THE CROSSING NUMBERS OF PRODUCTS OF PATH WITH GRAPHS OF ORDER SIX

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Abstract

The crossing numbers of Cartesian products of paths, cycles or stars with all graphs of order at most four are known. For the path $P_n$ of length $n$, the crossing numbers of Cartesian products $G \square P_n$ for all connected graphs $G$ on five vertices are also known. In this paper, the crossing numbers of Cartesian products $G \square P_n$ for graphs $G$ of order six are studied. Let $H$ denote the unique tree of order six with two vertices of degree three. The main contribution is that the crossing number of the Cartesian product $H \square P_n$ is $2(n - 1)$. In addition, the crossing numbers of $G \square P_n$ for forty graphs $G$ on six vertices are collected.

Keywords: graph, drawing, crossing number, Cartesian product, path, tree.

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1. Introduction

The crossing number $\text{cr}(G)$ of a simple graph $G$ with vertex set $V(G)$ and edge set $E(G)$ is defined as the minimum possible number of edge crossings in a drawing of $G$ in the plane. A drawing with minimum number of crossings (an optimal drawing) must be a good drawing; that is, each two edges have at most one point in common, which is either a common end-vertex or a crossing. Moreover, no three edges cross in a point. Let $D$ be a good drawing of the graph $G$. We

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denote the number of crossings in $D$ by $cr_D(G)$. Let $G_i$ and $G_j$ be edge-disjoint subgraphs of $G$. We denote by $cr_D(G_i, G_j)$ the number of crossings between edges of $G_i$ and edges of $G_j$, and by $cr_D(G_i)$ the number of crossings among edges of $G_i$ in $D$.

The investigation on the crossing number of graphs is a classical but very difficult problem. According to their special structure, Cartesian products of special graphs are one of few graph classes for which the exact values of crossing numbers were obtained. Let $G_1$ and $G_2$ be simple graphs with vertex sets $V(G_1)$ and $V(G_2)$, and edge sets $E(G_1)$ and $E(G_2)$, respectively. The Cartesian product $G_1 \Box G_2$ of the graphs $G_1$ and $G_2$ has vertex set $V(G_1 \Box G_2) = V(G_1) \times V(G_2)$ and two vertices $(u, u')$ and $(v, v')$ are adjacent in $G_1 \Box G_2$ if and only if either $u = v$ and $u'$ is adjacent with $v'$ in $G_2$, or $u' = v'$ and $u$ is adjacent with $v$ in $G_1$.

Let $C_n$ be the cycle of length $n$, $P_n$ be the path of length $n$, and $S_n$ be the star isomorphic to $K_{1,n}$. Beineke and Ringel in [1] started to study the crossing numbers of Cartesian products of cycles with all graphs of order at most four. In [3], [4], and [5], the crossing numbers of Cartesian products of cycles, paths and stars with all graphs of order four are given. The crossing numbers of Cartesian products of paths with all graphs of order five are collected in [8]. It seems natural to enquire about crossing numbers of Cartesian products of paths with other graphs. There are known the crossing numbers of products $G \Box P_n$ for some graphs $G$ on six vertices, see [9], [10], [11], [12], and [13]. In the paper, we extend these results by giving the exact values of crossing numbers for Cartesian products of paths with several graphs of order six. We consider graphs $G_i$, $i = 1, 2, \ldots, 40$, on six vertices which are collected in Table 1 in the last section of the paper. All known results concerning crossing numbers of Cartesian products of these graphs with paths are presented in Table 1.

![Figure 1. All trees of order six.](image-url)

2. Trees on Six Vertices

In this section, we give the crossing numbers of Cartesian products of paths with all trees on six vertices. There are six trees of order six shown in Figure 1. The graph $G_1 \Box P_n = P_5 \Box P_n$ is planar. The graph $G_6$ is isomorphic with the star.
It was proved in [2] that $\text{cr}(S_n \square P_n) = (n − 1)\lfloor \frac{m}{2} \rfloor \lfloor \frac{m-1}{2} \rfloor$. So, $\text{cr}(G_6 \square P_n) = 4(n − 1)$. As both graphs $G_2$ and $G_3$ contain the star $S_3$ as a subgraph, the Cartesian product $S_3 \square P_n$ is a subgraph of both graphs $G_2 \square P_n$ and $G_3 \square P_n$. Thus, $\text{cr}(G_2 \square P_n) \geq n − 1$ and $\text{cr}(G_3 \square P_n) \geq n − 1$, because $\text{cr}(S_3 \square P_n) = n − 1$, see [3].

