LOCAL COHOMOLOGY MODULES AND RELATIVE
COHEN-MACULAYNESS

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Abstract

Let \((R, \mathfrak{m})\) denote a commutative Noetherian local ring and let \(M\) be a finite \(R\)-module. In this paper, we study relative Cohen-Macaulay rings with respect to a proper ideal \(\mathfrak{a}\) of \(R\) and give some results on such rings in relation with Artinianness, Non-Artinianness of local cohomology modules and Lyubeznik numbers. We also present some related examples to this issue.

Keywords: local cohomology modules, Lyubeznik numbers, Non-Artinian modules, relative Cohen-Macaulayness.

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1. Introduction

Throughout this paper, we assume that \((R, \mathfrak{m})\) is a commutative Noetherian local ring with maximal ideal \(\mathfrak{m}\) and \(\mathfrak{a}\) an ideal of \(R\). For any non-zero \(R\)-module \(M\), the \(i\)th local cohomology module of \(M\) is defined as

\[
H^i_{\mathfrak{a}}(M) := \lim_{\to n \geq 1} \text{Ext}^i_R(R/\mathfrak{a}^n, M).
\]

\(V(\mathfrak{a})\) denotes the set of all prime ideals of \(R\) containing \(\mathfrak{a}\). For an \(R\)-module \(M\), the cohomological dimension of \(M\) with respect to \(\mathfrak{a}\) is defined as \(\text{cd}(\mathfrak{a}, M) := \sup\{i \in \mathbb{Z} \mid H^i_{\mathfrak{a}}(M) \neq 0\}\) which is known that for a local ring \((R, \mathfrak{m})\) and \(\mathfrak{a} = \mathfrak{m}\), this is equal to dimension of \(M\). In [14], an \(R\)-module \(M\) is called relative Cohen-Macaulay w.r.t \(\mathfrak{a}\) if there is precisely one non-vanishing local cohomology module
of $M$ w.r.t $a$, i.e., $\text{grade}(a, M) = \text{cd}(a, M)$. Recently, in [10], we have studied such modules. In the present paper, we will use this concept and derive some new results about local cohomology modules. It is well known that $H_{a}^{\dim M}(M)$ is an Artinian module. Artinianness and Non-Artinianness of local cohomology modules has been studied by many authors such as [1, 3], and [6]. As the first main result we prove that if $M$ is a finite relative Cohen-Macaulay local ring w.r.t $a$ with height $R_a = h$, then $\dim \text{Supp}_R H_{a}^{h}(M) = \dim M/aM$ (Proposition 2.1).

Proposition 2.1 opens the door for some interesting examples and corollaries. Consequently, if $(R, m)$ is a relative Cohen-Macaulay local ring w.r.t $a$ with height $R_a = h$ and $\dim R/a > 0$, then the local cohomology module $H_{a}^{h}(R)$ is not Artinian (see Corollary 2.6). This gives us two interesting results. As the first one, by assumptions of Corollary 2.6, we show that the inequality $f - \text{depth}(a, M) \leq \text{height}_M a$ proved in [12, Proposition 3.5] becomes an equality for the “ring” case, where $f - \text{depth}(a, M)$ is defined as the least integer $i$ such that $H_{a}^{i}(M)$ is not Artinian. We show that if $(R, m)$ is a relative Cohen-Macaulay local ring w.r.t $a$ and $\dim R/a > 0$, then $f - \text{depth}(a, R) = \text{height}_R a$ (see Corollary 2.8). As an another consequence of Corollary 2.6, we get the equality $f_a(R) = f - \text{depth}(a, R)$ (see Corollary 2.10), where the notion finiteness dimension of $M$ relative to $a$, $f_a(M)$, is defined by

$$f_a(M) := \inf \{ i \in \mathbb{N}_0 : H_{a}^{i}(M) \text{ is not finitely generated} \}.$$  

By convention, the infimum of the empty set of integers is interpreted by $\infty$.

Now, assume that $R$ is a local ring which admits a surjection from an $n$-dimensional regular local ring $S$ containing a field, $a$ be the kernel of surjection and $k = S/m$. Lyubeznik numbers defined in [11] as the Bass numbers $\lambda_{i,j}(R) = \dim_k \text{Ext}_S^j(k, H_{a}^{n-j}(S))$ depend only on $R$, $i$ and $j$ but neither on $S$ nor on the surjection $S \to R$. Lyubeznik numbers carry some topological and geometrical information and all are finite. For more applications of such invariants we refer the reader to [11]. We present the following result on Lyubeznik numbers.

