THE NICHE GRAPHS OF INTERVAL ORDERS

JEONGMI PARK

Department of Mathematics
Pusan National University
Busan 609-735, Korea

e-mail: jm1015@pusan.ac.kr

AND

YOSHIRO SANO

Division of Information Engineering
Faculty of Engineering, Information and Systems
University of Tsukuba
Ibaraki 305-8573, Japan

e-mail: sano@cs.tsukuba.ac.jp

Abstract

The niche graph of a digraph $D$ is the (simple undirected) graph which has the same vertex set as $D$ and has an edge between two distinct vertices $x$ and $y$ if and only if $N^+_D(x) \cap N^+_D(y) \neq \emptyset$ or $N^-_D(x) \cap N^-_D(y) \neq \emptyset$, where $N^+_D(x)$ (resp. $N^-_D(x)$) is the set of out-neighbors (resp. in-neighbors) of $x$ in $D$. A digraph $D = (V, A)$ is called a semiorder (or a unit interval order) if there exist a real-valued function $f : V \to \mathbb{R}$ on the set $V$ and a positive real number $\delta \in \mathbb{R}$ such that $(x, y) \in A$ if and only if $f(x) > f(y) + \delta$. A digraph $D = (V, A)$ is called an interval order if there exists an assignment $J$ of a closed real interval $J(x) \subset \mathbb{R}$ to each vertex $x \in V$ such that $(x, y) \in A$ if and only if $\min J(x) < \max J(y)$.

Kim and Roberts characterized the competition graphs of semiorders and interval orders in 2002, and Sano characterized the competition-common enemy graphs of semiorders and interval orders in 2010. In this note, we give characterizations of the niche graphs of semiorders and interval orders.

Keywords: competition graph, niche graph, semiorder, interval order.

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1. Introduction

Cohen [2] introduced the notion of competition graphs in 1968 in connection with a problem in ecology. The competition graph \( C(D) \) of a digraph \( D \) is the (simple undirected) graph which has the same vertex set as \( D \) and has an edge between two distinct vertices \( x \) and \( y \) if and only if \( N_D^+(x) \cap N_D^+(y) \neq \emptyset \), where \( N_D^+(x) = \{ v \in V(D) \mid (x, v) \in A(D) \} \) is the set of out-neighbors of \( x \) in \( D \). (For a digraph \( D \), we denote the vertex set and the arc set of \( D \) by \( V(D) \) and \( A(D) \), respectively.) It has been one of the important research problems in the study of competition graphs to characterize the competition graphs of digraphs satisfying some specified conditions.

A digraph \( D = (V, A) \) is called a semiorder (or a unit interval order) if there exist a real-valued function \( f : V \rightarrow \mathbb{R} \) on the set \( V \) and a positive real number \( \delta \in \mathbb{R} \) such that \( (x, y) \in A \) if and only if \( f(x) > f(y) + \delta \). A digraph \( D = (V, A) \) is called an interval order if there exists an assignment \( J \) of a closed real interval \( J(x) \subset \mathbb{R} \) to each vertex \( x \in V \) such that \( (x, y) \in A \) if and only if \( \min J(x) > \max J(y) \). We call \( J \) an interval assignment of \( D \). (See [3] for details on interval orders.)

A complete graph is a graph which has an edge between every pair of vertices. We denote the complete graph with \( n \) vertices by \( K_n \). An edgeless graph is a graph which has no edges. We denote the edgeless graph with \( n \) vertices by \( I_n \). The (disjoint) union of two graphs \( G \) and \( H \) is the graph \( G \cup H \) whose vertex set is the disjoint union of the vertex sets of \( G \) and \( H \) and whose edge set is the disjoint union of the edge sets of \( G \) and \( H \).

Kim and Roberts characterized the competition graphs of semiorders and interval orders as follows.

**Theorem 1** [4]. Let \( G \) be a graph. Then the following are equivalent.

(a) \( G \) is the competition graph of a semiorder,
(b) \( G \) is the competition graph of an interval order,
(c) \( G = K_r \cup I_q \) where if \( r \geq 2 \) then \( q \geq 1 \).