On the other hand, in Figure 2(a) and Figure 2(b) there are drawings of the graphs $G_2 \square P_n$ and $G_3 \square P_n$ with $n − 1$ crossings. This implies that $\text{cr}(G_2 \square P_n) \leq n − 1$ and $\text{cr}(G_3 \square P_n) \leq n − 1$ and therefore, $\text{cr}(G_2 \square P_n) = \text{cr}(G_3 \square P_n) = n − 1$. The drawing in Figure 2(c) shows the graph $G_4 \square P_n$ with $2(n − 1)$ crossings. As the graph $G_4 \square P_n$ contains $S_4 \square P_n$ as a subgraph and $\text{cr}(S_4 \square P_n) = 2(n − 1)$, see [4], the crossing number of the graph $G_4 \square P_n$ is $2(n − 1)$. The aim of the rest of this section is to establish the crossing number of the graph $G_5 \square P_n$.

We assume $n \geq 1$ and find it convenient to consider the graph $G_5 \square P_n$ in the following way: it has $6(n + 1)$ vertices and edges that are the edges in $n + 1$ copies $G_5^i$, $i = 0, 1, \ldots, n$, and in six paths of length $n$. For $i = 0, 1, \ldots, n$, let $a_i, b_i, c_i$, and $f_i$ be the vertices of $G_5^i$ of degree one, $c_i$ and $d_i$ the vertices of degree three (see Figure 3). Thus, for $x \in \{a, b, c, d, e, f\}$, the path $P_n^x$ is induced by the vertices $x_0, x_1, \ldots, x_n$. For $i = 1, 2, \ldots, n$, let $H^i$ denote the subgraph of $G_5 \square P_n$ containing the vertices of $G_5^i$ and the six edges joining $G_5^{i−1}$ to $G_5^i$. Let $Q^i$, $i = 1, 2, \ldots, n − 1$, denote the subgraph of $G_5 \square P_n$ induced by $V(G_5^{i−1}) \cup V(G_5^i) \cup V(G_5^{i+1})$. So, $Q^i = G_5^{i−1} \cup H^i \cup G_5^i \cup H^{i+1} \cup G_5^{i+1}$. Let us denote by $Q^i_{ab}$ the subgraph of $Q^i$ obtained from $Q^i$ by removing six vertices $e_j$ and $f_j$ for $j = i − 1, i, i + 1$ and two edges $\{c_{i−1}, c_i\}$ and $\{c_i, c_{i+1}\}$. Likewise, let $Q^i_{ej}$ be the subgraph of $Q^i$ obtained by removing six vertices $a_j$ and $b_j$ for $j = i − 1, i, i + 1$ and two edges $\{d_{i−1}, d_i\}$ and $\{d_i, d_{i+1}\}$. It is easy to see that both subgraphs $Q^i_{ah}$ and $Q^i_{eh}$ are subdivisions of the graph $K_{3,3}$.

The graph $G_5 \square P_1$ is planar. In the next lemma, the crossing number of the graph $G_5 \square P_2$ is determined.

**Lemma 1.** $\text{cr}(G_5 \square P_2) = 2$.

**Proof.** It can be seen from the drawing in Figure 3 that $\text{cr}(G_5 \square P_2) \leq 2$. To
prove the reverse inequality, assume that there is a drawing of the graph \( G_5 \Box P_2 \) with less than two crossings. The graph \( G_5 \Box P_2 \) can be considered as the graph \( Q^1 \) defined above. As the subgraph \( Q^1_{ab} \) of \( Q^1 \) is a subdivision of \( K_{3,3} \), at least one crossing appears among the edges of \( Q^1_{ab} \). This implies that \( \text{cr}(G_5 \Box P_2) \geq 1 \). Our assumption of the considered drawing with less than two crossings forces that none of the edges incident with the vertices \( e_i \) and \( f_i \), \( i = 0, 1, 2 \), is crossed. But the unique planar drawing of the subgraph induced by the edges incident with the vertices \( e_0, e_1, e_2, f_0, f_1, \) and \( f_2 \) divides the plane into two hexagonal regions and one octagonal region in such a way that at most two of the vertices \( d_0, d_1, \) and \( d_2 \) are contained on a boundary of one region. Hence, the edge \( \{c_1, d_1\} \) or at least one of the paths \( c_1 c_0 d_0 \) and \( c_1 c_2 d_2 \) joining the vertex \( c_1 \) with the vertices \( d_0, d_1, \) and \( d_2 \) crosses the edges incident with the vertices \( e_0, e_1, e_2, f_0, f_1, \) and \( f_2 \). Thus, at least two crossings appear in any drawing of the graph \( G_5 \Box P_2 \). This completes the proof.

