GENERALIZED DERIVATIONS WITH LEFT ANNIHILATOR CONDITIONS IN PRIME AND SEMIPRIME RINGS

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Abstract
Let \( R \) be a prime ring with its Utumi ring of quotients \( U \), \( C = Z(U) \) be the extended centroid of \( R \), \( H \) and \( G \) two generalized derivations of \( R \), \( L \) a noncentral Lie ideal of \( R \), \( I \) a nonzero ideal of \( R \). The left annihilator of \( S \subseteq R \) is denoted by \( l_R(S) \) and defined by \( l_R(S) = \{ x \in R \mid xS = 0 \} \). Suppose that \( S = \{ H(u^n)u^n + u^nG(u^n) \mid u \in L \} \) and \( T = \{ H(x^n)x^n + x^nG(x^n) \mid x \in I \} \), where \( n \geq 1 \) is a fixed integer. In the paper, we investigate the cases when the sets \( l_R(S) \) and \( l_R(T) \) are nonzero.

Keywords: prime ring, derivation, Lie ideal, generalized derivation, extended centroid, Utumi quotient ring.

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1. Introduction
Let \( R \) be an associative ring with center \( Z(R) \). For \( x, y \in R \), the commutator of \( x, y \) is denoted by \( [x, y] \) and defined by \( [x, y] = xy - yx \). By \( d \) we mean a derivation of \( R \). An additive mapping \( F \) from \( R \) to \( R \) is called a generalized derivation if there exists a derivation \( d \) from \( R \) to \( R \) such that \( F(xy) = F(x)y + xd(y) \) holds for all \( x, y \in R \). Throughout this paper, \( R \) will always present a prime ring with center \( Z(R) \), extended centroid \( C \) and \( U \) is its Utumi quotient ring. A well known result proved by Posner [20], states that if the commutators \( [d(x), x] \in Z(R) \) for all \( x \in R \), then either \( d = 0 \) or \( R \) is commutative. Then result of Posner was generalized in many

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directions by a number of authors. Posner’s theorem was extended to Lie ideals in prime rings by Lee [17] and then by Lanski [12].

On the other hand, authors generalized Posner’s theorem by considering two derivations. In [3], Brešar proved that if d and δ are two derivations of R such that \( d(x)x - xδ(x) \in Z(R) \) for all \( x \in R \), then either \( d = δ = 0 \) or R is commutative. Later Lee and Wong [18] consider the situation \( d(x)x - xδ(x) \in Z(R) \) for all \( x \) in some noncentral Lie ideal \( L \) of R and they proved that either \( d = δ = 0 \) or R satisfies \( s_4 \).

Recently in [22] Vukman proves that if \( d \) and \( δ \) are derivations on \( 2mn(m+n-1) \)-torsion free semiprime rings \( R \) such that \( d(x)x^n + x^nδ(x^m) = 0 \) for all \( x \in R \), where \( m,n \geq 1 \) are fixed integers, then both derivations \( d \) and \( δ \) map \( R \) into its center and \( d = -δ \).

In [23], Wei and Xiao studied the similar situation replacing derivations \( d \) and \( δ \) by generalized derivations \( G \) and \( H \). More precisely they proved the following:

Let \( m,n \) be fixed positive integers, \( R \) be a noncommutative \( 2(m+n)! \)-torsion free prime ring and \( G,H \) be a pair of generalized Jordan derivations on \( R \). If \( G(x^m)x^n + x^nH(x^m) \in Z(R) \) for all \( x \in R \), then \( G \) and \( H \) both are right (or left) multipliers.

In [14], Lee and Zhou studied the same situation of above result without considering torsion free restriction on \( R \). In this paper, Lee and Zhou [14] proved the following:

Let \( R \) be a prime ring that is not commutative and such that \( R \not\cong M_2(GF(2)) \), let \( G,H \) be two generalized derivations of \( R \), and let \( m,n \) be two fixed positive integers. Then \( G(x^m)x^n - x^nH(x^m) = 0 \) for all \( x \in R \) iff the following two conditions hold:

1. There exists \( w \in Q \) such that \( G(x) = xw \) and \( H(x) = wx \) for all \( x \in R \);
2. either \( w \in C \), or \( x^m \) and \( x^n \) are \( C \)-dependent for all \( x \in R \).

There are many papers in the literature which studied the identities of generalized derivations with left annihilator conditions.

For any subset \( S \) of \( R \), denote by \( r_R(S) \) the right annihilator of \( S \) in \( R \), that is, \( r_R(S) = \{ x \in R \mid Sx = 0 \} \) and \( l_R(S) \) the left annihilator of \( S \) in \( R \) that is, \( l_R(S) = \{ x \in R \mid xS = 0 \} \). If \( r_R(S) = l_R(S) \), then \( r_R(S) \) is called an annihilator ideal of \( R \) and is written as \( ann_R(S) \).

