Nonlinear $\ast$-Lie higher derivations of standard operator algebras

Mohammad Ashraf, Shakir Ali, Bilal Ahmad Wani

Abstract. Let $\mathcal{H}$ be an infinite-dimensional complex Hilbert space and $\mathfrak{A}$ be a standard operator algebra on $\mathcal{H}$ which is closed under the adjoint operation. It is shown that every nonlinear $\ast$-Lie higher derivation $D = \{\delta_n\}_{n \in \mathbb{N}}$ of $\mathfrak{A}$ is automatically an additive higher derivation on $\mathfrak{A}$. Moreover, $D = \{\delta_n\}_{n \in \mathbb{N}}$ is an inner $\ast$-higher derivation.

1 Introduction

Let $\mathfrak{A}$ be an algebra over a commutative ring $R$. Recall that an $R$-linear mapping $d: \mathfrak{A} \to \mathfrak{A}$ is called a derivation if $d(AB) = d(A)B + Ad(B)$ for all $A, B \in \mathfrak{A}$; in particular, $d$ is called an inner derivation if there exists some $X \in \mathfrak{A}$ such that $d(A) = AX -XA$ for all $A \in \mathfrak{A}$. An $R$-linear mapping $d: \mathfrak{A} \to \mathfrak{A}$ is called a Lie derivation if $d([A,B]) = [d(A),B] + [A,d(B)]$ for all $A, B \in \mathfrak{A}$, where $[A,B] = AB - BA$ is the usual Lie product. Furthermore, without linearity/additivity assumption, if $d$ satisfies $d([A,B]) = [d(A),B] + [A,d(B)]$ for all $A, B \in \mathfrak{A}$, then $d$ is called a nonlinear Lie derivation. The question of characterizing Lie derivations and revealing the relationship between derivations and Lie derivations have been studied by many authors (see [1], [2], [5], [6], [7], [8], [11], [12], [15], [18]).

2010 MSC: 47B47, 16W25, 46K15.

Key words: Nonlinear $\ast$-Lie derivation, nonlinear $\ast$-Lie higher derivation, additive $\ast$-higher derivation, standard operator algebra.

Affiliation:
Mohammad Ashraf – Department of Mathematics, Aligarh Muslim University, Aligarh-202002 India
E-mail: mashraf80@hotmail.com

Shakir Ali – Department of Mathematics, Aligarh Muslim University, Aligarh-202002 India
E-mail: shakir50@rediffmail.com

Bilal Ahmad Wani – Department of Mathematics, Aligarh Muslim University, Aligarh-202002 India
E-mail: bilalwanikmr@gmail.com
Let $\mathfrak{A}$ be an associative $*$-algebra over the complex field $\mathbb{C}$. A mapping $d: \mathfrak{A} \to \mathfrak{A}$ is said to be an additive $*$-derivation if it is an additive derivation and satisfies $d(A)^* = d(A^*)$ for all $A \in \mathfrak{A}$. Further, if $d: \mathfrak{A} \to \mathfrak{A}$ is a map (not necessarily linear) which satisfies $d([A,B]^*) = [d(A)^*, B] + [A, d(B)]^*$ for all $A, B \in \mathfrak{A}$, where $[A, B]^* = AB - BA^*$, then $d$ is known as a nonlinear $*$-Lie derivation of $\mathfrak{A}$.

In [16] Yu and Zhang showed that every nonlinear $*$-Lie derivation from a factor von Neumann algebra on an infinite-dimensional Hilbert space into itself is an additive $*$-derivation. It is to be noted that a factor von Neumann algebra is a von Neumann algebra whose centre is trivial. In [4] Wu Jing proved that every nonlinear $*$-Lie derivation on standard operator algebra is automatically linear. Moreover, it is an inner $*$-derivation.

Let us recall some basic facts related to Lie higher derivations and $*$-Lie higher derivations of an associative algebra. Many different kinds of higher derivations, which consist of a family of some additive mappings, have been widely studied in commutative and noncommutative rings. Let $\mathbb{N}$ be the set of non-negative integers and $D = \{d_n\}_{n \in \mathbb{N}}$ be a family of linear mappings $d_n: \mathfrak{A} \to \mathfrak{A}$ such that $d_0 = \text{id}_{\mathfrak{A}}$, the identity map on $\mathfrak{A}$. Then $D$ is called

(i) a higher derivation on $\mathfrak{A}$ if for every $n \in \mathbb{N}$,

$$d_n(AB) = \sum_{i+j=n} d_i(A)d_j(B)$$

for all $A, B \in \mathfrak{A}$.

(ii) a Lie higher derivation on $\mathfrak{A}$ if for every $n \in \mathbb{N}$,

$$d_n([A, B]) = \sum_{i+j=n} [d_i(A), d_j(B)]$$

for all $A, B \in \mathfrak{A}$.

(iii) a $*$-Lie higher derivation on $\mathfrak{A}$ if for every $n \in \mathbb{N}$,

$$d_n([A, B]^*) = \sum_{i+j=n} [d_i(A), d_j(B)]^*$$

for all $A, B \in \mathfrak{A}$.

