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Existence of solutions for a coupled system with ϕ-Laplacian operators and nonlinear coupled boundary conditions

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Abstract. We study the existence of solutions of the system

$$\begin{cases} (\phi_1(u_1'(t)))' = f_1(t, u_1(t), u_2(t), u_1'(t), u_2'(t)), & \text{a.e. } t \in [0,T], \\ (\phi_2(u_2'(t)))' = f_2(t, u_1(t), u_2(t), u_1'(t), u_2'(t)), & \text{a.e. } t \in [0,T], \end{cases}$$

submitted to nonlinear coupled boundary conditions on $[0,T]$ where $\phi_1, \phi_2\colon (-a,a) \to \mathbb{R}$, with $0 < a < +\infty$, are two increasing homeomorphisms such that $\phi_1(0) = \phi_2(0) = 0$, and $f_i : [0,T] \times \mathbb{R}^4 \to \mathbb{R}$, $i \in \{1,2\}$ are two L^1-Carathéodory functions. Using some new conditions and Schauder fixed point Theorem, we obtain solvability result.

1 Introduction

The aim of this paper is to study the existence of solutions for the quasilinear system of differential equations

$$\begin{cases} (\phi_1(u_1'(t)))' = f_1(t, u_1(t), u_2(t), u_1'(t), u_2'(t)), & \text{a.e. } t \in [0,T], \\ (\phi_2(u_2'(t)))' = f_2(t, u_1(t), u_2(t), u_1'(t), u_2'(t)), & \text{a.e. } t \in [0,T], \end{cases} \tag{1}$$

under nonlinear coupled boundary conditions

$$\begin{cases} g(u_1(0), u_2(0), u_1'(0), u_2'(0), u_1'(T), u_2'(T)) = (0,0) \\ (u_1(T), u_2(T)) = h(u_1(0), u_2(0)), \end{cases} \tag{2}$$

where $g\colon \mathbb{R}^6 \to \mathbb{R}^2$ and $h\colon \mathbb{R}^2 \to \mathbb{R}^2$ are two continuous functions, $f_i\colon [0,T] \times \mathbb{R}^4 \to \mathbb{R}$, $i \in \{1,2\}$ are two L^1-Carathéodory functions, and $\phi_1, \phi_2\colon (-a,a) \to \mathbb{R}$, with $0 < a < +\infty$, are two increasing homeomorphisms such that $\phi_1(0) = \phi_2(0) = 0$.

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The study of system (1)–(2) was justified by the applications of the nonlinear differential equations with singular ϕ-Laplacian operator to the areas of mechanics and physics (see [4], [5]).

In 2015, Naseer Ahmad Asif and Imran Talib studied in [1], problem (1)–(2), for f_i, $i \in \{1, 2\}$ continuous and $\phi_1(x) = \phi_2(x) = x$, $\forall x \in \mathbb{R}$. They proved under some monotony conditions upon g and h, an existence result using a new concept of coupled lower and upper solutions.

The concept of coupled lower and upper solutions was used by several authors (see [2], [6], [7], [8]).

In our study, we use some new conditions given only on the boundary conditions. These new conditions allow us to construct two coupled of ordered functions which do not check necessarily the definitions of lower and upper solutions, but which makes it possible to obtain a solution located between them. In addition monotonicity of g and h are not required.

After introducing notations and preliminaries in section 2, we prove in section 3, existence of solutions of the problem (1)–(2) using a new conditions.

In section 4, we give an example of application of this new result.