\[ \text{Lemma 2.} \quad \text{If} \ D \ \text{is a good drawing of the graph} \ G_5 \Box P_n, \ n \geq 3, \ \text{in which every} \ G_5^0 \cup H^1, \ G_5^n \cup H^n \ \text{and} \ G_5^i, \ i = 1, 2, \ldots, n - 1, \ \text{has at most one crossing on its edges, then in} \ D \ \text{there are at least} \ 2(n - 1) \ \text{crossings.} \]

\[ \text{Proof.} \quad \text{The proof is based on counting the total force of crossings in a drawing of a graph. This concept was introduced by Beineke and Ringel in [1]. Let us consider the following types of possible crossings on the edges of} \ Q^i \ \text{in a drawing of the graph} \ G_5 \Box P_n:} \]

1. a crossing of an edge in \( H^i \cup H^{i+1} \) with an edge in \( G_5^i \),

2. a crossing of an edge in \( G_5^{i-1} \cup H^i \) with an edge in \( G_5^{i+1} \cup H^{i+1} \).
(3) a crossing of an edge in \( G_{5}^{i-1} \cup G_{5}^{i+1} \) with an edge in \( G_{5}^{i} \),

(4) a crossing of an edge in \( G_{5}^{i-1} \cup H^{i} \) with an edge in \( G_{5}^{i+2} \cup H^{i+2} \),

(5) a crossing of an edge in \( G_{5}^{i+1} \cup H^{i+1} \) with an edge in \( G_{5}^{i-2} \cup H^{i-1} \).

It is readily seen that every crossing of types (1) and (2) appears in a good drawing of the graph \( G_{5}^{i} \cap P_{n} \) only on the edges of the subgraph \( Q^{i} \). For \( i \in \{2, 3, \ldots, n-1\} \), a crossing of type (3) in \( Q^{i} \) between an edge of \( G_{5}^{i-1} \) and an edge of \( G_{5}^{i} \) appears only in \( Q^{i-1} \) as a crossing of type (3), and a crossing of type (5) in \( Q^{i} \) appears only in \( Q^{i-1} \) as a crossing of type (4). For \( i \in \{1, 2, \ldots, n-2\} \), a crossing between an edge of \( G_{5}^{i+1} \) and an edge of \( G_{5}^{i} \) appears only in \( Q^{i+1} \) as a crossing of type (3), and a crossing of type (4) in \( Q^{i} \) appears only as a crossing of type (5) in \( Q^{i+1} \).

In a good drawing of \( G_{5}^{i} \cap P_{n} \), we define the force \( f(Q^{i}) \) of \( Q^{i} \) in the following way: every crossing of type (1) or (2) contributes the value 1 to \( f(Q^{i}) \) and every crossing of type (3), (4) or (5) contributes the value \( \frac{1}{2} \) to \( f(Q^{i}) \) (and \( \frac{1}{2} \) to \( Q^{i-1} \) or \( \frac{1}{2} \) to \( Q^{i+1} \)). The total force of the drawing is the sum of \( f(Q^{i}) \). As every crossing of type (1) or (2) is counted only once and every crossing of type (3), (4) or (5) is counted at most twice and no other crossing contributes to the total force of the drawing, the number of crossings in the drawing is not less than the total force of the drawing. So, the aim of this proof is to show that if every of the subgraphs \( G_{5}^{0} \cup H^{1} \), \( G_{5}^{i} \cup H^{n} \) and \( G_{5}^{i}, i = 1, 2, \ldots, n \), has at most one crossing on its edges, then \( f(Q^{i}) \geq 2 \) for all \( i = 1, 2, \ldots, n-1 \).