(iv) an inner higher derivation on $\mathfrak{A}$ if there exist two sequences $\{X_n\}_{n \in \mathbb{N}}$ and $\{Y_n\}_{n \in \mathbb{N}}$ in $\mathfrak{A}$ satisfying the conditions

$$X_0 = Y_0 = 1 \quad \text{and} \quad \sum_{i=0}^n X_iY_{n-i} = \delta_{n0} = \sum_{i=0}^n Y_iX_{n-i}$$

such that $d_n(A) = \sum_{i=0}^n X_iAY_{n-i}$, for all $A \in \mathfrak{A}$ and for every $n \in \mathbb{N}$, where $\delta_{n0}$ is the Kronecker sign.
If the linear assumption in the above definitions is dropped, then the corresponding higher derivation, Lie higher derivation and *-Lie higher derivation is said to be nonlinear higher derivation, nonlinear Lie higher derivation and nonlinear *-Lie higher derivation respectively. Moreover, if \( \mathcal{D} = \{d_n\}_{n \in \mathbb{N}} \) is assumed to be the family of additive mappings, then in the above definition higher derivation, Lie higher derivation and *-Lie higher derivation is said to be additive higher derivation, additive Lie higher derivation and additive *-Lie higher derivation respectively. Note that \( d_1 \) is always a derivation, Lie derivation and *-Lie derivation if \( \mathcal{D} = \{d_n\}_{n \in \mathbb{N}} \) is a higher derivation, Lie higher derivation and *-Lie higher derivation respectively.

The objective of this article is to investigate nonlinear *-Lie higher derivations on standard operator algebras which are closed under adjoint operation in infinite-dimensional complex Hilbert spaces. Many researchers have made important contributions to the related topics (see [3], [9], [13]). Xiao [14] proved that every nonlinear Lie higher derivation of triangular algebras is the sum of an additive higher derivation and a nonlinear functional vanishing on all commutators. Qi and Hou [10] gave a characterization of Lie higher derivations on nest algebras. Zhang et al., [17] showed that every nonlinear *-Lie higher derivation on factor von Neumann algebra is linear. Motivated by the above work in this article, we study nonlinear *-Lie higher derivations on standard operator algebras.

2 Nonlinear *-Lie higher derivations

Throughout this paper, \( \mathbb{R} \) and \( \mathbb{C} \) represents the set of real numbers and complex numbers respectively and \( \mathcal{H} \) represents a complex Hilbert space. By \( \mathcal{B}(\mathcal{H}) \) we mean the algebra of all bounded linear operators on \( \mathcal{H} \). Denote by \( \mathcal{F}(\mathcal{H}) \) the subalgebra of bounded finite rank operators. It is to be noted that \( \mathcal{F}(\mathcal{H}) \) forms a *-closed ideal in \( \mathcal{B}(\mathcal{H}) \). An algebra \( \mathfrak{A} \subset \mathcal{B}(\mathcal{H}) \) is said to be standard operator algebra in case \( \mathcal{F}(\mathcal{H}) \subset \mathfrak{A} \). An operator \( P \in \mathcal{B}(\mathcal{H}) \) is said to be a projection provided \( P^* = P \) and \( P^2 = P \). Note that, different from von Neumann algebras which are always weakly closed, a standard operator algebra is not necessarily closed. Recall that an algebra \( \mathfrak{A} \) is prime if \( \mathfrak{A} AB = 0 \) implies either \( A = 0 \) or \( B = 0 \). It is to be noted that any standard operator algebra is prime, which is a consequence of Hahn-Banach theorem. Motivated by the work of Jing [4], we have obtained the following main result.

**Theorem 1.** Let \( \mathcal{H} \) be an infinite-dimensional complex Hilbert space and \( \mathfrak{A} \) be a standard operator algebra on \( \mathcal{H} \) containing identity operator \( I \). If \( \mathfrak{A} \) is closed under the adjoint operation, then every nonlinear *-Lie higher derivation \( \mathcal{D} = \{d_n\}_{n \in \mathbb{N}} \) from \( \mathfrak{A} \) to \( \mathcal{B}(\mathcal{H}) \) is an additive *-higher derivation.

Now take a projection \( P_1 \in \mathfrak{A} \) and let \( P_2 = I - P_1 \). We write \( \mathfrak{A}_{jk} = P_j \mathfrak{A} P_k \) for \( j, k = 1, 2 \). Then by Peirce decomposition of \( \mathfrak{A} \) we have \( \mathfrak{A} = \mathfrak{A}_{11} \oplus \mathfrak{A}_{12} \oplus \mathfrak{A}_{21} \oplus \mathfrak{A}_{22} \). Note that any operator \( A \in \mathfrak{A} \) can be expressed as \( A = A_{11} + A_{12} + A_{21} + A_{22} \), and \( A_{jk} \in \mathfrak{A}_{jk} \) for any \( A_{jk} \in \mathfrak{A}_{jk} \).

We facilitate our discussion with the following known results.

**Lemma 1.** [4, Lemma 2.1] Let \( \mathfrak{A} \) be a standard operator algebra containing identity operator \( I \) in a complex Hilbert space which is closed under the adjoint operation. If \( AB = BA^* \) holds true for all \( B \in \mathfrak{A} \), then \( A \in \mathbb{R}I \).
Lemma 2. [4, Proposition 2.7] Let \( \mathfrak{A} \) be a standard operator algebra containing identity operator \( I \) in a complex Hilbert space which is closed under the adjoint operation. For any \( A \in \mathfrak{A} \),

(i) \( [iP_1, A]_* = 0 \) implies \( A_{11} = A_{12} = A_{21} = 0 \).

