

## THE UP-ISOMORPHISM THEOREMS FOR UP-ALGEBRAS<sup>1</sup>

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### Abstract

In this paper, we construct the fundamental theorem of UP-homomorphisms in UP-algebras. We also give an application of the theorem to the first, second, third and fourth UP-isomorphism theorems in UP-algebras.

**Keywords:** UP-algebra, fundamental theorem of UP-homomorphisms, first, second, third and fourth UP-isomorphism theorems.

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### 1. INTRODUCTION AND PRELIMINARIES

Among many algebraic structures, algebras of logic form an important class of algebras. Examples of these are BCK-algebras [6], BCI-algebras [7], BCH-algebras [3], KU-algebras [14], SU-algebras [9] and others. They are strongly connected with logic. For example, BCI-algebras introduced by Iséki [7] in 1966 have connections with BCI-logic being the BCI-system in combinatory logic which has application in the language of functional programming. BCK and BCI-algebras are two classes of logical algebras. They were introduced by Imai and Iséki [6, 7] in 1966 and have been extensively investigated by many researchers. It is known that the class of BCK-algebras is a proper subclass of the class of BCI-algebras.

The isomorphism theorems play an important role in a general logical algebra, which were studied by several researches such as: In 1998, Jun, Hong, Xin and Roh [8] proved isomorphism theorems by using Chinese Remainder Theorem in BCI-algebras. In 2001, Park, Shim and Roh [13] proved isomorphism theorems of IS-algebras. In 2004, Hao and Li [2] introduced the concept of ideals of

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an ideal in a BCI-algebra and some isomorphism theorems are obtained by using this concept. They obtained several isomorphism theorems of BG-algebras and related properties. In 2006, Kim [11] introduced the notion of KS-semigroups. He characterized ideals of a KS-semigroup and proved the first isomorphism theorem for KS-semigroups. In 2008, Kim and Kim [10] introduced the notion of BG-algebras which is a generalization of B-algebras. They obtained several isomorphism theorems of BG-algebras and related properties. In 2009, Paradero-Vilela and Cawi [12] characterized KS-semigroup homomorphisms and proved the isomorphism theorems for KS-semigroups. In 2011, Keawrahan and Leerawat [9] introduced the notion of SU-semigroups and proved the isomorphism theorems for SU-semigroups. In 2012, Asawasamrit [1] introduced the notion of KK-algebras and studied isomorphism theorems of KK-algebras.

Iampan [4] introduced a new algebraic structure, called a UP-algebra and a concept of UP-ideals, congruences and UP-homomorphisms in UP-algebras, and defined a congruence relation on a UP-algebra and a quotient UP-algebra. In this paper, we construct the fundamental theorem of UP-homomorphisms in UP-algebras. We also give an application of the theorem to the first, second, third and fourth UP-isomorphism theorems in UP-algebras.

Before we begin our study, we will introduce to the definition of a UP-algebra.

**Definition 1.1** [4]. An algebra  $A = (A, \cdot, 0)$  of type  $(2, 0)$  is called a *UP-algebra*, where  $A$  is a nonempty set,  $\cdot$  is a binary operation on  $A$ , and  $0$  is a fixed element of  $A$  (i.e., a nullary operation) if it satisfies the following assertions:

$$\text{(UP-1)} \quad (\forall x, y, z \in A)((y \cdot z) \cdot ((x \cdot y) \cdot (x \cdot z)) = 0),$$

$$\text{(UP-2)} \quad (\forall x \in A)(0 \cdot x = x),$$

$$\text{(UP-3)} \quad (\forall x \in A)(x \cdot 0 = 0), \text{ and}$$

$$\text{(UP-4)} \quad (\forall x, y \in A)(x \cdot y = 0, y \cdot x = 0 \Rightarrow x = y).$$

From [4], we know that a UP-algebra is a generalization of the concept of a KU-algebra.