Scott [6] introduced the competition-common enemy graphs of digraphs in 1987 as a variant of competition graphs. The competition-common enemy graph of a digraph \( D \) is the graph which has the same vertex set as \( D \) and has an edge between two distinct vertices \( x \) and \( y \) if and only if both \( N_D^+(x) \cap N_D^+(y) \neq \emptyset \) and \( N_D^-(x) \cap N_D^-(y) \neq \emptyset \) hold, where \( N_D^-(x) = \{ v \in V(D) \mid (v, x) \in A(D) \} \) is the set of in-neighbors of \( x \) in \( D \).

Sano characterized the competition-common enemy graphs of semiorders and interval orders as follows.

**Theorem 2** [5]. Let \( G \) be a graph. Then the following are equivalent.
(a) $G$ is the competition-common enemy graph of a semiorder,
(b) $G$ is the competition-common enemy graph of an interval order,
(c) $G = K_r \cup I_q$ where if $r \geq 2$ then $q \geq 2$.

Niche graphs are another variant of competition graphs, which were introduced by Cable, Jones, Lundgren and Seager [1]. The niche graph of a digraph $D$ is the graph which has the same vertex set as $D$ and has an edge between two distinct vertices $x$ and $y$ if and only if $N^+_D(x) \cap N^+_D(y) \neq \emptyset$ or $N^-_D(x) \cap N^-_D(y) \neq \emptyset$.

In this note, we characterize the niche graphs of semiorders and interval orders. As a consequence, it turns out that the class of the niche graphs of interval orders is larger than the class of the niche graphs of semiorders. In fact, the graph $P_3 \cup I_1$ (the union of a path with three vertices and an isolated vertex) is the niche graph of an interval order, but $P_3 \cup I_1$ is not the niche graph of a semiorder.

2. Main Results

To state our main results, we first recall basic terminology in graph theory. The vertex set and the edge set of a graph $G$ are denoted by $V(G)$ and $E(G)$, respectively. The complement of a graph $G$ is the graph $\overline{G}$ defined by $V(\overline{G}) = V(G)$ and $E(\overline{G}) = \{vv' \mid v, v' \in V(G), v \neq v', vv' \notin E(G)\}$. For positive integers $m$ and $n$, the complete bipartite graph $K_{m,n}$ is the graph defined by $V(K_{m,n}) = X \cup Y$, where $|X| = m$ and $|Y| = n$, and $E(K_{m,n}) = \{xy \mid x \in X, y \in Y\}$. We can observe that $K_{m,n} = K_m \cup K_n$.

The niche graphs of semiorders are characterized as follows.

**Theorem 3.** A graph $G$ is the niche graph of a semiorder if and only if $G$ is one of the following graphs.

(i) an edgeless graph $I_q$,
(ii) the union of two complete graphs $K_m \cup K_n$,
(iii) the union of two complete graphs and an edgeless graph $K_m \cup K_n \cup I_q$,
(iv) the complement of the union of a complete bipartite graph and an edgeless graph $K_{m,n} \cup I_q$,

where $m$, $n$, and $q$ are positive integers.

**Proof.** First, we show the “only if” part. Let $G$ be the niche graph of a semiorder $D$. Then there exist a function $f : V(D) \to \mathbb{R}$ and a positive real number $\delta \in \mathbb{R}_{>0}$ such that $A(D) = \{(x, y) \mid x, y \in V(D), f(x) > f(y) + \delta\}$. Let $r_1$ and $r_2$ be real numbers defined by

$$r_1 = \min_{x \in V(D)} f(x) \quad \text{and} \quad r_2 = \max_{x \in V(D)} f(x).$$
We consider the following three cases: Case 1. $r_1 + \delta \geq r_2$, Case 2. $r_1 + \delta < r_2 \leq r_1 + 2\delta$, Case 3. $r_1 + 2\delta < r_2$.