If $(R, m, k)$ is a regular local ring containing a field which is relative Cohen-Macaulay w.r.t $a$, then the Lyubeznik table of $R/a$ is trivial as follows:

$$\Lambda(R/a) = \begin{pmatrix} 0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 1 \end{pmatrix}$$

that is, $\lambda_{i,j}(R/a) = 1$ whenever $i = j = \dim R/a$ and otherwise $\lambda_{i,j}(R/a) = 0$ (Proposition 2.11).

In the process, in Proposition 2.15 we show that $H_{a}^{\text{cd}(a,R)}(R)$ is indecomposable, where $(R, m)$ is relative Cohen-Macaulay local ring w.r.t $a$ and $\text{Supp}_R(R/a) \subseteq V(m)$.
The notion of generalized local cohomology of two $R$-modules on a local ring $(R, m)$ introduced by Herzog in [8]. For each $i \in \mathbb{N}_0$, the $i$th generalized local cohomology module $H_i^a(M, N)$ of two $R$-modules $M$ and $N$ with respect to an ideal $a$ is defined by

$$H_i^a(M, N) = \lim_{n \to} \Ext_R^i(M/a^n M, N).$$

Clearly, $H_i^a(R, N)$ corresponds to the ordinary local cohomology module $H_i a(N)$ of $N$ with respect to $a$. By applying this notion and relative Cohen-Macaulay property, we prove the Artinianness of local cohomology modules as follows.

Let $M$ be a finite module of finite projective dimension $n$ over a local ring $(R, m)$ and $N$ be a non-zero relative Cohen-Macaulay $R$-module w.r.t $a$ with height $N a = h$ such that $\text{Supp} N/aN \subseteq V(m)$. Then $H_{a}^{n+h}(M, N)$ is Artinian. In particular, $H_{a}^{n+h}(N)$ is Artinian (Theorem 2.16).

Throughout, $(R, m)$ denotes a commutative Noetherian local ring. For unexplained notation and terminology about local cohomology modules, we refer the reader to [1].

2. Artinian and non-Artinian local cohomology modules

Recall that for a prime ideal $p \in \text{Supp}_R(M)$, the $M$-height of $p$ is defined by $\text{height}_M p = \dim M_p$. If $a$ is an ideal of $R$, the $M$-height of $a$ is defined to be $\text{height}_M a = \inf \{ \text{height}_M p | p \in \text{Supp}_R(M) \cap V(a) \}$. Notice that $\text{height}_M a \geq 0$ whenever $M \neq aM$. In [14], an $R$-module $M$ is called relative Cohen-Macaulay w.r.t $a$ if $H_i^a(M) = 0$ for all $i \neq \text{height}_M a$. In other words, this is the case if and only if $\text{grade}(a, M) = \text{cd}(a, M)$. We begin this section with the following result.

**Proposition 2.1.** Let $M$ be a finite relative Cohen-Macaulay local ring w.r.t $a$ with $\text{height}_R a = h$. Then

$$\dim \text{Supp}_R H_a^h(M) = \dim M/aM.$$

**Proof.** As $\text{Supp}_R H_a^h(M) \subseteq V(a) \cap \text{Supp}_R M$, we get $\dim \text{Supp}_R H_a^h(M) \leq \dim_R M/aM$. Where as, since $M$ is relative Cohen-Macaulay $R$-module w.r.t $a$, $M_p$ is Cohen-Macaulay for all $p \in V(a)$ with $\text{height}_R p = h$. In fact, in view of [1, Theorem 4.3.2 and Theorem 6.1.4], we have

$$(H_a^h(M))_p \cong H_a^h(M_p) \cong H_p^h(M_p) \neq 0.$$

Hence $p \in \text{Supp}_R H_a^h(M)$. Therefore,

$$\dim \text{Supp}_R H_a^h(M) \geq \dim M/aM,$$

which completes the proof. ■
Corollary 2.2. Let \((R, \mathfrak{m})\) be a relative Cohen-Macaulay local ring w.r.t \(\mathfrak{a}\) with height \(R\mathfrak{a} = h\). Then
\[
\dim \text{Supp}_R H^h_{\mathfrak{a}}(R) = \dim R - h.
\]

Proof. It follows easily by Proposition 2.1 and [6, Corollary 3.3]. \(\square\)

Example 2.3. Let \((R, \mathfrak{m})\) be a local ring and \(\mathfrak{a} \subseteq R\) an ideal such that \(\mathfrak{a} = (x_1, \ldots, x_i)\), where \(x_1, \ldots, x_i\) is regular. Then \(\dim \text{Supp}_R H^i_{(x_1, \ldots, x_i)}(R) = \dim R - i\).