In [4], Carini et al. studied the left annihilator of the set \( \{ H(u)u - uG(u) \mid u \in L \} \), where \( L \) is a noncentral Lie ideal of \( R \) and \( H,G \) two non-zero generalized derivations of \( R \). In case the annihilator is not zero, the conclusion is one of the following:

1. there exist \( b', c' \in U \) such that \( H(x) = b'x + xc' \), \( G(x) = c'x \) with \( ab' = 0 \);
2. \( R \) satisfies \( s_4 \) and there exist \( b', c', q' \in U \) such that \( H(x) = b'x + xc' \), \( G(x) = c'x + xq' \), with \( a(b' - q') = 0 \).
Recently, Carini and De Filippis proved the following theorem:

Let $R$ be a prime ring, $U$ the Utumi quotient ring of $R$, $C = Z(U)$ the extended centroid of $R$, $L$ a non-central Lie ideal of $R$, $H$ and $G$ non-zero generalized derivations of $R$. Suppose that there exists an integer $n \geq 1$ such that $H(u^n)u^n + u^nG(u^n) \in C$, for all $u \in L$, then either there exists $a \in U$ such that $H(x) = xa, G(x) = -ax$, or $R$ satisfies the standard identity $s_4$. Moreover, in the last case the structures of the maps $G, H$ are obtained.

In the present paper, we shall investigate the left annihilator of the sets $\{H(u^n)u^n + u^nG(u^n) \mid u \in L\}$ and $\{H(x^n)x^n + x^nG(x^n) \mid x \in I\}$, where $L$ is a noncentral Lie ideal of $R$, $I$ is a nonzero ideal of $R$, $n \geq 1$ is a fixed integer and $H, G$ two non-zero generalized derivations of $R$. More precisely, we prove the following theorems:

**Theorem 1.1.** Let $R$ be a prime ring with its Utumi ring of quotients $U$, $C = Z(U)$ be the extended centroid of $R$, $H$ and $G$ two generalized derivations of $R$, $L$ a noncentral Lie ideal of $R$ and $S = \{H(u^n)u^n + u^nG(u^n) \mid u \in L\}$, where $n \geq 1$ is a fixed integer. If $I_R(S) \neq \{0\}$, then either there exist $b', p \in U$ such that $H(x) = b'x - xp$ and $G(x) = px$ for all $x \in R$ with $ab' = 0$ or $R$ satisfies $s_4$. Moreover, in the last case, if $R$ satisfies $s_4$, then one of the following holds:

1. $n$ is even, there exist $b, p \in U$ and derivations $d, \delta$ of $R$ such that $H(x) = bx + d(x)$ and $G(x) = px + \delta(x)$ for all $x \in R$, with $a(b + p) = 0$;
2. $n$ is odd, there exist $b, p \in U$ and derivations $d, \delta$ of $R$ such that $H(x) = bx + d(x)$ and $G(x) = px + \delta(x)$ for all $x \in R$, with $a(b + p) = 0$.

**Theorem 1.2.** Let $R$ be a noncommutative prime ring with char $(R) \neq 2$, $U$ its Utumi ring of quotients, $C = Z(U)$ be the extended centroid of $R$, $H$ and $G$ two generalized derivations of $R$, $I$ a nonzero ideal of $R$ and $S = \{H(x^n)x^n + x^nG(x^n) \mid x \in I\}$, where $n \geq 1$ is a fixed integer. If $I_R(S) \neq \{0\}$, then there exist $b', p \in U$ such that $H(x) = b'x - xp$ and $G(x) = px$ for all $x \in R$ with $ab' = 0$.

As an immediate application of the Theorem 1.1, in particular when $G = -H$, then we have the following result which gives a particular result of Theorem 1.1 in [6].

**Corollary 1.3.** Let $R$ be a prime ring with its Utumi ring of quotients $U$, $C = Z(U)$ be the extended centroid of $R$, $H$ a generalized derivation of $R$ and $L$ a noncentral Lie ideal of $R$. Suppose that there exists $0 \neq a \in R$ such that $a[H(u^n), u^n] = 0$ for all $u \in L$, where $n \geq 1$ is a fixed integer. Then either there exists $\lambda \in C$ such that $H(x) = \lambda x$ for all $x \in R$ or $R$ satisfies $s_4$.

As an application of the Theorem 1.1, in particular when $G = 0$, then using
Theorem 2.2 in [8], we have the following result which gives a generalization of Theorem 1.1 in [21].

Corollary 1.4. Let $R$ be a prime ring of char $(R) \neq 2$ with its Utumi ring of quotients $U$, $C = Z(U)$ be the extended centroid of $R$, $H$ a generalized derivation of $R$ and $L$ a noncentral Lie ideal of $R$. Suppose that there exists $0 \neq a \in R$ such that $aH(u^n)u^n = 0$ for all $u \in L$, where $n \geq 1$ is a fixed integer. Then either there exist $b', p \in U$ such that $H(x) = b'x$ for all $x \in R$ with $ab' = 0$.

2. Proof of main results in prime rings

Let $R$ be a prime ring with extended centroid $C$. Let $H(x) = bx + xc$ and $G(x) = px + xq$ for all $x \in R$ and for some $b, c, p, q \in U$, be two inner generalized derivations of $R$ and $L$ be a noncentral Lie ideal of $R$. Then $a(H(x^n)x^n + x^nG(x^n)) = 0$ implies $a(bx^{2n} + x^n(c + p)x^n + x^{2n}q) = 0$ for all $x \in L$. We know that if char $(R) \neq 2$, by [2, Lemma 1] there exists a nonzero ideal $I$ of $R$ such that $0 \neq [I, R] \subseteq L$. If char $(R) = 2$ and $\dim_C RC > 4$ i.e., char $(R) = 2$ and $R$ does not satisfy $s_4$, then by [13, Theorem 13] there exists a nonzero ideal $I$ of $R$ such that $0 \neq [I, R] \subseteq L$. We assume that $R$ does not satisfy $s_4$. Then in any case of char $(R) = 2$ or char $(R) \neq 2$, we can conclude that there exists a nonzero ideal $I$ of $R$ such that $0 \neq [I, I] \subseteq L$. By hypothesis, we have

\[
(1) \quad a(b[x_1, x_2]^{2n} + x_1, x_2]^n(c + p)[x_1, x_2]^n + [x_1, x_2]^{2n}q) = 0
\]

for all $x_1, x_2 \in I$. Then following lemmas are immediate consequences:

Lemma 2.1. $R$ satisfies a nontrivial generalized polynomial identity (GPI) or $c, p, q \in C$ such that $a(b + c + p + q) = 0$.