**Example 1.2** [15]. Let  $X$  be a universal set and let  $\Omega \in \mathcal{P}(X)$  where  $\mathcal{P}(X)$  means the power set of  $X$ . Let  $\mathcal{P}_\Omega(X) = \{A \in \mathcal{P}(X) \mid \Omega \subseteq A\}$ . Define a binary operation  $\cdot$  on  $\mathcal{P}_\Omega(X)$  by putting  $A \cdot B = B \cap (A^C \cup \Omega)$  for all  $A, B \in \mathcal{P}_\Omega(X)$  where  $A^C$  means the complement of a subset  $A$ . Then  $(\mathcal{P}_\Omega(X), \cdot, \Omega)$  is a UP-algebra and we shall call it the *generalized power UP-algebra of type 1 with respect to  $\Omega$* . Let  $\mathcal{P}^\Omega(X) = \{A \in \mathcal{P}(X) \mid A \subseteq \Omega\}$ . Define a binary operation  $*$  on  $\mathcal{P}^\Omega(X)$  by putting  $A * B = B \cup (A^C \cap \Omega)$  for all  $A, B \in \mathcal{P}^\Omega(X)$ . Then  $(\mathcal{P}^\Omega(X), *, \Omega)$  is a UP-algebra and we shall call it the *generalized power UP-algebra of type 2 with respect to  $\Omega$* . In particular,  $(\mathcal{P}(X), \cdot, \emptyset)$  is a UP-algebra and we shall call it the

power UP-algebra of type 1, and  $(\mathcal{P}(X), *, X)$  is a UP-algebra and we shall call it the power UP-algebra of type 2.

**Example 1.3** [4]. Let  $A = \{0, 1, 2, 3\}$  be a set with a binary operation  $\cdot$  defined by the following Cayley table:

(1.1)	$\cdot$	0	1	2	3
	0	0	1	2	3
	1	0	0	0	0
	2	0	1	0	3
	3	0	1	2	0

Then  $(A, \cdot, 0)$  is a UP-algebra.

In a UP-algebra  $A = (A, \cdot, 0)$ , the following assertions are valid (see [4, 5]).

- (1.2)  $(\forall x \in A)(x \cdot x = 0)$ ,
- (1.3)  $(\forall x, y, z \in A)(x \cdot y = 0, y \cdot z = 0 \Rightarrow x \cdot z = 0)$ ,
- (1.4)  $(\forall x, y, z \in A)(x \cdot y = 0 \Rightarrow (z \cdot x) \cdot (z \cdot y) = 0)$ ,
- (1.5)  $(\forall x, y, z \in A)(x \cdot y = 0 \Rightarrow (y \cdot z) \cdot (x \cdot z) = 0)$ ,
- (1.6)  $(\forall x, y \in A)(x \cdot (y \cdot x) = 0)$ ,
- (1.7)  $(\forall x, y \in A)((y \cdot x) \cdot x = 0 \Leftrightarrow x = y \cdot x)$ ,
- (1.8)  $(\forall x, y \in A)(x \cdot (y \cdot y) = 0)$ ,
- (1.9)  $(\forall a, x, y, z \in A)((x \cdot (y \cdot z)) \cdot (x \cdot ((a \cdot y) \cdot (a \cdot z))) = 0)$ ,
- (1.10)  $(\forall a, x, y, z \in A)((((a \cdot x) \cdot (a \cdot y)) \cdot z) \cdot ((x \cdot y) \cdot z) = 0)$ ,
- (1.11)  $(\forall x, y, z \in A)((x \cdot y) \cdot z \cdot (y \cdot z) = 0)$ ,
- (1.12)  $(\forall x, y, z \in A)(x \cdot y = 0 \Rightarrow x \cdot (z \cdot y) = 0)$ ,
- (1.13)  $(\forall x, y, z \in A)((x \cdot y) \cdot z \cdot (x \cdot (y \cdot z)) = 0)$ , and
- (1.14)  $(\forall a, x, y, z \in A)((x \cdot y) \cdot z \cdot (y \cdot (a \cdot z)) = 0)$ .

**Definition 1.4** [4]. Let  $A$  be a UP-algebra. A nonempty subset  $B$  of  $A$  is called a UP-ideal of  $A$  if it satisfies the following properties:

- (1) the constant 0 of  $A$  is in  $B$ , and
- (2)  $(\forall x, y, z \in A)(x \cdot (y \cdot z) \in B, y \in B \Rightarrow x \cdot z \in B)$ .

Clearly,  $A$  and  $\{0\}$  are UP-ideals of  $A$ .