2 Notations and preliminaries

We denote:

- $\langle \cdot, \cdot \rangle$ the usual inner product in $\mathbb{R}^2$
- $\| \cdot \|$, the Euclidian norm of $\mathbb{R}^2$
- $\{e_1, e_2\}$ the canonical basis of $\mathbb{R}^2$
- $(x, y), (a, b) \in \mathbb{R}^2$, $(x, y) \le (a, b)$ if $x \le a$ and $y \le b$
- $(x, y), (a, b) \in \mathbb{R}^2$, $(x, y) \ge (a, b)$ if $x \ge a$ and $y \ge b$
- $C = C([0, T])$, the Banach space of continuous functions on $[0, T]$
- $\|u\|_C = \|u\|_\infty = \max\{|u(t)|; t \in [0, T]\}$, the norm of C
- $C^1 = C^1([0, T])$, the Banach space of continuous functions on $[0, T]$ having continuous first derivative on $[0, T]$
- $||u||_{C^1} = ||u||_C + ||u'||_C$, the norm of C^1
- $||(u_1, u_2)||_{C^1 \times C^1} = ||u_1||_{C^1} + ||u_2||_{C^1}$, the norm of $C^1 \times C^1$
- $AC = AC([0, T])$, the set of absolutely continuous functions on $[0, T]$
- $L^1 = L^1(0, T)$, the Banach space of functions Lebesgue integrable on $[0, T]$
- $|| \cdot ||_{L^1}$, the norm of L^1
- B_r, the corresponding open ball of $C^1 \times C^1$ of center 0 and radius r
- For $u = (u_1, u_2) \in C \times C$, $\forall t \in [0, T]$, $\int_0^t u(s)ds = \left(\int_0^t u_1(s)ds, \int_0^t u_2(s)ds\right)$

Definition 1. $(u_1, u_2) \in C^1 \times C^1$ is a solution of problem (1)–(2) if $\|u_1'\|_\infty < a$, $\|u_2'\|_\infty < a$, $(\phi_1 \circ u_1', \phi_2 \circ u_2') \in AC \times AC$ and satisfies (1)–(2).

Definition 2. $f : [0,T] \times \mathbb{R}^4 \to \mathbb{R}$ is L^1-Carathéodory if:

1. $f(\cdot, x_1, x_2, y_1, y_2) : [0,T] \to \mathbb{R}$ is measurable for all $(x_1, x_2, y_1, y_2) \in \mathbb{R}^4$;
2. $f(t, \cdot, \cdot, \cdot, \cdot) : \mathbb{R}^4 \to \mathbb{R}$ is continuous for a.e. $t \in [0,T]$;
3. For each compact set $A \subset \mathbb{R}^4$ there is a function $\mu_A \in L^1$ such that
$$|f(t, x_1, x_2, y_1, y_2)| \leq \mu_A(t)$$
for a.e. $t \in [0,T]$ and all $(x_1, x_2, y_1, y_2) \in A$.

3 Existence result

Theorem 1. *Assume that there exist* (α_1, α_2), $(\beta_1, \beta_2) \in C \times C$ *such that*
$$\alpha_i(0) < \beta_i(0), \text{ and } \alpha_i(T) < \beta_i(T), \quad \forall i \in \{1,2\},$$
$$\max\big\{|\beta_i(T) - \alpha_i(0)|; |\alpha_i(T) - \beta_i(0)|\big\} < aT, \quad \forall i \in \{1,2\},$$

$$\begin{aligned}
g(\alpha_1(0), t, w, x, y, z) &\geq (0,0), \quad \forall (t, w, x, y, z) \in [\alpha_2(0), \beta_2(0)] \times [-a,a]^4,\\
g(t, \alpha_2(0), w, x, y, z) &\geq (0,0), \quad \forall (t, w, x, y, z) \in [\alpha_1(0), \beta_1(0)] \times [-a,a]^4,\\
g(\beta_1(0), t, w, x, y, z) &\leq (0,0), \quad \forall (t, w, x, y, z) \in [\alpha_2(0), \beta_2(0)] \times [-a,a]^4,\\
g(t, \beta_2(0), w, x, y, z) &\leq (0,0), \quad \forall (t, w, x, y, z) \in [\alpha_1(0), \beta_1(0)] \times [-a,a]^4,\\
(\alpha_1(T), \alpha_2(T)) - h(x,y) &\leq (0,0), \quad \forall (x,y) \in [\alpha_1(0), \beta_1(0)] \times [\alpha_2(0), \beta_2(0)]
\end{aligned}$$

and

$$(\beta_1(T), \beta_2(T)) - h(x,y) \geq (0,0) \quad \forall (x,y) \in [\alpha_1(0), \beta_1(0)] \times [\alpha_2(0), \beta_2(0)].$$