Proof. Assume that $R$ does not satisfy any nontrivial GPI. Then $R$ must be noncommutative. Let $T = U \ast_C C\{x_1, x_2\}$, the free product of $U$ and $C\{x_1, x_2\}$, the free $C$-algebra in noncommuting indeterminates $x_1$ and $x_2$.

Then,

\[
a (b[x_1, x_2]^{2n} + [x_1, x_2]^n(c + p)[x_1, x_2]^n + [x_1, x_2]^{2n}q)
\]

is zero element in $T$. If $q \notin C$, then $q$ and 1 are linearly independent over $C$. Then from above

\[
a[x_1, x_2]^{2n}q = 0 \in T,
\]

implying $q = 0$, since $a \neq 0$, a contradiction. Therefore, we conclude that $q \in C$. 

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Then by hypothesis,
\[(2) \quad a((b + q)[x_1, x_2]^n + [x_1, x_2]^n(c + p))[x_1, x_2]^n = 0 \in T.\]

If \(c + p \notin C\), then by (2)
\[a([x_1, x_2]^n(c + p)) [x_1, x_2]^n = 0 \in T,\]
implying \(c + p = 0\), since \(a \neq 0\), a contradiction. Therefore, we have \(c + p \in C\) and hence
\[a(b + q + c + p)[x_1, x_2]^{2n} = 0 \in T.\]
This implies \(a(b + q + c + p) = 0\).

**Lemma 2.2.** \(c + p, q \in C\) with \(a(b + c + p + q) = 0\), unless \(R\) satisfies \(s_4\).

**Proof.** By hypothesis, \(R\) satisfies GPI
\[(3) \quad f(x_1, x_2) = a(b[x_1, x_2]^{2n} + [x_1, x_2]^n(c + p)[x_1, x_2]^n + [x_1, x_2]^{2n}q).\]

If \(R\) does not satisfy any nontrivial GPI, by Lemma 2.1, we obtain \(c, p, q \in C\) with \(a(b + c + p + q) = 0\) which gives the conclusion. So, we assume that \(R\) satisfies a nontrivial GPI. Since \(R\) and \(U\) satisfy the same generalized polynomial identities (see [5]), \(U\) satisfies \(f(x_1, x_2)\). In case \(C\) is infinite, we have \(f(x_1, x_2) = 0\) for all \(x_1, x_2 \in U \otimes_C \overline{C}\), where \(\overline{C}\) is the algebraic closure of \(C\). Moreover, both \(U\) and \(U \otimes_C \overline{C}\) are prime and centrally closed algebras [9]. Hence, replacing \(R\) by \(U\) or \(U \otimes_C \overline{C}\) according to \(C\) finite or infinite, without loss of generality we may assume that \(C = Z(R)\) and \(R\) is \(C\)-algebra centrally closed. By Martindale’s theorem [19], \(R\) is then a primitive ring having nonzero socle \(soc(R)\) with \(C\) as the associated division ring. Hence, by Jacobson’s theorem [10, p.75], \(R\) is isomorphic to a dense ring of linear transformations of a vector space \(V\) over \(C\).

If \(\dim_C V = 2\), then \(R \cong M_2(C)\), that is, \(R\) satisfies \(s_4\), a contradiction. So, let \(\dim_C V \geq 3\).

We show that for any \(v \in V\), \(v\) and \(qv\) are linearly \(C\)-dependent. Suppose that \(v\) and \(qv\) are linearly independent for some \(v \in V\). Since \(\dim_C V \geq 3\), there exists \(u \in V\) such that \(v, qv, u\) are linearly \(C\)-independent set of vectors. By density, there exists \(x_1, x_2 \in R\) such that
\[x_1v = v, \quad x_1qv = 0, \quad x_1u = qv; \quad x_2v = 0, \quad x_2qv = u, \quad x_2u = 0.\]
Then \(0 = a(b[x_1, x_2]^{2n} + [x_1, x_2]^n(c + p)[x_1, x_2]^n + [x_1, x_2]^{2n}q)v = aqv.\)
This implies that if for some \(v \in V\), \(aqv \neq 0\), then by contradiction, \(v\) and \(qv\) are linearly \(C\)-dependent.
Now choose $v \in V$ such that $v$ and $qv$ are linearly $C$-independent. Then $aqv = 0$. Let us consider a subspace $W = \{\alpha v + \beta v q | \alpha, \beta \in C\}$ of $V$. Let $aq \neq 0$. Then, there exists $w \in V$ such that $aqv \neq 0$. Then $aq(v - w) = aqw \neq 0$. Then by the above argument, $w, qw$ are linearly $C$-dependent and $(v - w), q(v - w)$ too. Thus there exist $\alpha, \beta \in C$ such that $qw = \alpha w$ and $q(v - w) = \beta(v - w)$. Then $qv = \beta(v - w) + qw = \beta(v - w) + \alpha w$ i.e., $(\alpha - \beta)w = qv - \beta v \in W$. Now $\alpha = \beta$ implies that $qv = \beta v$, a contradiction. Hence $\alpha \neq \beta$ and so $w \in W$.