Consider the good drawing \( D \) of \( G_{5}^{i} \cap P_{n} \) assumed in Lemma 2 and let \( D_{ab}^{i} \) be the subdrawing of the subgraph \( Q_{ab}^{i} \) induced by \( D \). Any drawing of \( K_{3,3} \) contains a pair of edges that cross each other and do not meet in a vertex. The graph \( Q_{ab}^{i} \) can be obtained by elementary subdivision of six edges of \( K_{3,3} \). So, in \( D_{ab}^{i} \), there is a forced crossing between an edge in \( G_{5}^{i-1} \cup H^{i} \) and an edge in \( G_{5}^{i+1} \cup H^{i+1} \), or between an edge in \( G_{5}^{i} \) and an edge in \( H^{i} \cup H^{i+1} \), or between an edge in \( G_{5}^{i} \) and an edge in \( G_{5}^{i-1} \cup G_{5}^{i+1} \). Every of the first two considered types of crossings contributes the value 1 to \( f(Q^{i}) \) and the last one contributes the value \( \frac{1}{2} \). Hence, the minimal contribution of the subdrawing \( D_{ab}^{i} \) to \( f(Q^{i}) \) is \( \frac{1}{2} \), but, an edge of \( G_{5}^{i} \) is crossed by an edge of \( G_{5}^{i-1} \) or by an edge of \( G_{5}^{i+1} \) if \( f(Q^{i}) = \frac{1}{2} \). The same consideration can be repeated for the subdrawing \( D_{ef}^{i} \) of \( Q_{ef}^{i} \). Only three edges \( \{c_{i-1}, d_{i-1}\}, \{c_{i}, d_{i}\} \) and \( \{c_{i+1}, d_{i+1}\} \) appear in both subgraphs \( Q_{ab}^{i} \) and \( Q_{ef}^{i} \). As two adjacent edges cannot cross each other in \( K_{3,3} \), a possible crossing between two of these three edges cannot be the forced crossing in the subdrawing \( D_{ab}^{i} \) of \( Q_{ab}^{i} \). The same holds for the subdrawing \( D_{ef}^{i} \) of \( Q_{ef}^{i} \). Thus, there are at least two forced crossings in the subdrawing \( D^{i} \) of \( Q^{i} \) induced by \( D \). As there is at most one crossing on the edges of \( G_{5}^{i} \), at least one forced crossing contributes 1 to \( f(Q^{i}) \). If both forced crossings among the edges of \( Q^{i} \) contribute 1 to \( f(Q^{i}) \), then \( f(Q^{i}) \geq 2 \).
For $i \in \{2, 3, \ldots, n-2\}$, assume that one of the forced crossings among the edges of $Q^i$, say in $D^i_{ab}$, contributes only $\frac{1}{2}$ to $f(Q^i)$. Without loss of generality suppose that an edge of $G^5_5$ is crossed by an edge of $G^5_{5-1}$ in $D^i_{ab}$. As no two adjacent edges cross in a drawing of the graph $K_{3,3}$, it is easy to see that neither a crossing between the edges $\{a_i, c_i\}$ and $\{a_{i-1}, c_{i-1}\}$ nor a crossing between the edges $\{b_i, c_i\}$ and $\{b_{i-1}, c_{i-1}\}$ is forced in $D^i_{ab}$. Thus, the forced crossing in $D^i_{ab}$ is one of the following: $\{a_i, c_i\}$ crosses $\{b_{i-1}, c_{i-1}\}$, $\{b_i, c_i\}$ crosses $\{a_{i-1}, c_{i-1}\}$, $\{d_i, c_i\}$ crosses $\{a_{i-1}, c_{i-1}\}$ or $\{b_{i-1}, c_{i-1}\}$, and $\{d_{i-1}, c_{i-1}\}$ crosses $\{a_i, c_i\}$ or $\{b_i, c_i\}$. Up to the symmetry it is enough to consider only three case: the edge $\{a_i, c_i\}$ is crossed by $\{b_{i-1}, c_{i-1}\}$, the edge $\{d_i, c_i\}$ is crossed by $\{b_{i-1}, c_{i-1}\}$, and the edge $\{d_{i-1}, c_{i-1}\}$ is crossed by $\{a_i, c_i\}$. Since in $D$ there is no other crossing either on the edges of $G^5_5$ or on the edges of $G^5_{5-1}$, one can find in Figure 3 that in all three cases two vertex disjoint cycles $a_{i+1+c_{i+1}}d_{i+1}e_{i+1}d_{i}c_{i}a_{i}$ and $b_{i-1}b_{i-2}c_{i-2}d_{i-2}e_{i-2}c_{i-1}d_{i-1}c_{i-1}b_{i-1}$ cross each other in $D$ at least two times in such a way that the path $a_{i+1+c_{i+1}}d_{i+1}e_{i+1}d_{i}c_{i}a_{i}$ crosses the path $b_{i-1}b_{i-2}c_{i-2}d_{i-2}e_{i-2}c_{i-1}$. This crossing of type (5) contributes $\frac{1}{2}$ to $f(Q^i)$, and hence, $f(Q^i) \geq 2$.