Then the problem (1)–(2) *admits at least one solution* (u_1, u_2), *with*
$$\alpha_i(0) - at \leq u_i(t) \leq \beta_i(0) + at, \quad \forall t \in [0,T] \quad \text{for each} \quad i \in \{1,2\}.$$

Consider the functions $\gamma_i \colon [0,T] \times \mathbb{R} \to \mathbb{R}$, $i \in \{1,2\}$, given by
$$\gamma_i(t,x) = \max\big\{\alpha_i(0) - at, \min\{x, \beta_i(0) + at\}\big\}.$$

Let us define the functions $\vartheta_i, \theta_i \colon \mathbb{R} \to \mathbb{R}$ $i \in \{1,2\}$ by
$$\vartheta_i(x) = \max\big\{\alpha_i(0), \min\{x, \beta_i(0)\}\big\} \quad \text{and} \quad \theta_i(x) = \max\big\{\alpha_i(T), \min\{x, \beta_i(T)\}\big\}$$

Consider $\delta \colon \mathbb{R} \to \mathbb{R}$ and $F \colon C^1 \times C^1 \to L^1 \times L^1$ defined by
$$\delta(x) = \max\big\{-a, \min\{x, a\}\big\}$$

and $\forall (u_1,u_2)\in C^1\times C^1$ and a.e. $t\in[0,T]$,

$$F(u_1,u_2)(t)=\Big(f_1(t,\gamma_1(t,u_1(t)),\gamma_2(t,u_2(t)),\delta(u_1'(t)),\delta(u_2'(t))),\\ f_2(t,\gamma_1(t,u_1(t)),\gamma_2(t,u_2(t)),\delta(u_1'(t)),\delta(u_2'(t)))\Big).$$

Let $\mathcal{A},\mathcal{B}: C^1\times C^1\to\mathbb{R}^2$ given by

$$\mathcal{A}(u_1,u_2)=\Big(\vartheta_1\Big(u_1(0)+\langle g\big(u_1(0),u_2(0),u_1'(0),u_2'(0),u_1'(T),u_2'(T)\big),e_1\rangle\Big),\\ \vartheta_2\Big(u_2(0)+\langle g\big(u_1(0),u_2(0),u_1'(0),u_2'(0),u_1'(T),u_2'(T)\big),e_2\rangle\Big)\Big)$$

and

$$\mathcal{B}(u_1,u_2)=\Big(\theta_1\Big(\frac{1}{2}u_1(T)+\frac{1}{2}\langle h(u_1(0),u_2(0)),e_1\rangle\Big),\\ \theta_2\Big(\frac{1}{2}u_2(T)+\frac{1}{2}\langle h(u_1(0),u_2(0)),e_2\rangle\Big)\Big).$$

$\mathcal{A}$ and $\mathcal{B}$ are bounded and continuous in $C^1\times C^1$.

Let ϕ defined from $(-a,a)\times(-a,a)$ onto $\mathbb{R}\times\mathbb{R}$ by

$$\phi(x_1,x_2)=(\phi_1(x_1),\phi_2(x_2)).$$

ϕ is an homeomorphism from $(-a,a)\times(-a,a)$ onto $\mathbb{R}\times\mathbb{R}$ and

$$\forall (y_1,y_2)\in\mathbb{R}^2,\quad \phi^{-1}(y_1,y_2)=(\phi_1^{-1}(y_1),\phi_2^{-1}(y_2)).$$

With the previous notations, we consider the modified problem

$$\begin{cases}(\phi(u_1'(t),u_2'(t)))'=F(u_1,u_2)(t), & \text{a.e. } t\in[0,T],\\ (u_1(0),u_2(0))=\mathcal{A}(u_1,u_2),\\ (u_1(T),u_2(T))=\mathcal{B}(u_1,u_2).\end{cases}\tag{3}$$

A solution of problem (3) is a couple of functions $(u_1,u_2)\in C^1\times C^1$ such that $\|u_1'\|_\infty<a$, $\|u_2'\|_\infty<a$, $(\phi_1\circ u_1',\phi_2\circ u_2')\in AC\times AC$, and satisfies (3).