**Example 1.5** [4]. Let  $A = \{0, 1, 2, 3, 4\}$  be a set with a binary operation  $\cdot$  defined by the following Cayley table:

$$(1.15) \quad \begin{array}{c|ccccc} \cdot & 0 & 1 & 2 & 3 & 4 \\ \hline 0 & 0 & 1 & 2 & 3 & 4 \\ 1 & 0 & 0 & 2 & 3 & 4 \\ 2 & 0 & 0 & 0 & 3 & 4 \\ 3 & 0 & 0 & 2 & 0 & 4 \\ 4 & 0 & 0 & 0 & 0 & 0 \end{array}$$

Then  $(A, \cdot, 0)$  is a UP-algebra and  $\{0, 1, 2\}$  and  $\{0, 1, 3\}$  are UP-ideals of  $A$ .

**Theorem 1.6** [4]. Let  $A$  be a UP-algebra and  $B$  a UP-ideal of  $A$ . Then the following statements hold: for any  $x, a, b \in A$ ,

- (1) if  $b \cdot x \in B$  and  $b \in B$ , then  $x \in B$ . Moreover, if  $b \cdot X \subseteq B$  and  $b \in B$ , then  $X \subseteq B$ ,
- (2) if  $b \in B$ , then  $x \cdot b \in B$ . Moreover, if  $b \in B$ , then  $X \cdot b \subseteq B$ , and
- (3) if  $a, b \in B$ , then  $(b \cdot (a \cdot x)) \cdot x \in B$ .

**Definition 1.7** [4]. Let  $A = (A, \cdot, 0)$  be a UP-algebra. A subset  $S$  of  $A$  is called a UP-subalgebra of  $A$  if the constant  $0$  of  $A$  is in  $S$ , and  $(S, \cdot, 0)$  itself forms a UP-algebra. Clearly,  $A$  and  $\{0\}$  are UP-subalgebras of  $A$ .

**Proposition 1.8** [4]. A nonempty subset  $S$  of a UP-algebra  $A = (A, \cdot, 0)$  is a UP-subalgebra of  $A$  if and only if  $S$  is closed under the  $\cdot$  multiplication on  $A$ .

**Definition 1.9** [4]. Let  $A$  be a UP-algebra and  $B$  a UP-ideal of  $A$ . Define the binary relation  $\sim_B$  on  $A$  as follows:

$$(1.16) \quad (\forall x, y \in A)(x \sim_B y \Leftrightarrow x \cdot y \in B, y \cdot x \in B).$$

**Definition 1.10** [4]. Let  $A$  be a UP-algebra. An equivalence relation  $\rho$  on  $A$  is called a congruence if

$$(\forall x, y, z \in A)(x \rho y \Rightarrow x \cdot z \rho y \cdot z, z \cdot x \rho z \cdot y).$$

**Proposition 1.11** [4]. Let  $A$  be a UP-algebra and  $B$  a UP-ideal of  $A$  with a binary relation  $\sim_B$  defined by (1.16). Then  $\sim_B$  is a congruence on  $A$ .

Let  $A$  be a UP-algebra and  $\rho$  a congruence on  $A$ . If  $x \in A$ , then the  $\rho$ -class of  $x$  is the  $(x)_\rho$  defined as follows:

$$(x)_\rho = \{y \in A \mid y \rho x\}.$$

Then the set of all  $\rho$ -classes is called the *quotient set of  $A$  by  $\rho$* , and is denoted by  $A/\rho$ . That is,

$$A/\rho = \{(x)_\rho \mid x \in A\}.$$

**Theorem 1.12** [4]. *Let  $A$  be a UP-algebra and  $B$  a UP-ideal of  $A$ . Then the following statements hold:*

- (1) *the  $\sim_B$ -class  $(0)_{\sim_B}$  is a UP-ideal and a UP-subalgebra of  $A$  which  $B = (0)_{\sim_B}$ ,*
- (2) *a  $\sim_B$ -class  $(x)_{\sim_B}$  is a UP-ideal of  $A$  if and only if  $x \in B$ ,*
- (3) *a  $\sim_B$ -class  $(x)_{\sim_B}$  is a UP-subalgebra of  $A$  if and only if  $x \in B$ , and*
- (4)  *$(A/\sim_B, *, (0)_{\sim_B})$  is a UP-algebra under the  $*$  multiplication defined by  $(x)_{\sim_B} * (y)_{\sim_B} = (x \cdot y)_{\sim_B}$  for all  $x, y \in A$ , called the quotient UP-algebra of  $A$  induced by the congruence  $\sim_B$ .*