Next assume that $u \in V$ such that $aqv = 0$. Then $aq(w + u) = aqw \neq 0$. By above argument, $aq(w + u) \neq 0$ implies $w + u \in W$. Since $w \in W$, we have $u \in W$. Thus it is observed that for any $v \in V$, $aqv \neq 0$ implies $v \in W$ and $aqv = 0$ implies $v \in W$. This implies that $V = W$ i.e., dim$_C V = 2$, a contradiction.

Thus up to now we have proved that $v$ and $qv$ are linearly $C$-dependent for all $v \in V$, unless $aq = 0$. If $aq \neq 0$, by standard argument, it follows that $qv = \lambda v$ for all $v \in V$ and $\lambda \in C$ fixed. Then $(q - \lambda)V = 0$, implying $q = \lambda \in C$.

Now let $aq = 0$. Since dim$_C V \geq 3$, there exists $w \in V$ such that $v, qv, w$ are linearly $C$-independent set of vectors. By density, there exists $x_1, x_2 \in R$ such that

$$x_1v = v, \ x_1qv = 0, \ x_1w = v + qv; \ x_2v = 0, \ x_2qv = w, \ x_2w = 0.$$ 

Then $0 = a(b[x_1, x_2]^{2n} + [x_1, x_2]^n(c + p)[x_1, x_2]^n + [x_1, x_2]^{2n}q)v = av$. Then by above argument, since $a \neq 0, q \in C$.

Therefore, we have proved that in any case $q \in C$. Hence our identity reduces to

$$a(b'[x_1, x_2]^{2n} + [x_1, x_2]^n c'[x_1, x_2]^n) = 0,$$

where $b' = b + q$ and $c' = c + p$.

Now we prove that $v$ and $c'v$ are linearly $C$-dependent. If possible let $v$ and $c'v$ be linearly independent for some $v \in V$. Then there exists $w \in V$ such that $v, c'v$ and $w$ are linearly independent over $C$. By density there exist $x_1, x_2 \in R$ such that

$$x_1v = 0, \ x_1c'v = v, \ x_1w = 2c'v; \ x_2v = c'v, \ x_2c'v = w, \ x_2w = 0.$$ 

Then $0 = a(b'[x_1, x_2]^{2n} + [x_1, x_2]^nc'[x_1, x_2]^n)v = a(b' + c')v$. As above, this implies either $a(b' + c') = 0$ or $c' \in C$. Let $a(b' + c') = 0$. Then we have that $R$ satisfies $0 = a[c', x_1, x_2]^n][x_1, x_2]^n$. By density there exist $x_1, x_2 \in R$ such that

$$x_1v = 0, \ x_1c'v = v, \ x_1w = c'v; \ x_2v = c'v, \ x_2c'v = w, \ x_2w = 0.$$ 

Thus $0 = a[c', x_1, x_2]^n][x_1, x_2]^nv = ac'v$. This implies either $ac' = 0$ or $c' \in C$. Let $ac' = 0$. Then we have that $R$ satisfies $0 = a[x_1, x_2]^n]c'[x_1, x_2]^n$. Again by density there exist $x_1, x_2 \in R$ such that

$$x_1v = 0, \ x_1c'v = v, \ x_1w = v + c'v; \ x_2v = c'v, \ x_2c'v = w, \ x_2w = 0.$$
Thus \(0 = a[x_1, x_2]^n]c'[x_1, x_2]^nv = av\). Since \(a \neq 0\), this implies \(c' \in C\). Thus in any case, we have \(c' \in C\). Hence \(R\) satisfies \(0 = a(b' + c')[x_1, x_2]^{2n}\), which implies \(a(b' + c') = 0\).

**Proof of Theorem 1.1.** Let \(0 \neq a \in l_R(S)\). Then \(a(H(u^n)u^n + u^nG(u^n)) = 0\) for all \(u \in L\). If \(\text{char } (R) = 2\) and \(R\) satisfies \(s_4\), then we obtain our conclusion (1). So we assume that either \(\text{char } (R) \neq 2\) or \(R\) does not satisfy \(s_4\). Then by [2, Lemma 1] and [13, Theorem 13], since \(L\) is a noncentral Lie ideal of \(R\), there exists a nonzero ideal \(I\) of \(R\) such that \([I, I] \subseteq L\). Hence, by our hypothesis, we have

\[
a(H([x_1, x_2]^n]x_1, x_2]^n + [x_1, x_2]^nG([x_1, x_2]^n]) = 0
\]

for all \(x_1, x_2 \in I\). Since \(I\), \(R\) and \(U\) satisfy the same generalized polynomial identities (see [5]) as well as the same differential identities (see [16]), they also satisfy the same generalized differential identities. Hence, by [15], \(U\) satisfies

\[
a(H([x_1, x_2]^n]x_1, x_2]^n + [x_1, x_2]^nG([x_1, x_2]^n)) = 0
\]

for all \(x_1, x_2 \in U\), where \(H(x) = bx + d(x)\) and \(G(x) = px + \delta(x)\), for some \(b, p \in U\) and derivations \(d\) and \(\delta\) of \(U\), that is, \(U\) satisfies

\[
a(b[x_1, x_2]^{2n} + d([x_1, x_2]^n])x_1, x_2]^n + [x_1, x_2]^n p[x_1, x_2]^n
\]

\[+ [x_1, x_2]^n \delta([x_1, x_2]^n)) = 0.\]