Remark 2.4 (cf. [5, Corollary 4.3]). Let \(R = k[x_1, \ldots, x_n]\) be a polynomial ring in \(n\) variables \(x_1, \ldots, x_n\) over a field \(k\) and \(\mathfrak{a}\) be a squarefree monomial ideal of \(R\). Then, the following are equivalent:

(i) \(H^h_{\mathfrak{a}}(R) = 0\) for all \(i \neq \text{height} R\mathfrak{a}\), i.e., \(\mathfrak{a}\) is cohomologically a complete intersection ideal.

(ii) \(R/\mathfrak{a}\) is a Cohen-Macaulay ring.

The above remark help us to bring the following example which has been calculated using CoCoA to provide an example to Corollary 2.2.

Example 2.5. Let \(R = k[x_1, \ldots, x_6]\) be a polynomial ring over a field \(k\) and \(\mathfrak{a} = (x_1x_2x_3, x_1x_2x_4, x_1x_3x_5, x_1x_4x_6, x_1x_5x_6, x_2x_3x_6, x_2x_4x_5, x_2x_5x_6, x_3x_4x_5, x_3x_4x_6)\) an ideal of \(R\). By using CoCoA [2], depth \(R/\mathfrak{a} = 3 = \dim R/\mathfrak{a}\), i.e., \(R/\mathfrak{a}\) is a Cohen-Macaulay ring. By virtue of Remark 2.4, \(H^i_{\mathfrak{a}}(R) = 0\) for all \(i \neq 3\). Therefore \(\dim H^3_{\mathfrak{a}}(R) = 3\) by Corollary 2.2.

As a consequence of Corollary 2.2, we give the following result.

Corollary 2.6. Let \((R, \mathfrak{m})\) be a relative Cohen-Macaulay local ring w.r.t \(\mathfrak{a}\) with height \(R\mathfrak{a} = h\) and \(\dim R/\mathfrak{a} > 0\). Then \(H^h_{\mathfrak{a}}(R)\) is not Artinian.

Proof. In view of Corollary 2.2, we have
\[
\dim H^h_{\mathfrak{a}}(R) = \dim \text{Supp}_R H^h_{\mathfrak{a}}(R) > 0.
\]
Thus \(H^h_{\mathfrak{a}}(R)\) is not Artinian. \(\square\)

Now, we recall the notion filter-depth and some results about it in order to turn out Corollary 2.8.

Definition 2.7 (see [9]). Let \((R, \mathfrak{m})\) be a local ring, \(\mathfrak{a} \subseteq R\) an ideal and \(M\) a finite \(R\)-module such that \(\text{Supp}_R M/\mathfrak{a}M \not\subseteq \{\mathfrak{m}\}\), then the filter-depth of \(M\) with respect to \(\mathfrak{a}\) is as
\[
f - \text{depth}(\mathfrak{a}, M) = \min \{ \text{depth}_{\mathfrak{a}p} M_p | p \in \text{Supp}_R M/\mathfrak{a}M \setminus \{\mathfrak{m}\} \}.
\]
In view of [9, Theorem 3.1], we have if \((R, \mathfrak{m})\) is a local ring, \(a \subseteq R\) an ideal and \(M\) a finite \(R\)-module such that \(\text{Supp}_R M/aM \not\subset \{\mathfrak{m}\}\), then
\[ f - \text{depth}(a, M) = \min \{ s \mid H^s_a(M) \text{ is not Artinian} \}. \]
Consequently, it follows from [12, Proposition 3.5] that if \(\dim(M/aM) > 0\), then
\[ \text{depth}(a, M) \leq f - \text{depth}(a, M) \leq \text{height}_M a. \]

We are now able to state our next result which is a consequence of Corollary 2.6 and it shows that the inequality \(f - \text{depth}(a, M) \leq \text{height}_M a\) from [12, Proposition 3.5] will becomes an equality for the "ring" case.

