq k$ and a trivial ear (an edge in $E'$) of $H_i$ with respect to $H_{i-1}$ for each $k + 1 \leq i \leq r$.

For the latter case, $H_1$ is a spanning subgraph of $G$. Then each edge in $E' = E(G) \setminus E(H_1)$ is a trivial ear of $G$ with respect to $C$. Since $G = H_1 + E'$, we are done.
Let \((G_0, G_1, \ldots, G_r)\) be an arbitrary ear decomposition of \(G\). Recall that \(G_0\) is \(K_2\) and \(G_1\) is an even cycle. To complete the proof, we show that for each \(1 \leq i \leq r - 1\), \(G_i\) is \(PM\)-compact. Note that \(G - V(G_i)\) either is null or has a perfect matching (which is unique). Thus either \(G_i\) is a spanning subgraph of \(G\) or \(G - V(G_i)\) has a unique perfect matching. Since \(G_i\) also has a perfect matching, by Lemma 4, \(G_i\) is \(PM\)-compact.

Note that in the proof of Theorem 5, we show a stronger assertion that for each ear decomposition of a \(PM\)-compact bipartite graph \(G\) with \(\delta(G) \geq 2\), each member in the decomposition sequence is \(PM\)-compact.

By Theorem 2 and Theorem 5, we get the following.

**Corollary 6.** Any \(PM\)-compact bipartite graph \(G\) with \(\delta(G) \geq 2\) is matching-covered.

**Acknowledgement**

The authors are grateful to referee for his/her helpful comments which have improved the presentation of this paper.

**References**


Received 19 July 2012
Revised 22 October 2012
Accepted 22 October 2012