**Definition 1.13** [4]. Let  $(A, \cdot, 0)$  and  $(A', \cdot', 0')$  be UP-algebras. A mapping  $f$  from  $A$  to  $A'$  is called a *UP-homomorphism* if

$$(\forall x, y \in A)(f(x \cdot y) = f(x) \cdot' f(y)).$$

A UP-homomorphism  $f: A \rightarrow A'$  is called a

- (1) *UP-epimorphism* if  $f$  is surjective,
- (2) *UP-monomorphism* if  $f$  is injective,
- (3) *UP-isomorphism* if  $f$  is bijective. Moreover, we say  $A$  is *UP-isomorphic* to  $A'$ , symbolically,  $A \cong A'$ , if there is a UP-isomorphism from  $A$  to  $A'$ .

Let  $f$  be a mapping from  $A$  to  $A'$ , and let  $B$  be a nonempty subset of  $A$ , and  $B'$  of  $A'$ . The set  $\{f(x) \mid x \in B\}$  is called the *image* of  $B$  under  $f$ , denoted by  $f(B)$ . In particular,  $f(A)$  is called the *image* of  $f$ , denoted by  $\text{Im}(f)$ . Dually, the set  $\{x \in A \mid f(x) \in B'\}$  is said the *inverse image* of  $B'$  under  $f$ , symbolically,  $f^{-1}(B')$ . Especially, we say  $f^{-1}(\{0'\})$  is the *kernel* of  $f$ , written by  $\text{Ker}(f)$ . That is,

$$\text{Im}(f) = \{f(x) \in A' \mid x \in A\}$$

and

$$\text{Ker}(f) = \{x \in A \mid f(x) = 0'\}.$$

**Theorem 1.14** [4]. *Let  $A$  be a UP-algebra and  $B$  a UP-ideal of  $A$ . Then the mapping  $\pi_B: A \rightarrow A/\sim_B$  defined by  $\pi_B(x) = (x)_{\sim_B}$  for all  $x \in A$  is a UP-epimorphism, called the natural projection from  $A$  to  $A/\sim_B$ .*

On a UP-algebra  $A = (A, \cdot, 0)$ , we define a binary relation  $\leq$  on  $A$  as follows:

$$(1.17) \quad (\forall x, y \in A)(x \leq y \Leftrightarrow x \cdot y = 0).$$

**Theorem 1.15** [4]. *Let  $(A, \cdot, 0_A)$  and  $(B, *, 0_B)$  be UP-algebras and let  $f: A \rightarrow B$  be a UP-homomorphism. Then the following statements hold:*

- (1)  $f(0_A) = 0_B$ ,
- (2) for any  $x, y \in A$ , if  $x \leq y$ , then  $f(x) \leq f(y)$ ,
- (3) if  $C$  is a UP-subalgebra of  $A$ , then the image  $f(C)$  is a UP-subalgebra of  $B$ . In particular,  $\text{Im}(f)$  is a UP-subalgebra of  $B$ ,
- (4) if  $D$  is a UP-subalgebra of  $B$ , then the inverse image  $f^{-1}(D)$  is a UP-subalgebra of  $A$ . In particular,  $\text{Ker}(f)$  is a UP-subalgebra of  $A$ ,
- (5) if  $C$  is a UP-ideal of  $A$  such that  $\text{Ker}(f) \subseteq C$ , then the image  $f(C)$  is a UP-ideal of  $f(A)$ ,
- (6) if  $D$  is a UP-ideal of  $B$ , then the inverse image  $f^{-1}(D)$  is a UP-ideal of  $A$ . In particular,  $\text{Ker}(f)$  is a UP-ideal of  $A$ , and
- (7)  $\text{Ker}(f) = \{0_A\}$  if and only if  $f$  is injective.

## 2. MAIN RESULTS

In this section, we construct the fundamental theorem of UP-homomorphisms in UP-algebras. We also give an application of the theorem to the first, second, third and fourth UP-isomorphism theorems in UP-algebras.