**Corollary 2.8.** Let \((R, \mathfrak{m})\) be a relative Cohen-Macaulay local ring w.r.t \(a\) and \(\dim R/a > 0\). Then
\[ f - \text{depth}(a, R) = \text{height}_R a. \]

**Proof.** Apply Corollary 2.6.

**Example 2.9.** Let \(R = k[x_1, \ldots, x_4]\) be a polynomial ring over a field \(k\), and \(S := k[x_1, \ldots, x_4]/(x_1, \ldots, x_4)\) be the local ring and \(a = (x_1 x_3, x_2 x_4)\) be an ideal of \(S\). By using CoCoA [2], we get \(S/a\) is Cohen-Macaulay ring and clearly \(\text{height}_S a = 2\). Then by Remark 2.4 and Corollary 2.8, we get \(f - \text{depth}(a, S) = 2\).

Recall the notion \(f_a(M)\), the finiteness dimension of \(M\) relative \(a\), is defined to be the least integer \(i\) such that \(H^i_a(M)\) is not finite, if there exist such \(i\)’s and \(\infty\) otherwise. Notice that if \(M\) is a relative Cohen-Macaulay \(R\)-module w.r.t \(a\), then obviously \(f_a(M) = \text{height}_M a\). Hence, in conjunction with Corollary 2.8, we get the following result.

**Corollary 2.10.** Let \((R, \mathfrak{m})\) be a relative Cohen-Macaulay local ring w.r.t \(a\) and \(\dim R/a > 0\). Then
\[ f_a(R) = f - \text{depth}(a, R). \]

Now recall the concept of Lyubeznik numbers due to [11]. Let \(R\) be a local ring which admits a surjection from an \(n\)-dimensional regular local ring \(S\) containing a field, \(a\) be the kernel of surjection and \(k = S/\mathfrak{m}\). The Bass numbers \(\lambda_{i,j}(R) = \dim_k \text{Ext}^j_S(k, H^{n-j}_a(S))\) known as Lyubeznik numbers of \(R\) which depend only on \(R\), \(i\) and \(j\) but neither on \(S\) nor on the surjection \(S \rightarrow R\). Let \(d = \dim(R)\). Lyubeznik numbers satisfy the following properties:
(a) \(\lambda_{i,j}(R) = 0\) for \(j > d\) or \(i > j\).
(b) \(\lambda_{d,d}(R) \neq 0\).
Therefore, we collect them in the so-called Lyubeznik table:

\[
\Lambda(R) = \begin{pmatrix}
\lambda_{0,0} & \cdots & \lambda_{0,d} \\
\vdots & & \vdots \\
\lambda_{d,0} & \cdots & \lambda_{d,d}
\end{pmatrix}
\]

and the Lyubeznik table is trivial if \( \lambda_{d,d} = 1 \) and the rest of these invariants vanish, where \( d = \dim(R) \) (see [11]).

We now state the following result.

**Proposition 2.11.** Let \((R, m, k)\) be a local regular ring containing a field which is relative Cohen-Macaulay w.r.t \( a \). Then \( \lambda_{i,j}(R/a) = 1 \) whenever \( i = j = \dim R/a \) and otherwise \( \lambda_{i,j}(R/a) = 0 \).

**Proof.** As \( H^i_m(H^d_{a,j}(R)) \Rightarrow H^{i+d-j}_m(R) \), from Corollary 2.2, if \( i = j = \dim R/a \), then

\[
H^{\dim R/a}_m(H^d_{a,d-j}(R/a)) \Rightarrow H^d_m(R) \neq 0.
\]

For \( i = j = \dim R/a \), we have

\[
\lambda_{i,j}(R/a) = \dim_k \text{Hom}_R(k, H^i_m(H^d_{a,d-j}(R)))) = \dim_k \text{Hom}_R(k, E) = 1,
\]

where \( E \) is the injective hull of \( k \). Otherwise \( \lambda_{i,j}(R/a) = 0 \). \( \blacksquare \)

In order to prove Proposition 2.15, we recall the following definitions.