Consider now the subgraph $Q^1$ induced on the vertices of $G^0_5$, $G^1_5$, and $G^2_5$. In the subdrawing $D^1$ of $Q^1$ induced by $D$ there are at least two forced crossings. If $\text{cr}_D(G^0_5, G^1_5) = \text{cr}_D(G^1_5, G^2_5) = 0$, then $f(Q^1) \geq 2$. If the subgraphs $G^1_5$ and $G^2_5$ cross each other, then the analysis in the previous paragraph implies that a crossing of type (4) between $G^0_5 \cup H^1$ and $G^2_5 \cup H^2$ is necessary. Hence, $f(Q^1) \geq 2$ in this case. As a crossing between the edges $\{c_0, d_0\}$ and $\{c_1, d_1\}$ is not a forced crossing in $D^1$, one of the edges $\{c_0, a_0\}$, $\{c_0, b_0\}$, $\{d_0, c_0\}$, $\{d_0, f_0\}$ must be crossed if $\text{cr}(G^0_5, G^1_5) = 1$. Without loss of generality let $\{c_0, a_0\}$ is crossed by an edge of $G^1_5$. Assume now the subgraph of $Q^1_{ef}$ induced by the edges incident with the vertices $c_0, d_0, c_0, f_0, c_1, d_1, e_1$ and $f_1$. As, by hypothesis, no other crossing appear on the edges of $G^0_5 \cup H^1 \cup G^1_5$, the unique planar subdrawing of the considered subgraph divides the plane into three hexagonal regions in such a way that at most two of the vertices $c_1, e_1$ and $f_1$ are placed on a boundary of one region. But, in this case, in the subdrawing $D^1_{ef}$ at least one of the paths $d_2e_2c_1$, $d_2f_2c_1$, and $d_2f_2f_1$ crosses an edge of $G^0_5 \cup H^1 \cup G^1_5$. This contradiction with the assumption of Lemma 2 implies that both forced crossings in $D^1$ contribute 1 to $f(Q^1)$ an therefore, $f(Q^1) \geq 2$. A similar analysis for the subgraph $Q^{n-1}$ gives that $f(Q^{n-1}) \geq 2$ as well. Hence, the total force of the drawing $D$ is at least $2(n-1)$, and in $D$ there are at least $2(n-1)$ crossings. This completes the proof.

\textbf{Theorem 3.} $\text{cr}(G_5 \Box P_n) = 2(n-1)$ for $n \geq 1$.

\textbf{Proof.} The drawing in Figure 3 shows that $\text{cr}(G_5 \Box P_n) \leq 2(n-1)$, because every copy of $G_5$, $i = 1, 2, \ldots, n - 1$, is crossed two times and there is no other crossings in the drawing. We prove the reverse inequality by induction on $n$. It
is easy to see that the graph $G_5 \Box P_1$ is planar and, by Lemma 1, $\text{cr}(G_5 \Box P_1) = 2$. So, the result is true for $n = 1$ and $n = 2$. Assume that it is true for $n = k$, $k \geq 2$, and suppose that there is a good drawing of $G_5 \Box P_{k+1}$ with fewer than $2k$ crossings. By Lemma 2, some of the subgraphs $G_5^0 \cup H^1$, $G_5^{k+1} \cup H^{k+1}$ and $G_5^i$, $i = 1, 2, \ldots, k$, must be crossed at least twice. If $G_5^0 \cup H^1$ has at least two crossings on its edges, the deletion of all vertices of $G_5^0$ results in a drawing of the graph $G_5 \Box P_k$ with fewer than $2(k-1)$ crossings. This contradicts the induction hypothesis. The same contradiction is obtained, if at least two crossings appear on the edges of $G_5^{k+1} \cup H^{k+1}$. If some $G_5^i$, $i = 1, 2, \ldots, k$, is crossed at least twice, by the removal of all edges of this $G_5^i$, a subdivision of $G_5 \Box P_k$ with fewer than $2(k-1)$ crossings is obtained. This contradiction with the induction hypothesis completes the proof.