**Theorem 2.1** (Fundamental Theorem of UP-homomorphisms). *Let  $(A, \cdot, 0_A)$  and  $(B, \bullet, 0_B)$  be UP-algebras, and  $f: A \rightarrow B$  a UP-homomorphism. Then there exists uniquely a UP-homomorphism  $\varphi$  from  $A / \sim_{\text{Ker}(f)}$  to  $B$  such that  $f = \varphi \circ \pi_{\text{Ker}(f)}$ . Moreover,*

- (1)  $\pi_{\text{Ker}(f)}$  is a UP-epimorphism and  $\varphi$  a UP-monomorphism, and
- (2)  $f$  is a UP-epimorphism if and only if  $\varphi$  is a UP-isomorphism.

As  $f$  makes the following diagram commute,

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ \pi_{\text{Ker}(f)} \downarrow & \nearrow \varphi & \\ A / \sim_{\text{Ker}(f)} & & \end{array}$$

**Proof.** Put  $K = \text{Ker}(f)$ . By Theorem 1.15 (6), we have  $K$  is a UP-ideal of  $A$ . It follows from Theorem 1.12 (4) that  $(A/\sim_K, *, (0_A)_{\sim_K})$  is a UP-algebra. Define

$$(2.1) \quad \varphi: A/\sim_K \rightarrow B, (x)_{\sim_K} \mapsto f(x).$$

Let  $(x)_{\sim_K}, (y)_{\sim_K} \in A/\sim_K$  be such that  $(x)_{\sim_K} = (y)_{\sim_K}$ . Then  $x \sim_K y$ , so  $x \cdot y \in K$  and  $y \cdot x \in K$ . Thus

$$f(x) \bullet f(y) = f(x \cdot y) = 0_B \text{ and } f(y) \bullet f(x) = f(y \cdot x) = 0_B.$$

By (UP-4), we have  $f(x) = f(y)$  and so  $\varphi((x)_{\sim_K}) = \varphi((y)_{\sim_K})$ . Thus  $\varphi$  is a mapping. For any  $x, y \in A$ , we see that

$$\varphi((x)_{\sim_K} * (y)_{\sim_K}) = \varphi((x \cdot y)_{\sim_K}) = f(x \cdot y) = f(x) \bullet f(y) = \varphi((x)_{\sim_K}) \bullet \varphi((y)_{\sim_K}).$$

Thus  $\varphi$  is a UP-homomorphism. Also, since

$$(\varphi \circ \pi_K)(x) = \varphi(\pi_K(x)) = \varphi((x)_{\sim_K}) = f(x) \text{ for all } x \in A,$$

we obtain  $f = \varphi \circ \pi_K$ . We have shown the existence. Let  $\varphi'$  be a mapping from  $A/\sim_K$  to  $B$  such that  $f = \varphi' \circ \pi_K$ . Then for any  $(x)_{\sim_K} \in A/\sim_K$ , we have

$$\varphi'((x)_{\sim_K}) = \varphi'(\pi_K(x)) = (\varphi' \circ \pi_K)(x) = f(x) = (\varphi \circ \pi_K)(x) = \varphi(\pi_K(x)) = \varphi((x)_{\sim_K}).$$

Hence,  $\varphi = \varphi'$ , showing the uniqueness.

(1) By Theorem 1.14, we have  $\pi_K$  is a UP-epimorphism. Also, let  $(x)_{\sim_K}, (y)_{\sim_K} \in A/\sim_K$  be such that  $\varphi((x)_{\sim_K}) = \varphi((y)_{\sim_K})$ . Then  $f(x) = f(y)$ , and it follows from (1.2) that

$$f(x \cdot y) = f(x) \bullet f(y) = f(y) \bullet f(x) = 0_B,$$

that is,  $x \cdot y \in K$ . Similarly,  $y \cdot x \in K$ . Hence,  $x \sim_K y$  and  $(x)_{\sim_K} = (y)_{\sim_K}$ . Therefore,  $\varphi$  is a UP-monomorphism.

(2) Assume that  $f$  is a UP-epimorphism. By (1), it suffices to prove  $\varphi$  is surjective. Let  $y \in B$ . Then there exists  $x \in A$  such that  $f(x) = y$ . Thus  $y = f(x) = \varphi((x)_{\sim_K})$ , so  $\varphi$  is surjective. Hence,  $\varphi$  is a UP-isomorphism.