**Definition 2.12** (see [13] and [15]). For a commutative local ring \( R \), let \( \sum R \) be the direct sum \( \oplus_{m \in \text{Max Spec}(R)} R/m \) of all simple \( R \)-modules, \( E \) be the injective hull of \( \sum R \), and \( D_R(\cdot) \) be the functor \( \text{Hom}_R(\cdot, E_R) \). (Note that \( D_R(\cdot) \) is a natural generalization of Matlis duality functor to non-local rings.)

**Definition 2.13** (see [7]). An \( R \)-module \( M \) is called \( a \)-cofinite if \( \text{Supp}_R(M) \subseteq V(a) \) and \( \text{Ext}_R^j(R/a, M) \) is finite for all \( j \geq 0 \).

**Remark 2.14** (see [4, Theorem 2.1]). For a finite \( R \)-module \( M \) and a non-negative integer \( c \) if \( H^i_a(M) \) is \( a \)-cofinite for all \( i < c \) then \( \text{Hom}_R(R/a, H^c_a(M)) \) is finite.

We now bring the following result.

**Proposition 2.15.** Let \((R, m)\) be a relative Cohen-Macaulay local ring w.r.t \( a \) with height \( h = h \) and \( \text{Supp}_R(R/a) \subseteq V(m) \). Then \( H^h_a(R) \) is indecomposable.
Proof. By assumption, $H_a^i(R) = 0$ for all $i < h$ and so $H_a^i(R)$ is a-cocofinite for all $i < h$. Hence, Hom$_R(R/a, H_a^h(R))$ is finite from Remark 2.14. Since Supp$_R(R/a) \subseteq V(m)$, it deduces Hom$_R(R/a, H_a^h(R))$ is Artinian. Thus, in view of [1, Theorem 7.1.2], $H_a^h(R)$ is Artinian over $R$. Without loss of generality, we may assume that $R$ is a complete ring too. We suppose that $H_a^h(R)$ is not indecomposable and we look for a contradiction. Let $H_a^h(R) = U \oplus V$, where $U$ and $V$ are non-zero Artinian $R$-modules. Hence, $D_R(H_a^h(R)) = D_R(U) \oplus D_R(V)$. Since $D_R(H_a^h(R))$ is indecomposable by [14, Corollary 4.9], it follows that $D(U) = 0$ or $D(V) = 0$. Therefore, $U = 0$ or $V = 0$ which is a contradiction. 

Recall that for each $i \in \mathbb{N}_0$, the $i$th generalized local cohomology module $H_a^i(M, N)$ of two $R$-modules $M$ and $N$ with respect to an ideal $a$ is defined by

$$H_a^i(M, N) = \lim_{n \to 1} \text{Ext}^i_R(M/a^n M, N).$$

It is clear that $H_a^i(R, N)$ is just the ordinary local cohomology module $H_a^i(N)$ of $N$ with respect to $a$.

The following theorem deals with the Artinianness of local cohomology modules.

Theorem 2.16. Let $M$ be a finite module of finite projective dimension $n$ over a local ring $(R, m)$ and $N$ be a non-zero relative Cohen-Macaulay $R$-module w.r.t $a$ with height$_N a = h$ such that Supp$_N aN \subseteq V(m)$. Then $H_a^{n+h}(M, N)$ is Artinian. In particular, if $M = R$, then $H_a^{n+h}(N)$ is Artinian.

Proof. We use induction on pd($M$). If pd($M$) = 0, then $M \oplus M' \cong R^t$ for some $R$-module $M'$ and some integer $t$. Thus

$$H_a^h(M, N) \oplus H_a^h(M', N) \cong H_a^h(R^t, N) \cong H_a^h(N)^t$$

Since Supp$_N aN \subseteq V(m)$, $H_a^h(N)$ is Artinian as we have seen in the proof of Proposition 2.15. Thus the assertion holds. Now, suppose that pd($M$) > 0 and the assertion is true for any finite $R$-module $K$ with pd($K$) < pd($M$). Consider the exact sequence $0 \to K \to F \to M \to 0$, where $F$ is free $R$-module of finite rank and $K$ is a finite $R$-module. Therefore, we get the following long exact sequence.

$$\ldots \to H_a^{n+h-1}(K, N) \to H_a^{n+h}(M, N) \to H_a^{n+h}(F, N) \to \ldots$$

But $H_a^{n+h-1}(K, N)$ is Artinian by induction hypothesis and $H_a^{n+h}(F, N)$ is Artinian by [16, 3.1]. Hence $H_a^{n+h}(M, N)$ is Artinian.
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