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3. THE COLLECTION OF $\text{cr}(G_i \Box P_n)$ FOR GRAPHS $G_i$ ON SIX VERTICES

The aim of this section is to collect Cartesian products of graphs of order six with paths for which the crossing numbers are known. As for a disconnected graph $G$, the Cartesian product $G \Box P_n$ is disconnected, we are interesting only of connected graphs on six vertices. There are 112 connected graphs on six vertices. At present, we are able to summarise the crossing numbers of $G_i \Box P_n$ for forty connected graphs $G_i$ of order six shown in the Table 1.

In the previous section, the crossing numbers of Cartesian products of paths with all trees on six vertices are collected. These results enable us to determine the exact values of crossing numbers for Cartesian products of paths with some other graphs. It is easy to see that the graph $C_6 \Box P_n$ is planar. As the graph $G_7$ is isomorphic to the cycle $C_6$, $\text{cr}(C_6 \Box P_n) = 0$. The graphs $G_8$, $G_9$, $G_{12}$, and $G_{18}$ contain $S_3$ as a subgraph. Thus, all Cartesian products $G_i \Box P_n$, $i = 8, 9, 12, 18$, contain $S_3 \Box P_n$ as a subgraph. It was proved in [3] that $\text{cr}(S_3 \Box P_n) = n - 1$. This implies that $\text{cr}(G_i \Box P_n) \geq n - 1$ for $i = 8, 9, 12, 18$. On the other hand, the graphs $G_8 \Box P_n$, $G_9 \Box P_n$, and $G_{12} \Box P_n$ are subgraphs of the graph $G_{18} \Box P_n$. In Figure 4(a) there is a drawing of the graph $G_{18} \Box P_n$ with $n - 1$ crossings and therefore, $\text{cr}(G_{18} \Box P_n) \leq n - 1$. Hence, $\text{cr}(G_i \Box P_n) = n - 1$ for the graphs $G_i$, $i = 8, 9, 12, 18$.

Figure 4(b) shows the drawing of the graph $G_{27} \Box P_n$ with $2(n-1)$ crossings. The graph $G_{27} \Box P_n$ contains $G_{11} \Box P_n$, $G_{15} \Box P_n$, $G_{16} \Box P_n$, $G_{19} \Box P_n$, and $G_{25} \Box P_n$ as subgraphs. Thus, $\text{cr}(G_i \Box P_n) \leq 2(n - 1)$ for $i = 11, 15, 16, 19, 25$, and 27. As $\text{cr}(S_4 \Box P_n) = 2(n-1)$, see [4], $\text{cr}(G_{11} \Box P_n) = \text{cr}(G_{15} \Box P_n) = \text{cr}(G_{16} \Box P_n) = \text{cr}(G_{19} \Box P_n) = \text{cr}(G_{25} \Box P_n) = \text{cr}(G_{27} \Box P_n) = 2(n - 1)$, because each of these graphs contains $S_4 \Box P_n$ as a subgraph.

By Theorem 3, $\text{cr}(G_5 \Box P_n) = 2(n - 1)$. The graph $G_5 \Box P_n$ is a subgraph of all
The crossing number of all these graphs is at least \(2(n-1)\). To show the reverse inequality, we need suitable drawings of two of the considered six graphs. Except of the graph \(G_{17}\square P_n\), all other graphs \(G_i\square P_n\), \(i=10, 14, 17, 21, 23, 31\) are subgraphs of the graph \(G_{31}\square P_n\). In Figure 4(c) and Figure 4(d) one can find the drawings of the graphs \(G_{31}\square P_n\) and \(G_{17}\square P_n\), respectively, both with \(2(n-1)\) crossings. This implies that for \(i=10, 14, 17, 21, 23, 31\), the crossing number of the graphs \(G_i\square P_n\) is \(2(n-1)\).