Now we divide the proof into two cases:

**Case I.** Let \(d(x) = [c, x]\) for all \(x \in U\) and \(\delta(x) = [q, x]\) for all \(x \in U\) i.e., \(d\) and \(\delta\) be inner derivations of \(U\). Then from (4), we obtain that \(U\) satisfies

\[
(5)\quad a((b + c)[x_1, x_2]^{2n} + [x_1, x_2]^n(p - c + q)[x_1, x_2]^n - [x_1, x_2]^{2n}q) = 0.
\]

By Lemma 2.2, when \(R\) does not satisfy \(s_4\), we have \(q, p - c + q \in C\) with \(a(b + p) = 0\). This implies \(p - c \in C\). Hence \(H(x) = bx + [c, x]\) = \(bx + [p, x] = b'x - xp\), \(G(x) = px\) for all \(x \in U\) and so for all \(x \in R\) with \(ab' = 0\), where \(b' = b + p\).

Moreover, when \(R\) satisfies \(s_4\) (in this case by assumption \(\text{char } (R) \neq 2\)), then \(R \subseteq M_2(F)\) and, \(R\) and \(M_2(F)\) satisfy the same GPI, where \(M_2(F)\) is a matrix ring over a field \(F\). Hence \(M_2(F)\) satisfies \(a((b + c)[x_1, x_2]^{2n} + [x_1, x_2]^n(p - c + q)[x_1, x_2]^n - [x_1, x_2]^{2n}q) = 0\). Since \([x, y]^2 \in Z(M_2(F))\) for all \(x, y \in M_2(F)\), \(M_2(F)\) satisfies

\[
(6)\quad a((b + c - q)[x_1, x_2]^{2n} + [x_1, x_2]^n(p - c + q)[x_1, x_2]^n) = 0.
\]

If \(n\) is even, then by choosing \(x_1 = e_{12}\), \(x_2 = e_{21}\), we have \(0 = a(b + p)\).

If \(n\) is odd, then \(M_2(F)\) satisfies \(a((b + c - q)[x_1, x_2] + [x_1, x_2]^n(p - c + q)) [x_1, x_2]^{2n-1} = 0\). By Lemma 2.7 in [7], we conclude that \(p - c + q \in Z(R)\) and \(a(b + p) = 0\).
Thus when $R$ satisfies $s_4$, one of the following holds:

(i) $n$ is even and $a(b + p) = 0$. In this case, $H(x) = bx + [c, x]$ and $G(x) = px + [q, x]$ for all $x \in R$, with $a(b + p) = 0$. This is our conclusion (2).

(ii) $n$ is odd and $p - c + q \in C$ and $a(b + p) = 0$. Hence $H(x) = bx + [c, x]$ and $G(x) = px + [q, x] = px - [p - c, x] = xp + [c, x]$ for all $x \in R$, with $a(b + p) = 0$. This is our conclusion (3).

Case II. Next assume that $d$ and $\delta$ are not both inner derivations of $U$, but they are $C$-dependent modulo inner derivations of $U$. Suppose $d = \lambda \delta + adc$, that is, $d(x) = \lambda \delta(x) + [c, x]$ for all $x \in U$, where $\lambda \in C$, $c \in U$. Then $d$ can not be inner derivation of $U$. From (4), we have that $U$ satisfies

$$a \left( b[x_1, x_2]^{2n} + \lambda \delta([x_1, x_2]^n)[x_1, x_2]^n + [c, [x_1, x_2]^n][x_1, x_2]^n 
+ [x_1, x_2]^n p[x_1, x_2]^n + [x_1, x_2]^n \delta([x_1, x_2]^n) \right) = 0. $$

This gives

$$a \left( b[x_1, x_2]^{2n} + \lambda \sum_{i=0}^{n-1} [x_1, x_2]^i \delta([x_1, x_2]^i)[x_1, x_2]^{n-i}[x_1, x_2]^n 
+ [x_1, x_2]^n p[x_1, x_2]^n + [x_1, x_2]^n \sum_{i=0}^{n-1} [x_1, x_2]^i \delta([x_1, x_2]^i)[x_1, x_2]^{n-i} \right) = 0. $$

Then by Kharchenko’s theorem [11], we have that $U$ satisfies

$$a \left( b[x_1, x_2]^{2n} + \lambda \sum_{i=0}^{n-1} [x_1, x_2]^i ([y_1, x_2] + [x_1, y_2])[x_1, x_2]^{n-i}[x_1, x_2]^n 
+ [x_1, x_2]^n p[x_1, x_2]^n 
+ [x_1, x_2]^n \sum_{i=0}^{n-1} [x_1, x_2]^i ([y_1, x_2] + [x_1, y_2])[x_1, x_2]^{n-i} \right) = 0. $$

In particular $U$ satisfies blended component

$$a \left( b[x_1, x_2]^{2n} + [c, [x_1, x_2]^n][x_1, x_2]^n + [x_1, x_2]^n p[x_1, x_2]^n \right) = 0$$

and

$$a \left( \lambda \sum_{i=0}^{n-1} [x_1, x_2]^i ([y_1, x_2] + [x_1, y_2])[x_1, x_2]^{n-i}[x_1, x_2]^n 
+ [x_1, x_2]^n \sum_{i=0}^{n-1} [x_1, x_2]^i ([y_1, x_2] + [x_1, y_2])[x_1, x_2]^{n-i} \right) = 0. $$
For \( y_1 = [q, x_1] \) and \( y_2 = [q, x_2] \), where \( q \notin C \) we have that \( U \) satisfies
\[
(10) \quad a(\lambda q, [x_1, x_2]^n)[x_1, x_2]^n + [x_1, x_2]^n[q, [x_1, x_2]^n]) = 0.
\]