Conversely, assume that  $\varphi$  is a UP-isomorphism. Then  $\varphi$  is surjective. Let  $y \in B$ . Then there exists  $(x)_{\sim_K} \in A/\sim_K$  such that  $\varphi((x)_{\sim_K}) = y$ . Thus  $f(x) = \varphi((x)_{\sim_K}) = y$ , so  $f$  is surjective. Hence,  $f$  is a UP-epimorphism.  $\blacksquare$

**Theorem 2.2** (First UP-isomorphism Theorem). *Let  $(A, \cdot, 0_A)$  and  $(B, \bullet, 0_B)$  be UP-algebras, and  $f: A \rightarrow B$  a UP-homomorphism. Then*

$$A/\sim_{\text{Ker}(f)} \cong \text{Im}(f).$$

**Proof.** By Theorem 1.15 (3), we have  $\text{Im}(f)$  is a UP-subalgebra of  $B$ . Thus  $f: A \rightarrow \text{Im}(f)$  is a UP-epimorphism. Applying Theorem 2.1 (2), we obtain  $A/\sim_{\text{Ker}(f)} \cong \text{Im}(f)$ . ■

**Lemma 2.3.** *Let  $(A, \cdot, 0)$  be a UP-algebra,  $H$  a UP-subalgebra of  $A$ , and  $K$  a UP-ideal of  $A$ . Denote  $HK = \bigcup_{h \in H} (h)_{\sim_K}$ . Then  $HK$  is a UP-subalgebra of  $A$ .*

**Proof.** Clearly,  $\emptyset \neq HK \subseteq A$ . Let  $a, b \in HK$ . Then  $a \in (x)_{\sim_K}$  and  $b \in (y)_{\sim_K}$  for some  $x, y \in H$ , so  $(a)_{\sim_K} = (x)_{\sim_K}$  and  $(b)_{\sim_K} = (y)_{\sim_K}$ . Thus

$$(a \cdot b)_{\sim_K} = (a)_{\sim_K} * (b)_{\sim_K} = (x)_{\sim_K} * (y)_{\sim_K} = (x \cdot y)_{\sim_K},$$

so  $a \cdot b \in (x \cdot y)_{\sim_K}$ . Since  $x, y \in H$ , it follows from Proposition 1.8 that  $x \cdot y \in H$ . Thus  $a \cdot b \in (x \cdot y)_{\sim_K} \subseteq HK$ . Hence,  $HK$  is a UP-subalgebra of  $A$ . ■

**Theorem 2.4** (Second UP-isomorphism Theorem). *Let  $(A, \cdot, 0)$  be a UP-algebra,  $H$  a UP-subalgebra of  $A$ , and  $K$  a UP-ideal of  $A$ . Denote  $HK/\sim_K = \{(x)_{\sim_K} \mid x \in HK\}$ . Then*

$$H/\sim_{H \cap K} \cong HK/\sim_K.$$

**Proof.** By Lemma 2.3, we have  $HK$  is a UP-subalgebra of  $A$ . Then it is easy to check that  $HK/\sim_K$  is a UP-subalgebra of  $A/\sim_K$ , thus  $(HK/\sim_K, *, (0)_{\sim_K})$  itself is a UP-algebra. Also, it is obvious that  $H \subseteq HK$ , then

$$(2.2) \quad f: H \rightarrow HK/\sim_K, x \mapsto (x)_{\sim_K},$$

is a mapping. For any  $x, y \in H$ , we have

$$f(x \cdot y) = (x \cdot y)_{\sim_K} = (x)_{\sim_K} * (y)_{\sim_K} = f(x) * f(y).$$