The drawing of the graph \(G_{35}\square P_n\) with \(4(n-1)\) crossings is shown in Figure 4(e). Thus, \(cr(G_{35}\square P_n) \leq 4(n-1)\). As \(G_{35}\square P_n\) contains all graphs \(G_i\square P_n\), \(i=13, 22, 24, 26, 28\), as subgraphs, the value \(4(n-1)\) is the upper bound for crossing numbers of these graphs. On the other hand, each of the graphs \(G_i\square P_n\), \(i=13, 22, 24, 26, 28, 35\), contains \(S_5\square P_n\) as a subgraph. Bokal in [2] proved that \(cr(S_5\square P_n) = 4(n-1)\). Hence, \(cr(G_{13}\square P_n) = cr(G_{22}\square P_n) = cr(G_{24}\square P_n) = cr(G_{26}\square P_n) = cr(G_{28}\square P_n) = cr(G_{35}\square P_n) = 4(n-1)\).

In [6], the crossing number of the Cartesian product \(K_{2,3}\square P_n\) is given. Namely, \(cr(K_{2,3}\square P_n) = 2n\). We use these result and we give the values of crossing numbers of two other Cartesian products of paths with graphs of order six. The graph \(G_{20}\) is a subdivision of the complete bipartite graph \(K_{2,3}\) and the graph \(G_{29}\) contains a subdivision of \(K_{2,3}\) as a subgraph. Hence, the crossing number of both Cartesian products \(G_{20}\square P_n\) and \(G_{29}\square P_n\) is at least \(2n\). In Figure 5(a) there is a drawing of \(G_{29}\square P_n\) with \(2n\) crossings. Thus, \(cr(G_{29}\square P_n) \leq 2n\) and therefore, \(cr(G_{29}\square P_n) = 2n\). Moreover, as \(G_{20}\square P_n\) is a subgraph of \(G_{29}\square P_n\), the
Let $H_5$ be the graph obtained from the complete graph on five vertices $K_5$ by deleting three edges incident with the same vertex. It was shown in [7] that $\text{cr}(H_5 \square P_n) = 3n - 1$. Both graphs $G_{30}$ and $G_{36}$ contain a subdivision of the graph $H_5$ as a subgraph. This implies that the crossing number of both Cartesian products $G_{30} \square P_n$ and $G_{36} \square P_n$ is greater or equal $3n - 1$, which is the crossing number of the graph $H_5 \square P_n$. The graph $G_{30} \square P_n$ is a subgraph of $G_{36} \square P_n$ and therefore, $\text{cr}(G_{30} \square P_n) \leq \text{cr}(G_{36} \square P_n)$. In the drawing of the graph $G_{36} \square P_n$ in Figure 5(b) it is easy to see that $\text{cr}(G_{36} \square P_n) \leq 3n - 1$. Thus, $\text{cr}(G_{30} \square P_n) = \text{cr}(G_{36} \square P_n) = 3n - 1$.

Recently, some few results concerning crossing numbers of Cartesian products of paths with graphs on six vertices were obtained. For the graph $G_{33} = P(3, 1)$, Peng and Yiew proved that the Cartesian product $G_{33} \square P_n$ has crossing number $4n$, see [10]. The graph $G_{37}$ is isomorphic with the second power of the path of length five denoted by $P_5^2$. It was proved in [9] that $\text{cr}(P_5^2 \square P_n) = \text{cr}(G_{37} \square P_n) = 4(n - 1)$. For two other graphs, namely for $G_{38}$ and $G_{39}$, the crossing numbers are also known. In [12] one can find that $\text{cr}(G_{38} \square P_n) = 4n$ and $\text{cr}(G_{39} \square P_n) = 6n$. 

Figure 5. The graphs $G_{29} \square P_n$, $G_{30} \square P_n$, and $G_{31} \square P_n$. 

(a) (b) (c)
For the complete graph on six vertices, it is shown in [13] that the crossing number of its Cartesian product with the path \( P_n \) is \( 15n + 3 \). Thus, we have that \( \text{cr}(K_6 \square P_n) = \text{cr}(G_{40} \square P_n) = 15n + 3 \). The last known result one can find in [11].