Lemma 1. *For each $(k,q)\in C\times(-aT,aT)$, $\forall i\in\{1,2\}$, there exists a unique $b_i=\Sigma_{\phi_i}(k,q)$ such that*

$$\int_0^T \phi_i^{-1}(k(t)-b_i)\,\mathrm{d}t=q.$$

Moreover, the function $\Sigma_{\phi_i}: C\times(-aT,aT)\to\mathbb{R}$ is continuous, and, for each fixed $q\in(-aT,aT)$, $\Sigma_{\phi_i}(\cdot,q)$ takes bounded sets of C into bounded sets of $\mathbb{R}$.

Proof. See [3] proof of Lemma 1. □

Lemma 2. *For every* $(v_1, v_2) \in C^1 \times C^1$ *there exists a unique* $(\tau_{v_1}, \tau_{v_2}) \in \mathbb{R} \times \mathbb{R}$ *such that*

$$\int_0^T \Big(\phi^{-1}\big[(\tau_{v_1}, \tau_{v_2}) + \int_0^t F(v_1, v_2)(s)\,\mathrm{d}s\big]\Big)\,\mathrm{d}t = \mathcal{B}(v_1, v_2) - \mathcal{A}(v_1, v_2).$$

Proof. It is an easy consequence of Lemma 1. In fact, $\forall i \in \{1, 2\}$, let

$$k_i(t) = \int_0^t \langle F(v_1, v_2)(s), e_i\rangle\,\mathrm{d}s$$

and $q_i = \langle \mathcal{B}(v_1, v_2) - \mathcal{A}(v_1, v_2), e_i\rangle$. We have, $\forall i \in \{1, 2\}$, $k_i \in C$ and

$$|q_i| \leq \big|\langle \mathcal{B}(v_1, v_2) - \mathcal{A}(v_1, v_2), e_i\rangle\big| \leq \max\big\{|\beta_i(T) - \alpha_i(0)|; |\alpha_i(T) - \beta_i(0)|\big\} < aT.$$

By Lemma (1), we get the existence of (τ_{v_1}, τ_{v_2}). □

Lemma 3. *Any solution* (u_1, u_2) *of (3) is such that* $\alpha_i(0) - at < u_i(t) < \beta_i(0) + at$, $\forall t \in [0, T]$ *and for each* $i \in \{1, 2\}$.

Proof. Let (u_1, u_2) be a solution of (3). By definition of the boundary conditions, we have

$$\alpha_i(0) \leq u_i(0) \leq \beta_i(0) \quad \text{and} \quad \alpha_i(T) \leq u_i(T) \leq \beta_i(T) \quad \forall i \in \{1, 2\}. \tag{4}$$

As $\|u_1'\|_\infty < a$ and $\|u_2'\|_\infty < a$, we have

$$u_i(0) - at < u_i(t) < u_i(0) + at, \ \forall t \in [0, T] \text{ and for each } i \in \{1, 2\}\,. \tag{5}$$

Therefore, by (4) and (5), we have $\alpha_i(0) - at \leq u_i(t) \leq \beta_i(0) + at$, $\forall t \in [0, T]$ and for each $i \in \{1, 2\}$. □

Lemma 4. *If* (u_1, u_2) *is a solution of* (3) *then*

$$g(u_1(0), u_2(0), u_1'(0), u_2'(0), u_1'(T), u_2'(T)) = (0, 0)$$

and

$$(u_1(T), u_2(T)) - h(u_1(0), u_2(0)) = (0, 0).$$

Proof. Let (u_1, u_2) be a solution of problem (3).
Step 1: $(u_1(T), u_2(T)) - h(u_1(0), u_2(0)) = (0, 0)$.
Suppose that for some $j \in \{1, 2\}$,

$$\beta_j(T) < \frac{1}{2}u_j(T) + \frac{1}{2}\big\langle h(u_1(0), u_2(0)), e_j\big\rangle.$$