Case 1. Consider the case where $r_1 + \delta \geq r_2$. In this case, we can observe that $D$ has no arcs. Therefore $G$ is an edgeless graph.

Case 2. Consider the case where $r_1 + \delta < r_2 \leq r_1 + 2\delta$. Note that $r_1 < r_2 - \delta \leq r_1 + \delta < r_2$. Let $V_1$, $V_2$, and $V_3$ be subsets of $V(D)$ defined by

$$V_1 = \{v \in V(D) \mid r_1 \leq f(v) < r_2 - \delta\},$$
$$V_2 = \{v \in V(D) \mid r_2 - \delta \leq f(v) \leq r_1 + \delta\},$$
$$V_3 = \{v \in V(D) \mid r_1 + \delta < f(v) \leq r_2\}.$$

Then it follows that $V(G) = V_1 \cup V_2 \cup V_3$ and $V_i \cap V_j = \emptyset$ if $i \neq j$. Note that $V_1 \neq \emptyset$ since there exists a vertex $x \in V(D)$ such that $f(x) = r_1$, and that $V_3 \neq \emptyset$ since there exists a vertex $x \in V(D)$ such that $f(x) = r_2$. The set $V_1$ forms a clique in $G$ since any vertex in $V_1$ has a common in-neighbor which belongs to $V_3$ in $D$. The set $V_3$ forms a clique in $G$ since any vertex in $V_3$ has a common out-neighbor which belongs to $V_1$ in $D$. Any vertex in $V_1$ and any vertex in $V_3$ are not adjacent in $G$ since any vertex in $V_1$ has no out-neighbor in $D$ and any vertex in $V_3$ has no in-neighbor in $D$. Furthermore, any vertex in the set $V_2$ is an isolated vertex in $G$ since it has neither an in-neighbor nor an out-neighbor in $D$. That is, the set $V_2$ induces an edgeless graph if $V_2 \neq \emptyset$. Thus, $G$ is the union of two complete graphs, or $G$ is the union of two complete graphs and an edgeless graph.

Case 3. Consider the case where $r_1 + 2\delta < r_2$. Note that $r_1 < r_1 + \delta < r_2 - \delta < r_2$. Let $V_1$, $V_2$, and $V_3$ be subsets of $V(D)$ defined by

$$V_1 = \{v \in V(D) \mid r_1 \leq f(v) \leq r_1 + \delta\},$$
$$V_2 = \{v \in V(D) \mid r_1 + \delta < f(v) < r_2 - \delta\},$$
$$V_3 = \{v \in V(D) \mid r_2 - \delta \leq f(v) \leq r_2\}.$$

Then it follows that $V(G) = V_1 \cup V_2 \cup V_3$ and $V_i \cap V_j = \emptyset$ if $i \neq j$. Note that $V_1 \neq \emptyset$ and $V_3 \neq \emptyset$. The set $V_2 \cup V_3$ forms a clique in $G$ since any vertex in $V_2 \cup V_3$ has a common out-neighbor which belongs to $V_1$ in $D$. The set $V_1 \cup V_2$ forms a clique in $G$ since any vertex in $V_1 \cup V_2$ has a common in-neighbor which belongs to $V_3$ in $D$. Any vertex in $V_1$ and any vertex in $V_3$ are not adjacent in $G$ since any vertex in $V_1$ has no out-neighbor in $D$ and any vertex in $V_3$ has no in-neighbor in $D$. Therefore, $G = K_{m,n} \cup T_q$ where $m = |V_1|$, $n = |V_3|$, and $q = |V_2|$. Thus, $G$ is the union of two complete graphs if $V_2 = \emptyset$, and $G$ is the complement of the union of a complete bipartite graph and an edgeless graph if $V_2 \neq \emptyset$.

Second, we show the “if” part.
Case (i). Let $G$ be an edgeless graph. We define a function $f : V(G) \rightarrow \mathbb{R}$ by $f(x) = 1$ for all $x \in V(G)$, and let $\delta = 1$. Then $f$ and $\delta$ gives a semiorder $D = (V, A)$ where $V = V(G)$ and $A = \emptyset$, and the niche graph of the semiorder $D$ is the graph $G$.