### Table 1. The known crossing numbers of \( G_i \square P_n \) for graphs \( G_i \) on six vertices.

<table>
<thead>
<tr>
<th>( G_i )</th>
<th>( \text{cr}(G_i \square P_n) )</th>
<th>( G_i )</th>
<th>( \text{cr}(G_i \square P_n) )</th>
<th>( G_i )</th>
<th>( \text{cr}(G_i \square P_n) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( G_1 )</td>
<td>0</td>
<td>( G_{15} )</td>
<td>( 2(n - 1) )</td>
<td>( G_{29} )</td>
<td>( 2n )</td>
</tr>
<tr>
<td>( G_2 )</td>
<td>( n - 1 )</td>
<td>( G_{16} )</td>
<td>( 2(n - 1) )</td>
<td>( G_{30} )</td>
<td>( 3n - 1 )</td>
</tr>
<tr>
<td>( G_3 )</td>
<td>( n - 1 )</td>
<td>( G_{17} )</td>
<td>( 2(n - 1) )</td>
<td>( G_{31} )</td>
<td>( 2(n - 1) )</td>
</tr>
<tr>
<td>( G_4 )</td>
<td>( 2(n - 1) )</td>
<td>( G_{18} )</td>
<td>( n - 1 )</td>
<td>( G_{32} )</td>
<td>( 4n )</td>
</tr>
<tr>
<td>( G_5 )</td>
<td>( 2(n - 1) )</td>
<td>( G_{19} )</td>
<td>( 2(n - 1) )</td>
<td>( G_{33} )</td>
<td>( 4n )</td>
</tr>
<tr>
<td>( G_6 )</td>
<td>( 4(n - 1) )</td>
<td>( G_{20} )</td>
<td>( 2n )</td>
<td>( G_{34} )</td>
<td>( 4n )</td>
</tr>
<tr>
<td>( G_7 )</td>
<td>0</td>
<td>( G_{21} )</td>
<td>( 2(n - 1) )</td>
<td>( G_{35} )</td>
<td>( 4(n - 1) )</td>
</tr>
<tr>
<td>( G_8 )</td>
<td>( n - 1 )</td>
<td>( G_{22} )</td>
<td>( 4(n - 1) )</td>
<td>( G_{36} )</td>
<td>( 3n - 1 )</td>
</tr>
<tr>
<td>( G_9 )</td>
<td>( n - 1 )</td>
<td>( G_{23} )</td>
<td>( 2(n - 1) )</td>
<td>( G_{37} )</td>
<td>( 4(n - 1) )</td>
</tr>
<tr>
<td>( G_{10} )</td>
<td>( 2(n - 1) )</td>
<td>( G_{24} )</td>
<td>( 4(n - 1) )</td>
<td>( G_{38} )</td>
<td>( 4n )</td>
</tr>
<tr>
<td>( G_{11} )</td>
<td>( 2(n - 1) )</td>
<td>( G_{25} )</td>
<td>( 2(n - 1) )</td>
<td>( G_{39} )</td>
<td>( 6n )</td>
</tr>
<tr>
<td>( G_{12} )</td>
<td>( n - 1 )</td>
<td>( G_{26} )</td>
<td>( 4(n - 1) )</td>
<td>( G_{40} )</td>
<td>( 15n + 3 )</td>
</tr>
<tr>
<td>( G_{13} )</td>
<td>( 4(n - 1) )</td>
<td>( G_{27} )</td>
<td>( 2(n - 1) )</td>
<td>[</td>
<td>( G_{14} )</td>
</tr>
</tbody>
</table>
It is shown that the crossing number of the graph $G_{32} \square P_n = K_{2,4} \square P_n$ is $4n$. This result we use to establish the crossing number of the Cartesian product $G_{34} \square P_n$. The graph $G_{34}$ contains a subgraph $K_{2,4}$ and therefore, $\text{cr}(G_{34} \square P_n) \geq 4n$. On the other hand, in Figure 5(c) there is a drawing of the graph $G_{34} \square P_n$ with $4n$ crossings. This confirms that $\text{cr}(G_{34} \square P_n) = 4n$. All known results concerning crossing numbers of Cartesian products of paths with graphs on six vertices are collected in Table 1.

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