Then

$$u_j(T) = \theta_j\Big(\frac{1}{2}u_j(T) + \frac{1}{2}\Big\langle h(u_1(0), u_2(0)), e_j\Big\rangle\Big) = \beta_j(T). \tag{6}$$

Therefore we obtain the contradiction

$$\begin{aligned} 0 &= \beta_j(T) - u_j(T) \\ &< \frac{1}{2}\langle h(u_1(0), u_2(0)), e_j\rangle + \frac{1}{2}u_j(T) - u_j(T) \\ &\leq \frac{1}{2}\langle h(u_1(0), u_2(0)), e_j\rangle - \frac{1}{2}\beta_j(T) \leq 0. \end{aligned}$$

It follows that

$$\big(\beta_1(T), \beta_2(T)\big) \geq \frac{1}{2}\big(u_1(T), u_2(T)\big) + \frac{1}{2}h\big(u_1(0), u_2(0)\big)\,. \tag{7}$$

Suppose that for some $j \in \{1, 2\}$,

$$\alpha_j(T) > \frac{1}{2}u_j(T) + \frac{1}{2}\big\langle h(u_1(0), u_2(0)), e_j\big\rangle.$$

Then

$$u_j(T) = \theta_j\Big(\frac{1}{2}u_j(T) + \frac{1}{2}\Big\langle h(u_1(0), u_2(0)), e_j\Big\rangle\Big) = \alpha_j(T).$$

Therefore we obtain the contradiction

$$\begin{aligned} 0 &= \alpha_j(T) - u_j(T) \\ &> \frac{1}{2}\big\langle h(u_1(0), u_2(0)), e_j\big\rangle + \frac{1}{2}u_j(T) - u_j(T) \\ &\geq \frac{1}{2}\big\langle h(u_1(0), u_2(0)), e_j\big\rangle - \frac{1}{2}\alpha_j(T) \geq 0. \end{aligned}$$

It follows that

$$\big(\alpha_1(T), \alpha_2(T)\big) \leq \frac{1}{2}\big(u_1(T), u_2(T)\big) + \frac{1}{2}h\big(u_1(0), u_2(0)\big)\,. \tag{8}$$

(7) and (8) show that

$$\big(u_1(T), u_2(T)\big) = \frac{1}{2}\big(u_1(T), u_2(T)\big) + \frac{1}{2}h\big(u_1(0), u_2(0)\big).$$

Therefore $\big(u_1(T), u_2(T)\big) - h\big(u_1(0), u_2(0)\big) = (0, 0)$.
Step 2: $g(u_1(0), u_2(0), u_1'(0), u_2'(0), u_1'(T), u_2'(T)) = (0, 0)$.
Suppose that for some $j \in \{1, 2\}$,

$$\beta_j(0) < u_j(0) + \big\langle g(u_1(0), u_2(0), u_1'(0), u_2'(0), u_1'(T), u_2'(T)), e_j\big\rangle.$$

Then

$$u_j(0) = \vartheta_j\Big(u_j(0) + \Big\langle g\Big(u_1(0), u_2(0), u_1'(0), u_2'(0), u_1'(T), u_2'(T)\Big), e_j\Big\rangle\Big) = \beta_j(0).$$

Therefore we obtain the contradiction

$$0 = \beta_1(0) - u_1(0) < \Big\langle g\big(\beta_1(0), u_2(0), u_1'(0), u_2'(0), u_1'(T), u_2'(T)\big), e_1\Big\rangle \leq 0.$$

or

$$0 = \beta_2(0) - u_2(0) < \Big\langle g\big(u_1(0), \beta_2(0), u_1'(0), u_2'(0), u_1'(T), u_2'(T)\big), e_2 \Big\rangle \leq 0.$$

It follows that

$$\big(\beta_1(0), \beta_2(0)\big) \geq \big(u_1(0), u_2(0)\big) + g\big(u_1(0), u_2(0), u_1'(0), u_2'(0), u_1'(T), u_2'(T)\big). \quad (9)$$