Thus  $f$  is a UP-homomorphism. We shall show that  $f$  is a UP-epimorphism with  $\text{Ker}(f) = H \cap K$ . For any  $(x)_{\sim_K} \in HK/\sim_K$ , we have  $x \in HK = \bigcup_{h \in H} (h)_{\sim_K}$ . Then there exists  $h \in H$  such that  $x \in (h)_{\sim_K}$  and so  $(x)_{\sim_K} = (h)_{\sim_K}$ . Thus  $f(h) = (h)_{\sim_K} = (x)_{\sim_K}$ . Therefore,  $f$  is a UP-epimorphism. Also, for any  $h \in H$ , if  $h \in \text{Ker}(f)$ , then  $f(h) = (0)_{\sim_K}$ . Since  $f(h) = (h)_{\sim_K}$ , we obtain  $(h)_{\sim_K} = (0)_{\sim_K}$ . By (UP-2) and (1.16), we have  $h = 0 \cdot h \in K$ . Thus  $h \in H \cap K$ , that is,  $\text{Ker}(f) \subseteq H \cap K$ . On the other hand, if  $h \in H \cap K$ , by  $h \in H$ ,  $f(h)$  is well-defined, by  $h \in K$  and  $0 \in K$ ,  $h \cdot 0 \in K$  and  $0 \cdot h \in K$ . By (1.16), we have  $h \sim_K 0$  and so  $(h)_{\sim_K} = (0)_{\sim_K}$ . Thus  $f(h) = (h)_{\sim_K} = (0)_{\sim_K}$ . So,  $h \in \text{Ker}(f)$ , that is,  $H \cap K \subseteq \text{Ker}(f)$ . Therefore,  $\text{Ker}(f) = H \cap K$ . Now, Theorem 2.2 gives  $H/\sim_{H \cap K} \cong HK/\sim_K$ . ■

**Theorem 2.5** (Third UP-isomorphism Theorem). *Let  $(A, \cdot, 0)$  be a UP-algebra, and  $H$  and  $K$  UP-ideals of  $A$  with  $H \subseteq K$ . Then*

$$(A/\sim_H)/\sim_{(K/\sim_H)} \cong A/\sim_K.$$



**Proof.** By Theorem 1.12 (4), we obtain  $(A/\sim_K, *, (0)_{\sim_K})$  and  $(A/\sim_H, *', (0)_{\sim_H})$  are UP-algebras. Define

$$(2.3) \quad f: A/\sim_H \rightarrow A/\sim_K, (x)_{\sim_H} \mapsto (x)_{\sim_K}.$$

For any  $x, y \in A$ , if  $(x)_{\sim_H} = (y)_{\sim_H}$ , then  $x \cdot y, y \cdot x \in H$ . Since  $H \subseteq K$ , we obtain  $x \cdot y, y \cdot x \in K$ . Thus  $(x)_{\sim_K} = (y)_{\sim_K}$ , so  $f((x)_{\sim_H}) = f((y)_{\sim_H})$ . Thus  $f$  is a mapping. Also, for any  $x, y \in A$ , we see that

$$f((x)_{\sim_H} *' (y)_{\sim_H}) = f((x \cdot y)_{\sim_H}) = (x \cdot y)_{\sim_K} = (x)_{\sim_K} * (y)_{\sim_K} = f((x)_{\sim_H}) * f((y)_{\sim_H}).$$

Thus  $f$  is a UP-homomorphism. Clearly,  $f$  is surjective. Hence,  $f$  is a UP-epimorphism. We shall show that  $\text{Ker}(f) = K/\sim_H$ . In fact,

$$\begin{aligned} \text{Ker}(f) &= \{(x)_{\sim_H} \in A/\sim_H \mid f((x)_{\sim_H}) = (0)_{\sim_K}\} \\ &= \{(x)_{\sim_H} \in A/\sim_H \mid (x)_{\sim_K} = (0)_{\sim_K}\} \\ ((\text{UP-2})) \quad &= \{(x)_{\sim_H} \in A/\sim_H \mid x = 0 \cdot x \in K\} \\ &= K/\sim_H. \end{aligned}$$

Now, Theorem 2.2 gives  $(A/\sim_H)/\sim_{(K/\sim_H)} \cong A/\sim_K$ . ■

**Theorem 2.6** (Fourth UP-isomorphism Theorem). *Let  $(A, \cdot, 0_A)$  and  $(B, \bullet, 0_B)$  be UP-algebras, and  $f: A \rightarrow B$  a UP-epimorphism. Denote  $\mathcal{A} = \{X \mid X \text{ is a UP-ideal of } A \text{ containing } \text{Ker}(f)\}$  and  $\mathcal{B} = \{Y \mid Y \text{ is a UP-ideal of } B\}$ . Then the following statements hold:*