(ii) \( [iP_2, A]_* = 0 \) implies \( A_{12} = A_{21} = A_{22} = 0 \).

(iii) \( [i(P_2 - P_1), A]_* = 0 \) implies \( A_{11} = A_{22} = 0 \).

Now we shall use the hypothesis of Theorem 1 freely without any specific mention in proving the following lemmas.

Lemma 3. \( d_n(0) = 0 \) for each \( n \in \mathbb{N} \).

Proof. We proceed by induction on \( n \in \mathbb{N} \) with \( n \geq 1 \). If \( n = 1 \), by [4, Lemma 2.2], the result is true. Now assume that the result is true for \( k < n \), i.e., \( d_k(0) = 0 \). Our aim is to show that \( d_n \) satisfies the similar property. Observe that

\[
d_n(0) = d_n([0,0]_*) = \sum_{i+j=n} [d_i(0),d_j(0)]_* = [d_n(0),0]_* + [0,d_n(0)]_* = 0.
\]

Lemma 4. \( d_n \) has the following properties:

(i) For any \( \lambda \in \mathbb{R} \), \( d_n(\lambda I) \in \mathbb{R}I \).

(ii) For any \( A \in \mathfrak{A} \) with \( A = A^* \), \( d_n(A) = d_n(A^*) = d_n(A)* \).

(iii) For any \( \lambda \in \mathbb{C} \), \( d_n(\lambda I) \in \mathbb{C}I \).

Proof. We proceed by induction on \( n \in \mathbb{N} \) with \( n \geq 1 \). By Lemmas 2.3, 2.4 & 2.5 of [4] the result is true for \( n = 1 \).

Assume that the result is true for \( k < n \), i.e.,

\[
d_k(\lambda I) \in \mathbb{R}I, \ d_k(A) = d_k(A^*) = d_k(A)*, \ d_k(\lambda I) \in \mathbb{C}I.
\]

Our aim is to show that \( d_n \) satisfies the similar property. By the induction hypothesis;

(i) For any \( \lambda \in \mathbb{R} \), since \( d_k(\lambda I) \in \mathbb{R}I \), i.e., \( d_k(\lambda I) = d_k(\lambda I)^* \in \mathbb{R}I \)

\[
0 = d_n([\lambda I,A]_*) = [d_n(\lambda I),A]_* + [\lambda I,d_n(A)]_* + \sum_{0<i,j\leq n-1} [d_i(\lambda I),d_j(A)]_*
\]

\[
= d_n(\lambda I)A - Ad_n(\lambda I)^*.
\]

This gives us that \( d_n(\lambda I)A = Ad_n(\lambda I)^* \). By Lemma 1, we have \( d_n(\lambda I) \in \mathbb{R}I \).
Using (i), we have for $A = A^*$

$$0 = d_n([A, I]_*) = [d_n(A), I]_* + [A, d_n(I)]_* + \sum_{0 < i, j \leq n-1} [d_i(A), d_j(I)]_*$$

$$= d_n(A) - d_n(A)^*.$$

(iii) For any $\lambda \in \mathbb{C}$ and $A \in \mathfrak{A}$ with $A = A^*$, applying (ii), we see that

$$0 = d_n([A, \lambda I]_*) = [d_n(A), \lambda I]_* + [A, d_n(\lambda I)]_* + \sum_{0 < p, q \leq n-1} [d_p(1/2iI), d_q(1/2iI)]_*$$

$$= Ad_n(\lambda I) - d_n(\lambda I)A.$$

This yields that $d_n(\lambda I)A = Ad_n(\lambda I)$ for all $A \in \mathfrak{A}$ with $A = A^*$, and hence $d_n(\lambda I) \in \mathbb{C}I$.

\[\square\]

**Lemma 5.** $d_n(1/2iI) = 0$ for each $n \in \mathbb{N}$ with $n \geq 1$ and $d_n(iA) = id_n(A)$ for all $A \in \mathfrak{A}$.

**Proof.** The result is true for $n = 1$ by [4, Lemma 2.6]. Assume that the result is true for $k < n$, i.e., $d_k(1/2iI) = 0$. Now we compute

$$d_n(-1/2I) = d_n\left(\left[\frac{1}{2}iI, \frac{1}{2}iI\right]_*\right)$$

$$= [d_n(1/2iI), 1/2iI]_* + \sum_{0 < p, q \leq n-1} [d_p(1/2iI), d_q(1/2iI)]_*$$

$$= id_n(1/2iI) + \frac{1}{2}i\left\{d_n\left(\frac{1}{2}iI\right) - d_n\left(\frac{1}{2}iI\right)^*\right\}^*.$$