Suppose that for some $j \in \{1, 2\}$,

$$\alpha_j(0) > u_j(0) + \Big\langle g\big(u_1(0), u_2(0), u_1'(0), u_2'(0), u_1'(T), u_2'(T)\big), e_j \Big\rangle.$$

Then

$$u_j(0) = \vartheta_j\Big(u_j(0) + \Big\langle g\big(u_1(0), u_2(0), u_1'(0), u_2'(0), u_1'(T), u_2'(T)\big), e_j \Big\rangle\Big) = \alpha_j(0). \quad (10)$$

Therefore we obtain the contradiction

$$0 = \alpha_1(0) - u_1(0) > \Big\langle g\big(\alpha_1(0), u_2(0), u_1'(0), u_2'(0), u_1'(T), u_2'(T)\big), e_1 \Big\rangle \geq 0.$$

or

$$0 = \alpha_2(0) - u_2(0) > \Big\langle g\big(u_1(0), \alpha_2(0), u_1'(0), u_2'(0), u_1'(T), u_2'(T)\Big), e_2 \Big\rangle \geq 0.$$

It follows that

$$\big(\alpha_1(0), \alpha_2(0)\big) \leq \big(u_1(0), u_2(0)\big) + g\big(u_1(0), u_2(0), u_1'(0), u_2'(0), u_1'(T), u_2'(T)\big). \quad (11)$$

(9) and (11) show that

$$\big(u_1(0), u_2(0)\big) = \big(u_1(0), u_2(0)\big) + g\big(u_1(0), u_2(0), u_1'(0), u_2'(0), u_1'(T), u_2'(T)\big).$$

Therefore

$$g(u_1(0), u_2(0), u_1'(0), u_2'(0), u_1'(T), u_2'(T)) = (0, 0)\,.$$

□

Proof. [Proof of Theorem 1] Let $\Theta : C^1 \times C^1 \to C^1 \times C^1$ defined by

$$\Theta(u_1, u_2)(t) = \mathcal{A}(u_1, u_2) + \int_0^t \phi^{-1}\Big[(\tau_{u_1}, \tau_{u_2}) + \int_0^s F(u_1, u_2)(x)\,\mathrm{d}x\Big]\,\mathrm{d}s, \quad \forall t \in [0, T]\,,$$

where (τ_{u_1}, τ_{u_2}) is the value associated to (u_1, u_2) in Lemma 2. We can see that problem (3) is equivalent to the fixed point problem $(u_1, u_2) = \Theta(u_1, u_2)$.

Using Lemma 1, Lemma 2 and Arzelá-Ascoli's Theorem, we can see that Θ is completely continuous. Moreover, for all (u_1, u_2) in $C^1 \times C^1$, we have

$$\|\Theta(u_1, u_2)\|_{C^1 \times C^1} < \max\big\{\|\alpha_1\|_\infty, \|\beta_1\|_\infty\big\} + \max\big\{\|\alpha_2\|_\infty, \|\beta_2\|_\infty\big\} + 2a(1 + T)\,.$$

By a straightforward application of Schauder fixed point Theorem and the properties of the degree, Θ has a fixed point (U_1, U_2) on $\overline{B}_\mu$ with

$$\mu > \max\big\{\|\alpha_1\|_\infty, \|\beta_1\|_\infty\big\} + \max\big\{\|\alpha_2\|_\infty, \|\beta_2\|_\infty\big\} + 2a(1 + T)\,.$$

Consequently (U_1, U_2) is a solution of (3). Therefore, using Lemmas 3 and 4, (U_1, U_2) is also a solution of problem (1)–(2). □

Remark 1. The approach used in this paper can be:

- generalized to n equations $n \geq 3$;
- used when we have only one equation.