By Lemma 2.2, if \( R \) does not satisfy \( s_4 \), then \( q \in C \), a contradiction. Hence we conclude that \( R \) satisfies \( s_4 \). Now the relations (8) and (10) are similar to the relation (5). Thus by same argument as given in Case I, when \( R \) satisfies \( s_4 \) (in this case \( \text{char}(R) \) must be not equal to 2), one of the following holds:

(i) Let \( n \) be even. Then by (8), \( a(b + p) = 0 \). Thus \( H(x) = bx + d(x) \) and \( G(x) = px + \delta(x) \) for all \( x \in R \), with \( a(b + p) = 0 \). This is our conclusion (2).

(ii) Let \( n \) be odd. Then by (8), \( p - c \in C \) and \( a(b + p) = 0 \). Again by (10), \( q - \lambda q = q(1 - \lambda) \in C \). Since \( q \notin C \), we have \( \lambda = 1 \). Then replacing \( y_1 = x_1 \) and \( y_2 = 0 \), (9) gives \( na(\lambda + 1)[x_1, x_2]^{2n} = 0 \), implying \( 2na = 0 \). Since \( \text{char}(R) \neq 2 \), \( na = 0 \). Hence \( H(x) = bx + \lambda \delta(x) + [c, x] = bx + \delta(x) + [c, x] \) and \( G(x) = px + \delta(x) = (p-c)x + cx + \delta(x) = xp + \delta(x) + [c, x] \) for all \( x \in R \). This is our conclusion (3).

The situation when \( \delta = \lambda d + ad_c \) is similar.

Next assume that \( d \) and \( \delta \) are \( C \)-independent modulo inner derivations of \( U \).

Since neither \( d \) nor \( \delta \) is inner, by Kharchenko’s Theorem [11], we have from (4) that \( U \) satisfies
\[
(11) \quad a\left(b[x_1, x_2]^{2n} + \sum_{i=0}^{n-1} [x_1, x_2]^i([u_1, x_2] + [x_1, u_2])[x_1, x_2]^{n-1-i}[x_1, x_2]^n + [x_1, x_2]^n p[x_1, x_2]^n + [x_1, x_2]^n \sum_{i=0}^{n-1} [x_1, x_2]^i([v_1, x_2] + [x_1, v_2])[x_1, x_2]^{n-1-i}\right) = 0.
\]

Then \( U \) satisfies blended component
\[
(12) \quad a\left(b[x_1, x_2]^{2n} + [x_1, x_2]^n p[x_1, x_2]^n\right) = 0
\]
and
\[
(13) \quad a\left([x_1, x_2]^n \sum_{i=0}^{n-1} [x_1, x_2]^i([v_1, x_2] + [x_1, v_2])[x_1, x_2]^{n-1-i}\right) = 0.
\]
Replacing \( v_1 \) with \( [q, x_1] \) and \( v_2 \) with \( [q, x_2] \) for some \( q \notin C \) in (13), we obtain that \( U \) satisfies
\[
(14) \quad a([x_1, x_2]^n[q, [x_1, x_2]^n]) = 0.
\]

By Lemma 2.2, we have \( q \in C \), a contradiction, unless \( R \) satisfies \( s_4 \). So we consider the case when \( R \) satisfies \( s_4 \). In this case by same argument of Case I, (12) and (14) together implies that \( n \) is even and \( a(b + p) = 0 \). This gives our conclusion (2). Hence the theorem is proved. \( \blacksquare \)
Corollary 2.3. Let $R$ be a prime ring with its Utumi ring of quotients $U$, $C = Z(U)$ be the extended centroid of $R$, $H$ and $G$ two generalized derivations of $R$ and $L$ a noncentral Lie ideal of $R$. Suppose that there exists $0 \neq a \in R$ such that $a(H(u^2)u^2 + u^2G(u^2)) = 0$ for all $u \in L$. Then either there exist $b', p \in U$ such that $H(x) = b'x - xp$ and $G(x) = px$ for all $x \in R$ with $ab' = 0$ or $R$ satisfies $s_4$. Moreover, if $R$ satisfies $s_4$, then one of the following holds:

1. char $(R) = 2$;
2. there exist $b, p \in U$ and derivations $d, \delta$ of $R$ such that $H(x) = bx + d(x)$ and $G(x) = px + \delta(x)$ for all $x \in R$, with $a(b + p) = 0$.

Proof of Theorem 1.2. Let $0 \neq a \in l_R(S)$. Then $a(H(x^n)x^n + x^nG(x^n)) = 0$ for all $x \in I$. By Theorem 1.1, we have only to consider the case when $R$ satisfies $s_4$. In this case $R$ is a PI-ring, and so there exists a field $K$ such that $R \subseteq M_2(K)$ and, $R$ and $M_2(K)$ satisfy the same GPI. First we assume that $H$ and $G$ are inner generalized derivations of $R$, that is, $H(x) = bx + xc$ for all $x \in R$ and $G(x) = px + xq$ for all $x \in R$, for some $b, c, p, q \in R$. Since $M_2(F)$ is a simple ring, by our hypothesis, $M_2(F)$ satisfies

$$a(bx^{2n} + x^n(c + p)x^n + x^{2n}q) = 0.$$  

Moreover, $R$ is a dense ring of $K$-linear transformations over a vector space $V$. Let $aq \neq 0$. Assume there exists $v \neq 0$, such that $\{v, qv\}$ is linear $K$-independent. By the density of $R$, there exists $r \in R$ such that $rv = 0; \ r(qv) = qv$.