Since both $d_n(-1/2I)$ and $\frac{1}{2}i\left\{d_n\left(\frac{1}{2}iI\right) - d_n\left(\frac{1}{2}iI\right)^*\right\}^*$ are self-adjoint, $id_n(1/2iI)$ is also self-adjoint, and hence it follows that

$$d_n\left(\frac{1}{2}iI\right) = -d_n\left(\frac{1}{2}iI\right)^*.$$

Thus, the above computation gives that

$$d_n\left(-\frac{1}{2}I\right) = 2id_n\left(\frac{1}{2}iI\right). \quad (1)$$

Similarly, we can obtain from the fact $[-\frac{1}{2}iI, -\frac{1}{2}iI] = 1/2I$ that $d_n(-\frac{1}{2}iI)^* = -d_n(-\frac{1}{2}iI)$ and $d_n(-\frac{1}{2}I) = -2id_n(-\frac{1}{2}iI)$. Thus $d_n(-\frac{1}{2}iI) = -d_n(\frac{1}{2}iI)$. Now we
compute

\[ d_n\left(\frac{1}{2}iI\right) = d_n\left(\left[-\frac{1}{2}iI, -\frac{1}{2}I\right]\right) \]

\[ = d_n\left(\left[-\frac{1}{2}iI, -\frac{1}{2}I\right]\right) + \sum_{0 < p, q \leq n-1} d_p\left(-\frac{1}{2}iI\right), d_q\left(\frac{1}{2}I\right) \]

\[ = -id_n\left(-\frac{1}{2}iI\right) - id_n\left(-\frac{1}{2}I\right) = d_n\left(\frac{1}{2}iI\right) - id_n\left(-\frac{1}{2}I\right). \]

It follows that \( d_n\left(-\frac{1}{2}I\right) = 0 \), and so, by the equality (1), we have \( d_n\left(\frac{1}{2}iI\right) = 0 \).

Now, for any \( A \in \mathfrak{A} \), we have by induction hypothesis

\[ d_n(iA) = d_n\left(\left[\frac{1}{2}iI, A\right]\right) \]

\[ = \left[d_n\left(\frac{1}{2}iI\right), -A\right] + \left[\frac{1}{2}iI, d_n(A)\right] + \sum_{0 < p, q \leq n-1} d_p\left(\frac{1}{2}iI\right), d_q(A) \]

\[ = id_n(A). \]

\[ \square \]

**Lemma 6.** For any \( A_{12} \in \mathfrak{A}_{12} \) and \( B_{21} \in \mathfrak{A}_{21} \),

\[ d_n(A_{12} + B_{21}) = d_n(A_{12}) + d_n(B_{21}). \]

**Proof.** We proceed by induction on \( n \in \mathbb{N} \) with \( n \geq 1 \). By [4, Lemma 2.8] the result is true for \( n = 1 \).

Assume that the result is true for \( k < n \), i.e., \( d_k(A_{12} + B_{21}) = d_k(A_{12}) + d_k(B_{21}) \).

Let \( M = d_n(A_{12} + B_{21}) - d_n(A_{12}) - d_n(B_{21}) \). We now show that \( M = 0 \).

By the induction hypothesis, we have

\[ 0 = d_n([i(P_2 - P_1), A_{12} + B_{21}],) \]

\[ = [d_n(i(P_2 - P_1)), A_{12} + B_{21}] + [i(P_2 - P_1), d_n(A_{12} + B_{21})] \]

\[ + \sum_{0 < r, s \leq n-1} [d_r(i(P_2 - P_1)), d_s(A_{12} + B_{21})] \]

\[ = [d_n(i(P_2 - P_1)), A_{12} + B_{21}] + [i(P_2 - P_1), d_n(A_{12} + B_{21})] \]

\[ + \sum_{0 < r, s \leq n-1} [d_r(i(P_2 - P_1)), d_s(A_{12}) + d_s(B_{21})]. \]
On the other hand,

\[
0 = d_n([i(P_2 - P_1), A_{12}]_*) + d_n([i(P_2 - P_1), B_{21}]_*)
\]

\[
= [d_n(i(P_2 - P_1)), A_{12}]_* + [i(P_2 - P_1), d_n(A_{12})]_*
\]

\[
+ \sum_{r+s=n, 0<r,s\leq n-1} [d_r(i(P_2 - P_1)), d_s(A_{12})]_* + [d_n(i(P_2 - P_1)), B_{21}]_*
\]

\[
+ [i(P_2 - P_1), d_n(B_{21})]_* + \sum_{r+s=n, 0<r,s\leq n-1} [d_r(i(P_2 - P_1)), d_s(B_{21})]_*
\]

\[
= [d_n(i(P_2 - P_1)), A_{12} + B_{21}]_* + [i(P_2 - P_1), d_n(A_{12}) + d_n(B_{21})]_*
\]

\[
+ \sum_{r+s=n, 0<r,s\leq n-1} [d_r(i(P_2 - P_1)), d_s(A_{12}) + d_s(B_{21})]_*.
\]

Comparing the above two equations, we arrive at \([i(P_2 - P_1), M]_* = 0\). It follows from Lemma 2 that \(M_{11} = M_{22} = 0\). Now we calculate \(d_n(A_{12} - A_{12}^*)\) in two ways

\[
d_n(A_{12} - A_{12}^*) = d_n([A_{12} + B_{21}, P_2]_*)
\]

\[
= [d_n(A_{12} + B_{21}), P_2]_* + [A_{12} + B_{21}, d_n(P_2)]_*
\]

\[
+ \sum_{r+s=n, 0<r,s\leq n-1} [d_r(A_{12} + B_{21}), d_s(P_2)]_*
\]

\[
= [d_n(A_{12} + B_{21}), P_2]_* + [A_{12} + B_{21}, d_n(P_2)]_*
\]

\[
+ \sum_{r+s=n, 0<r,s\leq n-1} [d_r(A_{12}) + d_r(B_{21}), d_s(P_2)]_*.
\]