4 Example

Consider the system

$$\begin{cases} \left(\frac{u_1'(t)}{\sqrt{1-(u_1'(t))^2}}\right)' = \frac{\max\{0,u_1(t)\}}{\sqrt{t}} + \big(u_2(t)\big)^2 + \frac{-|u_1'(t)|}{2\sqrt{t}} - \sqrt{t} & \text{for a.e. } t \in [0,1];\\ \left(\frac{u_2'(t)}{\sqrt{1-|u_2'(t)|}}\right)' = \big(u_1(t)\big)^3 + \frac{u_2(t)}{\sqrt{t}} + \sin\big(u_1'(t)\big) - e^{u_2'(t)} & \text{for a.e. } t \in [0,1];\\ \Big(u_1(0)u_2(0)\Big(u_1'(0)+u_2'(0)+u_1'(1)+u_2'(1)-4\Big), \\ \qquad u_1(0)u_2(0)\Big(u_1'(0)-u_2'(0)-u_1'(1)+u_2'(1)-4\Big)\Big) = (0,0); \\ \big(u_1(1),u_2(1)\big) = \big(u_1(0),u_2(0)\big). \end{cases} \tag{12}$$

We have $a=1$, $T=1$, $\phi_1(x) = \frac{x}{\sqrt{1-(x)^2}}$, $\phi_2(x) = \frac{x}{\sqrt{1-|x|}}$,

$$f_1(t,x_1,x_2,y_1,y_2) = \frac{\max\{0,x_1\}}{\sqrt{t}} + (x_2)^2 + \frac{-|y_1|}{2\sqrt{t}} - \sqrt{t}\,,$$
$$f_2(t,x_1,x_2,y_1,y_2) = (x_1)^3 + \frac{x_2}{\sqrt{t}} + \sin(y_1) - e^{y_2}\,,$$
$$g(u,v,w,x,y,z) = \big(uv(w+x+y+z-4), uv(w-x-y+z-4)\big)$$

and

$$h(x,y) = (x,y)\,,$$

where $f_i\colon [0,1]\times\mathbb{R}^4 \to \mathbb{R}$, $i\in\{1,2\}$ are two L^1-Carathéodory functions and $g\colon \mathbb{R}^6 \to \mathbb{R}^2$ and $h\colon \mathbb{R}^2\to\mathbb{R}^2$ are two continuous functions. Taking

$$\alpha_1(t) = \alpha_2(t) = 0 \text{ and } \beta_1(t) = \beta_2(t) = \frac{1}{2},\ \forall t\in[0,1]\,,$$

we have

$$(\alpha_1,\alpha_2),(\beta_1,\beta_2) \in C\times C\,,$$
$$\alpha_i(0) < \beta_i(0) \text{ and } \alpha_i(T) < \beta_i(T), \quad \forall i\in\{1,2\},$$
$$\max\big\{|\beta_i(T)-\alpha_i(0)|;|\alpha_i(T)-\beta_i(0)|\big\} = \frac{1}{2} < 1 = aT, \quad \forall i\in\{1,2\},$$
$$g(\alpha_1(0),t,w,x,y,z) = (0,0), \quad \forall (t,w,x,y,z)\in\big[\alpha_2(0),\beta_2(0)\big]\times[-a,a]^4,$$
$$g(t,\alpha_2(0),w,x,y,z) = (0,0), \quad \forall (t,w,x,y,z)\in\big[\alpha_1(0),\beta_1(0)\big]\times[-a,a]^4,$$
$$g(\beta_1(0),t,w,x,y,z) \leq (0,0), \quad \forall (t,w,x,y,z)\in\big[\alpha_2(0),\beta_2(0)\big]\times[-a,a]^4,$$

$$g(t,\beta_2(0),w,x,y,z) \leq (0,0), \quad \forall (t,w,x,y,z) \in \big[\alpha_1(0),\beta_1(0)\big] \times [-a,a]^4,$$

$$\big(\alpha_1(T),\alpha_2(T)\big) - h(x,y) \leq (0,0), \quad \forall (x,y) \in \big[\alpha_1(0),\beta_1(0)\big] \times \big[\alpha_2(0),\beta_2(0)\big]$$

and

$$\big(\beta_1(T),\beta_2(T)\big) - h(x,y) \geq (0,0), \quad \forall (x,y) \in \big[\alpha_1(0),\beta_1(0)\big] \times \big[\alpha_2(0),\beta_2(0)\big]\,.$$

By Theorem 1, we deduce existence of at least one solution of the system (12).

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