Cases (ii) and (iii). Let $G$ be the union of two complete graphs $K$ and $K'$ and an edgeless graph $I$, where $I$ may possibly be the graph with no vertices. We define a function $f : V(G) \rightarrow \mathbb{R}$ by $f(x) = 1$ if $x \in V(K)$, $f(x) = 4$ if $x \in V(K')$, $f(x) = 2$ if $x \in V(I)$, and let $\delta = 2$. Then $f$ and $\delta$ gives a semiorder $D = (V, A)$ where $V = V(G)$ and $A = \{(x, y) \mid x \in V(K'), y \in V(K)\}$, and the niche graph of the semiorder $D$ is the graph $G$.

Case (iv). Let $G = K_{m,n} \cup I_q$. Let $X$ and $Y$ be the partite sets of the complete bipartite graph $K_{m,n}$ and let $Z$ be the vertex set of the edgeless graph $I_q$. Then $(X, Y, Z)$ is a tripartition of the vertex set of $G$ and $E(G) = \{vv' \mid v, v' \in V(G), v \neq v'\} \setminus \{xy \mid x \in X, y \in Y\}$. Now, we define a function $f : V(G) \rightarrow \mathbb{R}$ by $f(x) = 1$ if $x \in X$, $f(z) = 3$ if $z \in Z$, $f(y) = 5$ if $y \in Y$, and let $\delta = 1$. Then $f$ and $\delta$ gives a semiorder $D = (V, A)$ where $V = V(G)$ and $A = \{(y, x) \mid x \in X, y \in Y\} \cup \{(z, x) \mid x \in X, z \in Z\} \cup \{(y, z) \mid z \in Z, y \in Y\}$, and the niche graph of the semiorder $D$ is the graph $G$. Hence the theorem holds. $\blacksquare$

The next theorem characterizes the niche graphs of interval orders.

**Theorem 4.** A graph $G$ is the niche graph of an interval order if and only if $G$ is one of the following graphs:

(i) an edgeless graph $I_q$,
(ii) the union of two complete graphs $K_m \cup K_n$,
(iii) the union of two complete graphs and an edgeless graph $K_m \cup K_n \cup I_r$,
(iv) the complement of the union of a complete bipartite graph and an edgeless graph $\overline{K_{m,n}} \cup I_q$,
(v) the union of an edgeless graph and the complement of the union of a complete bipartite graph and an edgeless graph $I_r \cup \overline{K_{m,n}} \cup I_q$,
where $m$, $n$, $q$, and $r$ are positive integers.

**Proof.** For positive integers $m$ and $n$ and non-negative integers $q$ and $r$, let

$$\Gamma(m, n, q, r) = \overline{K_{m,n}} \cup I_q \cup I_r.$$

We remark that $\Gamma(m, n, 0, 0) = K_m \cup K_n$, $\Gamma(m, n, 0, r) = K_m \cup K_n \cup I_r$, and $\Gamma(m, n, q, 0) = \overline{K_{m,n}} \cup I_q$.

First, we show the “only if” part. Let $G$ be the niche graph of an interval order $D$. Then there exists an interval assignment $J$ of $D$. Let $r_1$ and $r_2$ be real numbers defined by

$$r_1 = \min_{x \in V(D)} \max J(x) \quad \text{and} \quad r_2 = \max_{x \in V(D)} \min J(x).$$
If \( r_1 \geq r_2 \), then we can observe that \( D \) has no arcs and therefore \( G \) is an edgeless graph. Now, we consider the case where \( r_1 < r_2 \). Note that \(|V(G)| \geq 2\) since \( r_1 \) and \( r_2 \) are attained by different vertices. Let \( V_1, V_2, V_3, \) and \( V_4 \) be subsets of \( V(D) \) defined by

\[
V_1 = \{ v \in V(D) \mid \min J(v) \leq r_1 \leq \max J(v) < r_2 \}, \\
V_2 = \{ v \in V(D) \mid r_1 < \min J(v), \max J(v) < r_2 \}, \\
V_3 = \{ v \in V(D) \mid r_1 < \min J(v) \leq r_2 \leq \max J(v) \}, \\
V_4 = \{ v \in V(D) \mid \min J(v) \leq r_1, r_2 \leq \max J(v) \}.
\]