Hence $0 = a(br^{2n} + r^n(c + p)r^n + r^{2n}q)v = aqv$.

Of course for any $w \in V$ such that $\{w, v\}$ are linearly $K$-dependent implies $aqw = 0$. Since $aq \neq 0$, there exists $w \in V$ such that $aqw \neq 0$. Then $\{w, v\}$ must be linearly $K$-independent. By the above argument it follows that $w$ and $qw$ are linearly $K$-dependent, as are $\{w + v, q(w + v)\}$ and $\{w - v, q(w - v)\}$. Therefore there exist $\alpha_w, \alpha_{w+v}, \alpha_{w-v} \in K$ such that

$$qw = \alpha_ww, \quad q(w + v) = \alpha_{w+v}(w + v), \quad q(w - v) = \alpha_{w-v}(w - v).$$

In other words we have

$$\alpha_ww + qv = \alpha_{w+v}w + \alpha_{w+v}v$$

and

$$\alpha_ww - qv = \alpha_{w-v}w - \alpha_{w-v}v.$$
By comparing (16) with (17) we get both

\[(2\alpha_w - \alpha_{w+v} - \alpha_{w-v})w + (\alpha_{w-v} - \alpha_{w+v})v = 0\]

and

\[2qv = (\alpha_{w+v} - \alpha_{w-v})w + (\alpha_{w+v} + \alpha_{w-v})v.\]

By (18) and since \{w, v\} is \(K\)-independent and \char(K) \(\neq 2\), we have \(\alpha_w = \alpha_{w+v} = \alpha_{w-v}\). Thus by (19) it follows \(2qv = 2\alpha_wv\). Since \{qv, v\} is \(K\)-independent, the conclusion \(\alpha_w = \alpha_{w+v} = 0\) follows, that is \(qw = 0\) and \(q(w+v) = 0\), which implies the contradiction \(qv = 0\).

Hence we conclude that for any \(v \in V\), \{\(v, qv\}\} is linearly \(K\)-dependent. Thus there exists a suitable \(\alpha_v \in K\) such that \(qv = \alpha_vv\), and standard argument shows that there is \(\alpha \in K\) such that \(qv = \alpha v\) for all \(v \in V\). Now let \(r \in R, v \in V\). Since \(qv = \alpha v\),

\[(q, r)v = (qr)v - (rq)v = q(rv) - r(qv) = \alpha(rv) - r(\alpha v) = 0.\]

Thus \([q, r]v = 0\) for all \(v \in V\) i.e., \([q, r]V = 0\). Since \([q, r]\) acts faithfully as a linear transformation on the vector space \(V\), \([q, r] = 0\) for all \(r \in R\). Therefore, \(q \in C\).

Thus up to now, we have proved that either \(aq = 0\) or \(q \in C\).

Let \(aq = 0\). In this case, assume that there exists \(v \neq 0\), such that \(\{v, qv\}\) is linear \(K\)-independent. By the density of \(R\), there exists \(r \in R\) such that

\(rv = 0; \quad r(qv) = v + qv.\)

Hence

\(0 = a(br^{2n} + r^n(c + p)r^n + r^{2n}q)v = av.\)

Thus by the same argument as above, this implies either \(a = 0\) or \(q \in C\). Since \(a \neq 0\), \(q \in C\).

Thus in any case we conclude that \(q \in C\).

Then (15) reduces to

\[a((b + q)x^n + x^n(c + p))x^n = 0.\]

Let there exists \(v \neq 0\), such that \(\{v, (c + p)v\}\) is linear \(K\)-independent. By the density of \(R\), there exists \(r \in R\) such that

\(rv = 0; \quad r((c + p)v) = (c + p)v.\)

Hence

\(0 = a((b + q)r^n + r^n(c + p)r^n)v = a(c + p)v.\)
Then again by same argument, \( c + p \in C \). Then (21) reduces to
\[
\begin{align*}
(22) \\
a(b + c + p + q)x^{2n} &= 0
\end{align*}
\]
for all \( x \in R \). This implies \( a(b + c + p + q) = 0 \), where \( q, c + p \in C \). Hence
\[
H(x) = bx + xc = bx + x(c + p) - xp = (b + c + p)x - xp = (b + c + p + q)x - x(p + q)
\]
for all \( x \in R \) and \( G(x) = (p + q)x \) for all \( x \in R \). This gives our conclusion.