On the other hand,

\[
d_n(A_{12} - A_{12}^*) = d_n([A_{12}, P_2]_*) + d_n([B_{21}, P_2]_*)
\]

\[
= [d_n(A_{12}), P_2]_* + [A_{12}, d_n(P_2)]_*
\]

\[
+ \sum_{r+s=n, 0<r,s\leq n-1} [d_r(A_{12}), d_s(P_2)]_*
\]

\[
+ [d_n(B_{21}), P_2]_* + [B_{21}, d_n(P_2)]_*
\]

\[
+ \sum_{r+s=n, 0<r,s\leq n-1} [d_r(B_{21}), d_s(P_2)]_*
\]

\[
= [d_n(A_{12}) + d_n(B_{21}), P_2]_* + [A_{12} + B_{21}, d_n(P_2)]_*
\]

\[
+ \sum_{r+s=n, 0<r,s\leq n-1} [d_r(A_{12}) + d_r(B_{21}), d_s(P_2)]_*.
\]
The above two identities give us that $[M, P_2]_* = 0$. But

$$[M, P_2]_* = MP_2 - P_2 M^* = (M_{12} + M_{21}) P_2 - P_2 (M_{12}^* + M_{21}^*) = M_{12} - M_{12}^*.$$ 

Hence it follows that $M_{12} = 0$.

Similarly, using the fact that

$$d_n(B_{12} - B_{21}^*) = d_n([A_{12} + B_{21}, P_1])_* = d_n([A_{12}, P_1])_* + d_n([B_{21}, P_1])_*,$$

one can show that $M_{21} = 0$.

$\square$

**Lemma 7.** For any $A_{11} \in \mathfrak{A}_{11}$, $B_{12} \in \mathfrak{A}_{12}$, $C_{21} \in \mathfrak{A}_{21}$ and $D_{22} \in \mathfrak{A}_{22}$;

(i) $d_n(A_{11} + B_{12} + C_{21}) = d_n(A_{11}) + d_n(B_{12}) + d_n(C_{21})$.

(ii) $d_n(B_{12} + C_{21} + D_{22}) = d_n(B_{12}) + d_n(C_{21}) + d_n(D_{22})$.

**Proof.** (i) We proceed by induction on $n \in \mathbb{N}$ with $n \geq 1$. By [4, Lemma 2.9] the result is true for $n = 1$.

Assume that the result is true for $k < n$, that is,

$$d_k(A_{11} + B_{12} + C_{21}) = d_k(A_{11}) + d_k(B_{12}) + d_k(C_{21}).$$

Let

$$M = d_n(A_{11} + B_{12} + C_{21}) - d_n(A_{11}) - d_n(B_{12}) - d_n(C_{21}).$$

We now show that $M = 0$.

By the induction hypothesis, we have by Lemma 6,

$$d_n(iB_{12}) + d_n(iC_{21}) = d_n(iB_{12} + iC_{21}) = d_n([iP_2, A_{11} + B_{12} + C_{21}]_*),$$

$$= [d_n(iP_2), A_{11} + B_{12} + C_{21}]_* + [iP_2, d_n(A_{11} + B_{12} + C_{21})]_* + \sum_{r+s=n}^{0<r,s\leq n-1} [d_r(iP_2), d_s(A_{11} + B_{12} + C_{21})]_*$$

$$= [d_n(iP_2), A_{11} + B_{12} + C_{21}]_* + [iP_2, d_n(A_{11} + B_{12} + C_{21})]_* + \sum_{r+s=n}^{0<r,s\leq n-1} [d_r(iP_2), d_s(A_{11}) + d_s(B_{12}) + d_s(C_{21})]_*.$$
Proof. Assume that the result is true for \(k < n\), i.e.,
\[
d_k(A_{11} + B_{12} + C_{21} + D_{22}) = d_k(A_{11}) + d_k(B_{12}) + d_k(C_{21}) + d_k(D_{22}).
\]

Our aim is to show that the result is true for every \(n \in \mathbb{N}\). Let
\[
M = d_n(A_{11} + B_{12} + C_{21} + D_{22}) - d_n(A_{11}) - d_n(B_{12}) - d_n(C_{21}) - d_n(D_{22}).
\]

On the other hand, we have
\[
d_n(iB_{12}) + d_n(iC_{21}) = d_n([iP_2, A_{11}]) + d_n([iP_2, B_{21}]) + d_n([iP_2, C_{21}])
\]
\[
= [d_n(iP_2), A_{11}] + [iP_2, d_n(A_{11})] + \sum_{0 < r, s \leq n-1} [d_r(iP_2), d_s(A_{11})]
\]
\[
+ [d_n(iP_2), B_{12}] + [iP_2, d_n(B_{12})] + \sum_{0 < r, s \leq n-1} [d_r(iP_2), d_s(B_{12})]
\]
\[
+ [d_n(iP_2), C_{21}] + [iP_2, d_n(C_{21})] + \sum_{0 < r, s \leq n-1} [d_r(iP_2), d_s(C_{21})]
\]
\[
= [d_n(iP_2), A_{11} + B_{12} + C_{21}] + [iP_2, d_n(A_{11}) + d_n(B_{12}) + d_n(C_{21})]
\]
\[
+ \sum_{0 < r, s \leq n-1} [d_r(iP_2), d_s(A_{11}) + d_s(B_{12}) + d_s(C_{21})].
\]

Comparing the above two equalities, we have \([iP_2, M] = 0\) and hence it follows from Lemma 2 (ii), that \(M_{12} = M_{21} = M_{22} = 0\).