(1) *there is an inclusion preserving bijection*

$$(2.4) \quad \varphi: \mathcal{A} \rightarrow \mathcal{B}, X \mapsto f(X),$$

*with inverse given by  $Y \mapsto f^{-1}(Y)$ , and*

(2) *for any  $X \in \mathcal{A}$ ,*

$$A/\sim_X \cong B/\sim_{f(X)}.$$

**Proof.** (1) For any  $X \in \mathcal{A}$ , it follows from Theorem 1.15 (5) that  $f(X)$  is a unique UP-ideal of  $B$  such that  $\varphi(X) = f(X)$ . Thus  $\varphi$  is a mapping. For any  $X_1, X_2 \in \mathcal{A}$ , if  $\varphi(X_1) = \varphi(X_2)$ , then  $f(X_1) = f(X_2)$ . Since  $\text{Ker}(f) \subseteq X_1$ , we obtain  $X_1 = f^{-1}(f(X_1))$ . Indeed, let  $x \in f^{-1}(f(X_1))$ . Then  $f(x) \in f(X_1)$ , so  $f(x) = f(x_1)$  for some  $x_1 \in X_1$ . Applying (1.2), we have  $f(x_1 \cdot x) = f(x_1) \bullet f(x) = f(x_1) \bullet f(x_1) = 0_B$ . Thus  $x_1 \cdot x \in \text{Ker}(f) \subseteq X_1$ , it follows from Theorem 1.6 (1) that  $x \in X_1$ . So,  $f^{-1}(f(X_1)) \subseteq X_1$ . Clearly,  $X_1 \subseteq f^{-1}(f(X_1))$ . Similarly, since  $\text{Ker}(f) \subseteq X_2$ , we obtain  $X_2 = f^{-1}(f(X_2))$ . Thus  $X_1 = f^{-1}(f(X_1)) = f^{-1}(f(X_2)) = X_2$ . Hence,  $\varphi$  is injective. Also, for any  $Y \in \mathcal{B}$ , we obtain

$Y = f(f^{-1}(Y))$  because  $f$  is surjective. Applying Theorem 1.15 (6), we have  $f^{-1}(Y)$  is a UP-ideal of  $A$  with  $\text{Ker}(f) \subseteq f^{-1}(Y)$ . Thus  $f^{-1}(Y) \in \mathcal{A}$  is such that  $\varphi(f^{-1}(Y)) = f(f^{-1}(Y)) = Y$ . Hence,  $\varphi$  is surjective. Therefore,  $\varphi$  is bijective. Finally, for any  $Y \in \mathcal{B}$ , we get that  $Y = \varphi(f^{-1}(Y))$ . Hence,  $\varphi^{-1}(Y) = f^{-1}(Y)$ .

(2) By Theorem 1.15 (5) and Theorem 1.12 (4), we have  $f(X)$  is a UP-ideal of  $B$  and  $(B/\sim_{f(X)}, *, (0_B)_{\sim_{f(X)}})$  is a UP-algebra. It follows from Theorem 1.14 that  $\pi_{f(X)}: B \rightarrow B/\sim_{f(X)}$  is a UP-epimorphism. Thus  $\pi_{f(X)} \circ f: A \rightarrow B/\sim_{f(X)}$  is a UP-epimorphism. We shall show that  $\text{Ker}(\pi_{f(X)} \circ f) = X$ . In fact,

$$\begin{aligned}
 \text{Ker}(\pi_{f(X)} \circ f) &= \{a \in A \mid (\pi_{f(X)} \circ f)(a) = (0_B)_{\sim_{f(X)}}\} \\
 &= \{a \in A \mid \pi_{f(X)}(f(a)) = (0_B)_{\sim_{f(X)}}\} \\
 &= \{a \in A \mid (f(a))_{\sim_{f(X)}} = (0_B)_{\sim_{f(X)}}\} \\
 ((\text{UP-2})) \quad &= \{a \in A \mid f(a) = 0_B \bullet f(a) \in f(X)\} \\
 &= f^{-1}(f(X)) \\
 &= X.
 \end{aligned}$$

Applying Theorem 2.2, we have  $A/\sim_X \cong B/\sim_{f(X)}$ . ■

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