Next assume that \( H(x) = bx + d(x) \) and \( G(x) = px + \delta(x) \), where \( d, \delta \) are not both inner derivations of \( R \). In this case by our hypothesis, \( R \) satisfies
\[
(23) \\
a(bx^{2n} + d(x^n)x^n + x^npx^n + x^n\delta(x^n)) = 0.
\]
If \( d \) and \( \delta \) are \( C \)-dependent modulo inner derivations of \( R \), then \( d = \lambda \delta + ad_c \) for some \( \lambda \in C \). In this case (23) reduces to
\[
(24) \\
a(bx^{2n} + \lambda \delta(x^n)x^n + [c, x^n]x^n + x^npx^n + x^n\delta(x^n)) = 0.
\]

By Kharchenko’s Theorem [11], \( R \) satisfies
\[
(25) \\
a\left(bx^{2n} + \lambda \sum_i x^iyx^{n-i}x^n + [c, x^n]x^n + x^npx^n + x^n\sum_i x^iyx^{n-i-1}\right) = 0.
\]
Replacing \( y \) with \([p, x]\) for some \( p \notin C \), we have from (25) that
\[
(26) \\
a\left(bx^{2n} + \lambda[p, x^n]x^n + [c, x^n]x^n + x^npx^n + x^n[p, x^n]\right) = 0.
\]
Then this implies as above (for inner derivation case) that \( p \in C \), a contradiction.

The case when \( \delta = \lambda d + ad_c \) for some \( \lambda \in C \), is similar.

Next assume that \( d \) and \( \delta \) are \( C \)-independent modulo inner derivations of \( R \). Then by Kharchenko’s Theorem [11], \( R \) satisfies
\[
(27) \\
a\left(bx^{2n} + \sum_i x^iyx^{n-i}x^n + x^npx^n + x^n\sum_i x^izx^{n-i-1}\right) = 0.
\]
Replacing \( y \) with \([p, x]\) and \( z \) with \([p', x]\) for some \( p, p' \notin C \), we have
\[
(28) \\
a\left(bx^{2n} + [p, x^n]x^n + x^npx^n + x^n[p', x^n]\right) = 0.
\]
Then by same argument as above, it yields that \( p' \in C \), a contradiction.

In particular, when \( H \) and \( G \) are two derivations of \( R \), we have the following:

**Corollary 2.4.** Let \( R \) be a noncommutative prime ring with char \((R) \neq 2 \) and \( C \) the extended centroid of \( R \). Let \( d \) and \( \delta \) be two derivations of \( R \). If there exists \( 0 \neq a \in R \) such that \( a(d(x^n)x^n + x^n\delta(x^n)) = 0 \) for all \( x \in R \), where \( n \geq 1 \) is a fixed integer, then \( d = \delta = 0 \).
3. Results on semiprime rings

In this section we extend the Corollary 2.4 to semiprime rings. Let $R$ be a semiprime ring and $U$ the left Utumi ring of quotients of $R$. Then $C = Z(U)$, center of $U$, is called extended centroid of $R$. It is well known that $C$ is a Von Neumann regular ring. It is known that $C$ is a field if and only if $R$ is a prime ring. The set of all idempotents of $C$ is denoted by $E$. The elements of $E$ are called central idempotents.

We know that any derivation of $R$ can be uniquely extended to a derivation of $U$ (see [16, Lemma 2]).

By using the standard theory of orthogonal completions for semiprime rings, we prove the following:

**Theorem 3.1.** Let $R$ be a noncommutative 2-torsion free semiprime ring, $U$ the left Utumi quotient ring of $R$ and $d$, $\delta$ be two derivations of $R$. If there exists $0 \neq a \in R$ such that $a(d(x^n)x^n + x^n\delta(x^n)) = 0$ for all $x \in R$, where $n \geq 1$ is a fixed integer, then there exist orthogonal central idempotents $e_1, e_2, e_3 \in U$ with $e_1 + e_2 + e_3 = 1$ such that $(d + \delta)(e_1U) = 0$, $e_2a = 0$, and $e_3U$ is commutative.

**Proof.** Since any derivation $d$ can be uniquely extended to a derivation in $U$, and $U$ and $R$ satisfy the same differential identities (see [16]), $a(d(x^n)x^n + x^n\delta(x^n)) = 0$ for all $x \in U$.

Let $B$ be the complete Boolean algebra of $E$. We choose a maximal ideal $P$ of $B$ such that $U/P$ is 2-torsion free. Then $PU$ is a prime ideal of $U$, which is $d$-invariant. Denote $\overline{U} = U/PU$ and $\overline{d}, \overline{\delta}$ be the canonical pair of derivations on $\overline{U}$ induced by $d$ and $\delta$ respectively. Then by hypothesis, $\overline{a}(\overline{d}(\overline{x^n})\overline{x^n} + \overline{x^n}\overline{\delta(x^n)}) = 0$ for all $x \in \overline{U}$. Since $\overline{U}$ is a prime ring, by Corollary 2.4, either $\overline{d} = \overline{\delta} = 0$ or $[\overline{U}, \overline{U}] = 0$ or $\overline{a} = 0$. In any case, we have $ad(U)[U, U] \subseteq PU$ and $ad(U)[U, U] \subseteq PU$ for all $P$, that is, $aD(U)[U, U] \subseteq PU$ for all $P$, where $D = d + \delta$. Since $\bigcap \{ PU : P \text{ is any maximal ideal in } B \}$ is the set of all central idempotents $e_1, e_2, e_3 \in U$ with $e_1 + e_2 + e_3 = 1$ such that $D(e_1U) = 0$, $e_2a = 0$, and $e_3U$ is commutative.

By using the theory of orthogonal completion for semiprime rings (see, [1, Chapter 3]), it follows that there exist orthogonal central idempotents $e_1, e_2, e_3 \in U$ with $e_1 + e_2 + e_3 = 1$ such that $D(e_1U) = 0$, $e_2a = 0$, and $e_3U$ is commutative.

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