We now show that \(M_{11} = 0\). Note that
\[
[i(P_2 - P_1), B_{12}] = [i(P_2 - P_1), C_{21}] = 0.
\]

We have
\[
d_n([i(P_2 - P_1), A_{11} + B_{12} + C_{21}]) = d_n([i(P_2 - P_1), A_{11}])
\]
\[
+ d_n([i(P_2 - P_1), B_{12}]) + d_n([i(P_2 - P_1), C_{21}]).
\]

Using the similar arguments as used above, we get \([i(P_2 - P_1), M] = 0\). Therefore by Lemma 2, \(M_{11} = 0\). Hence we are done.

(ii) Considering \(d_n([iP_1, B_{12} + C_{21} + D_{22}])\) and \(d_n([i(P_2 - P_1), B_{12} + C_{21} + D_{22}])\), with the similar argument as in (i), one can obtain
\[
d_n(B_{12} + C_{21} + D_{22}) = d_n(B_{12}) + d_n(C_{21}) + d_n(D_{22}).
\]
Note that \([iP_1, D_{22}]_* = 0\), by induction hypothesis, we have
\[
d_n([iP_1, A_{11} + B_{12} + C_{21} + D_{22}]_*) = [d_n(iP_1), A_{11} + B_{12} + C_{21} + D_{22}]_*
\]
\[
+ [iP_1, d_n(A_{11} + B_{12} + C_{21} + D_{22})]_*
\]
\[
+ \sum_{r+s=n}^{r+s<n} [d_r(iP_1), d_s(A_{11} + B_{12} + C_{21} + D_{22})]_*
\]
\[
= [d_n(iP_1), A_{11} + B_{12} + C_{21} + D_{22}]_*
\]
\[
+ [iP_1, d_n(A_{11} + B_{12} + C_{21})]_*
\]
\[
+ \sum_{r+s<n}^{r+s=n} [d_r(iP_1), d_s(A_{11} + B_{12} + C_{21})]_*
\]
\[
+ [d_n(iP_1), D_{22}]_* + [iP_1, d_n(D_{22})]_*
\]
\[
+ \sum_{r+s<n}^{r+s=n} [d_r(iP_1), d_s(D_{22})]_*
\]
\[
= [d_n(iP_1), A_{11} + B_{12} + C_{21}]_*
\]
\[
+ [iP_1, d_n(A_{11}) + d_n(B_{12}) + d_n(C_{21})]_*
\]
\[
+ \sum_{r+s<n}^{r+s=n} [d_r(iP_1), d_s(A_{11}) + d_s(B_{12}) + d_s(C_{21})]_*
\]
\[
+ [d_n(iP_1), D_{22}]_* + [iP_1, d_n(D_{22})]_*
\]
\[
+ \sum_{r+s<n}^{r+s=n} [d_r(iP_1), d_s(D_{22})]_*
\]

Comparing the above two equalities, it follows that \([iP_1, M] = 0\), and hence by Lemma 2, \(M_{11} = M_{12} = M_{21} = 0\). Using the fact that \([iP_2, A_{11}] = 0\) and the above similar arguments, we obtain \([iP_2, M]_* = 0\) which leads to \(M_{22} = 0\). This completes the proof. 

**Lemma 9.** For any \(A_{jk}, B_{jk} \in \mathcal{A}_{jk}\), where \(j, k \in 1, 2\), we have
\[
d_n(A_{jk} + B_{jk}) = d_n(A_{jk}) + d_n(B_{jk})
\]
Proof. We separate the proof in two distinct cases.

**Case I:** \( j \neq k \)

On one side, by Lemma 8, we have

\[
d_n(iA_{jk} + iB_{jk} + iA^*_{jk} + iB_{jk}A^*_{jk}) = d_n(iA_{jk} + iB_{jk}) + d_n(iA^*_{jk}) + d_n(iB_{jk}A^*_{jk}).
\]

On the other hand, using Lemmas 6 and 8, by induction, we have

\[
d_n(iA_{jk} + iB_{jk} + iA^*_{jk} + iB_{jk}A^*_{jk}) = d_n([iP_j + iA_{jk}, P_k + B_{jk}]_*)
\]

\[
= [d_n(iP_j + iA_{jk}), P_k + B_{jk}]_* + [iP_j + iA_{jk}, d_iP_k + B_{jk}]_*
\]

\[
= [d_n(iP_j) + d_n(iA_{jk}), P_k + B_{jk}]_*
\]

\[
+ [iP_j + iA_{jk}, d_iP_k + d_n(B_{jk})]^* 
\]

\[
+ \sum_{r+s=n \atop 0 < r, s \leq n-1} [d_r(iP_j + iA_{jk}), d_s(P_k) + d_s(B_{jk})]^* 
\]

\[
= d_n([iP_j, P_k]_*) + d_n([iP_j, B_{jk}]_*)
\]

\[
+ d_n([iA_{jk}, P_k]_* ) + d_n([iA_{jk}, B_{jk}]_*)
\]

\[
= d_n(iB_{jk}) + d_n(iA_{jk} + iA^*_{jk} + d_n(iB_{jk}A^*_{jk})
\]

\[
= d_n(iB_{jk}) + d_n(iA_{jk}) + d_n(iA^*_{jk}) + d_n(iB_{jk}A^*_{jk}).
\]