Then it follows that \( V(G) = V_1 \cup V_2 \cup V_3 \cup V_4 \) and \( V_i \cap V_j = \emptyset \) if \( i \neq j \). Note that \( V_1 \neq \emptyset \) since there exists a vertex \( x \in V(D) \) such that \( \max J(x) = r_1 \), and that \( V_3 \neq \emptyset \) since there exists a vertex \( x \in V(D) \) such that \( \min J(x) = r_2 \). The set \( V_2 \cup V_3 \) forms a clique in \( G \) since any vertex in \( V_2 \cup V_3 \) has a common out-neighbor which belongs to \( V_1 \) in \( D \). The set \( V_1 \cup V_2 \) forms a clique in \( G \) since any vertex in \( V_1 \cup V_2 \) has a common in-neighbor which belongs to \( V_3 \) in \( D \). Any vertex in \( V_1 \) and any vertex in \( V_3 \) are not adjacent in \( G \) since any vertex in \( V_3 \) has no out-neighbor in \( D \) and any vertex in \( V_3 \) has no in-neighbor in \( D \). Therefore, the set \( V_1 \cup V_2 \cup V_3 \) induces the graph \( K_{m,n} \cup T_q \) where \( m = |V_1|, n = |V_3|, \) and \( q = |V_2| \). Furthermore, any vertex in the set \( V_4 \) is an isolated vertex in \( G \) since it has neither an in-neighbor nor an out-neighbor in \( D \). That is, the set \( V_4 \) induces an edgeless graph. Thus \( G \) is the graph \( \Gamma(m,n,q,r) \) with \( m = |V_1|, n = |V_3|, q = |V_2|, \) and \( r = |V_4| \).

Second, we show the “if” part.

Case (i). Let \( G \) be an edgeless graph. We define an interval assignment \( J \) by \( J(x) = [1,2] \) for all \( x \in V(G) \), where \([a,b]\) denotes the closed real interval \( \{ r \in \mathbb{R} \mid a \leq r \leq b \} \). Then \( J \) gives an interval order \( D = (V, A) \) where \( V = V(G) \) and \( A = \emptyset \), and the niche graph of the semiorder \( D \) is the graph \( G \).

Cases (ii)–(v). Let \( G \) be the graph \( \Gamma(m,n,q,r) \) for some positive integers \( m \) and \( n \) and non-negative integers \( q \) and \( r \). Then, there exists a partition \( \{ U_1, U_2, U_3, U_4 \} \) of the vertex set of \( G \) such that \( E(G) = \{ vv' \mid v, v' \in U_1 \cup U_2 \cup U_3, v \neq v' \} \) \( \cup \{ u_1u_3 \mid u_1 \in U_1, u_3 \in U_3 \} \). Note that \( \{ |U_1|, |U_3| \} = \{ m, n \}, |U_2| = q, \) and \( |U_4| = r \). Now, we define an interval assignment \( J \) as follows: \( J(x) = [1,2] \) if \( x \in U_1 \); \( J(x) = [3,4] \) if \( x \in U_2 \); \( J(x) = [5,6] \) if \( x \in U_3 \); \( J(x) = [1,6] \) if \( x \in U_4 \). Then \( J \) gives an interval order \( D = (V, A) \) where \( V = V(G) \) and \( A = \{(x,y) \mid x \in U_i, y \in U_j, (i,j) \in \{(3,2),(3,1),(2,1)\}\} \), and the niche graph of the interval order \( D \) is the graph \( G \). Hence the theorem holds. \( \blacksquare \)
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