Comparing the above two equalities, we can conclude that

\[
d_n(A_{jk} + B_{jk}) = d_n(A_{jk}) + d_n(A^*_{jk}).
\]

**Case II:** \( j = k \)

Let \( A_{jj}, B_{jj} \in \mathfrak{A}_{jj} \) and \( n \in \{1, 2\} \) with \( n \neq j \). We have

\[
0 = d_n([iP_n, A_{jj} + B_{jj}]_*)
\]

\[
= [d_n(iP_n), A_{jj} + B_{jj}]_* + [iP_n, d_n(A_{jj} + B_{jj})]_*
\]

\[
+ \sum_{r+s=n \atop 0 < r, s \leq n-1} [d_r(iP_n), d_s(A_{jj} + B_{jj})]_*
\]

\[
= [d_n(iP_n), A_{jj} + B_{jj}]_* + [iP_n, d_n(A_{jj} + B_{jj})]_*
\]

\[
+ \sum_{r+s=n \atop 0 < r, s \leq n-1} [d_r(iP_n), d_s(A_{jj} + B_{jj})]_*.
\]
On the other hand we have,

\[
0 = d_n([iP_n, A_{jj}],_*) + d_n([iP_n, B_{jj}],_*) \\
= [d_n(iP_n), A_{jj}]_* + [iP_n, d_n(A_{jj})]_* + \sum_{0 < r, s \leq n - 1} [d_r(iP_n), d_s(A_{jj})]_* \\
+ [d_n(iP_n), B_{jj}]_* + [iP_n, d_n(B_{jj})]_* + \sum_{0 < r, s \leq n - 1} [d_r(iP_n), d_s(B_{jj})]_* \\
= [d_n(iP_n), A_{jj} + B_{jj}]_* + [iP_n, d_n(A_{jj}) + d_n(B_{jj})]_* \\
+ \sum_{0 < r, s \leq n - 1} [d_r(iP_n), d_s(A_{jj}) + d_s(B_{jj})]_*.
\]

Take \( M = d_n(A_{jj} + B_{jj}) - d_n(A_{jj}) - d_n(B_{jj}) \). The above computation yields that \([iP_n, M]_* = 0\). By Lemma 2, we have \( M_{nj} = M_{jn} = M_{nn} = 0\). We now show that \( M_{jj} = 0\). For any \( C_{jn} \in \mathfrak{A}_{jn} \), using Case I, we compute

\[
d_n([A_{jj} + B_{jj}, C_{jn}],_*) = [d_n(A_{jj} + B_{jj}), C_{jn}]_* + [A_{jj} + B_{jj}, d_n(C_{jn})]_* \\
+ \sum_{0 < r, s \leq n - 1} [d_r(A_{jj} + B_{jj}), d_s(C_{jn})]_* \\
= [d_n(A_{jj} + B_{jj}), C_{jn}]_* + [A_{jj} + B_{jj}, d_n(C_{jn})]_* \\
+ \sum_{0 < r, s \leq n - 1} [d_r(A_{jj}) + d_r(B_{jj}), d_s(C_{jn})]_*.
\]

On the other hand, we have

\[
d_n([A_{jj} + B_{jj}, C_{jn}],_*) = d_n(A_{jj} C_{jn} + B_{jj} C_{jn}) \\
= d_n(A_{jj} C_{jn}) + d_n(B_{jj} C_{jn}) \\
= d_n([A_{jj}, C_{jn}],_*) + d_n([B_{jj}, C_{jn}],_*) \\
= [d_n(A_{jj}), C_{jn}]_* + [A_{jj} + B_{jj}, d_n(C_{jn})]_* \\
+ \sum_{0 < r, s \leq n - 1} [d_r(A_{jj}), d_s(C_{jn})]_* \\
+ [d_n(B_{jj}), C_{jn}]_* + [B_{jj} + d_n(C_{jn})]_* \\
+ \sum_{0 < r, s \leq n - 1} [d_r(B_{jj}), d_s(C_{jn})]_*.
\]

Comparing the above two equalities, we obtain \([M, C_{jn}]_* = 0\) which leads to \(M_{jj} C_{jn} = 0\). Since \( \mathfrak{A} \) is prime, we see that \( M_{jj} = 0\), which completes the proof.

\[\square\]

**Lemma 10.** \( d_n \) is an additive \(*\)-higher derivation on \( \mathfrak{A} \).
We now show that \( d_n \) is additive. For arbitrary \( A, B \in \mathfrak{A} \), we write \( A = \sum_{j,k=1}^{2} A_{jk} \) and \( B = \sum_{j,k=1}^{2} B_{jk} \). It follows from Lemmas 8 and 9 that
\[
d_n(A + B) = d_n \left\{ \sum_{j,k=1}^{2} (A_{jk} + B_{jk}) \right\}
\]
\[
= \sum_{j,k=1}^{2} d_n(A_{jk} + B_{jk})
\]
\[
= \sum_{j,k=1}^{2} \left( d_n(A_{jk}) + d_n(B_{jk}) \right)
\]
\[
= d_n \left( \sum_{j,k=1}^{2} A_{jk} \right) + d_n \left( \sum_{j,k=1}^{2} B_{jk} \right)
\]
\[
= d_n(A) + d_n(B).
\]

We now show that \( d_n(A^*) = d_n(A)^* \).

For any \( A \in \mathfrak{A} \), it follows from Lemmas 4 and 5 that
\[
d_n(A^*) = d_n(\mathfrak{R}A - i\mathfrak{S}A) = d_n(\mathfrak{R}A) - d_n(i\mathfrak{S}A)
\]
\[
= d_n(\mathfrak{R}A) - id_n(\mathfrak{S}A) = d_n(\mathfrak{R}A)^* - id_n(\mathfrak{S}A)^*
\]
\[
= d_n(\mathfrak{R}A)^* + (id_n(\mathfrak{S}A))^* = d_n(\mathfrak{R}A)^* + d_n(i\mathfrak{S}A)^*
\]
\[
= (d_n(\mathfrak{R}A + i\mathfrak{S}A))^* = d_n(A)^*.
\]

To complete the proof, we need to show that \( d_n \) is a higher derivation on \( \mathfrak{A} \).

Since \( d_n \) is additive, it follows from Lemma 5, that \( d_n(iI) = 0 \). It is to be noted that \( [iI + A, B]_* = 2iB + AB - BA^* \).

\[
d_n(2iB) + d_n(AB) - d_n(BA^*) = d_n([iI + A, B]_*)
\]
\[
= \left[ d_n(iI + A), B \right]_* + \left[ iI + A, d_n(B) \right]_* + \sum_{r+s=n \atop 0 < r, s \leq n-1} \left[ d_r(iI + A), d_s(B) \right]_*
\]
\[
= \left[ d_n(iI) + d_n(A), B \right]_* + \left[ iI + A, d_n(B) \right]_* + \sum_{r+s=n \atop 0 < r, s \leq n-1} \left[ d_r(iI) + d_r(A), d_s(B) \right]_*
\]
\[
= \left[ d_n(A), B \right]_* + \left[ iI + A, d_n(B) \right]_* + \sum_{r+s=n \atop 0 < r, s \leq n-1} \left[ d_r(A), d_s(B) \right]_*
\]
\[
= d_n(A)B - Bd_n(A)^* + 2id_n(B) + Ad_n(B) - d_n(B)A^* + \sum_{r+s=n \atop 0 < r, s \leq n-1} (d_r(A)d_s(B) - d_s(B)d_r(A)^*)..
\]
It follows that

\[ d_n(AB) - d_n(BA^*) = d_n(A)B - Bd_n(A)^* + Ad_n(B) - d_n(B)A^* + \sum_{0 < r, s \leq n-1} (d_r(A)d_s(B) - d_s(B)d_r(A)^*). \]

Replacing \( A \) by \( iA \) in the above equality, we get

\[ d_n(AB) + d_n(BA^*) = d_n(A)B + Bd_n(A)^* + Ad_n(B) + d_n(B)A^* + \sum_{0 < r, s \leq n-1} (d_r(A)d_s(B) + d_s(B)d_r(A)^*). \]

Thus we have,

\[ d_n(AB) = d_n(A)B + Ad_n(B) + \sum_{0 < r, s \leq n-1} d_r(A)d_s(B) = \sum_{r + s = n} d_r(A)d_s(B). \]

This shows that \( d_n \) is an additive higher derivation with \( d_n(A^*) = d_n(A)^* \). Hence \( d_n \) is an additive \(*\)-higher derivation on \( \mathfrak{A} \), which completes the proof.

Note that every additive derivation \( d: \mathfrak{A} \to B(H) \) is an inner derivation (see [12]). Nowicki [9] proved that if every additive (linear) derivation of \( \mathfrak{A} \) is inner, then every additive (linear) higher derivation of \( \mathfrak{A} \) is inner (see also [13]). So by Theorem 1, the following corollary is immediate.

**Corollary 1.** Let \( H \) be an infinite-dimensional complex Hilbert space and \( \mathfrak{A} \) be a standard operator algebra on \( H \) containing identity operator \( I \). If \( \mathfrak{A} \) is closed under the adjoint operation, then every nonlinear \(*\)-Lie higher derivation \( \mathcal{D} = \{ d_n \}_{n \in \mathbb{N}} \) is inner with \( d_n(A^*) = d_n(A)^* \) for each \( A \in \mathfrak{A} \) and every \( n \in \mathbb{N} \).

**Acknowledgement**

The authors are highly indebted to the referee for his/her valuable remarks which have improved the paper immensely.

**References**


Nonlinear $\ast$-Lie higher derivations of standard operator algebras


Received: 13 July, 2017
Accepted for publication: 2 February, 2018
Communicated by: Stephen Glasby