APPLICATION OF THE PERTURBATION METHOD FOR DETERMINATION OF EIGENVALUES AND EIGENVECTORS FOR THE ASSUMED STATIC STRAIN

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Abstract

The paper presents the perturbation method which was used for computation of eigenvalues and eigenvectors for the assumed homogeneous state of strain in the hyperelastic Murnaghan material. The values calculated might be used for determination of the rate of propagation of unit vectors of wave amplitude for other non-linear

Keywords:

Perturbation method;
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Eigenvalue;
Eigenvector.

1. Introduction

The perturbation method is used for determination of the physical values from theoretical mathematical models based on expansion of these values into an analytical series of certain small parameter which is termed disturbance.

The perturbation procedure [1, 2] needs introduction of the following assumptions:

- eigenvalues and eigenvectors of the matrix $[\overset{\circ}{\mathbf{Q}}]$ and eigenvalues and eigenvectors of the matrix

$$[\mathbf{Q}] = \left([\overset{\circ}{\mathbf{Q}}] + \varepsilon [\overset{1}{\mathbf{Q}}] \right) \text{ have similar values,}$$

- eigenvalues and eigenvectors can be presented in a form of a power series of a small parameter ε .

We also assume that the matrices $[\overset{\circ}{\mathbf{Q}}]$ and $[\overset{1}{\mathbf{Q}}]$ are symmetrical [1].

The computations will be carried out in the Cartesian coordinate system. Let us denote eigenvectors as $\mathbf{W}^i$, corresponding to the eigenvalues μ_i of the matrix $[\mathbf{Q}]$, whereas $\mathbf{D}^i$ means eigenvectors corresponding to the eigenvalues λ_i of the matrix $[\overset{\circ}{\mathbf{Q}}]$, $i = 1, 2, 3$.

According to the assumptions adopted, eigenvalues and eigenvectors can be expanded to power series of the small parameter ε .

$$\mu_i = \lambda_i + \varepsilon \lambda_{i1} + \varepsilon^2 \lambda_{i2} + \dots,$$

$$\mathbf{W}^i = \mathbf{D}^i + \varepsilon \mathbf{D}^{i1} + \varepsilon^2 \mathbf{D}^{i2} + \dots \quad (1)$$

In order to determine the unknown scalars $\lambda_{i1}, \lambda_{i2}, \dots$ and unknown vectors $\mathbf{D}^{i1}, \mathbf{D}^{i2}, \dots$, the expansions (1) are substituted to the equation:

$$\left([\overset{\circ}{\mathbf{Q}}] + \varepsilon [\overset{1}{\mathbf{Q}}] \right) \mathbf{W}^i = \mu_i \mathbf{W}^i, \quad (2)$$

and the coefficients at the respective powers of the parameter ε are equated. This operation yields:

$$[\overset{\circ}{\mathbf{Q}}] \mathbf{D}^i = \lambda_i \mathbf{D}^i,$$

$$[\overset{\circ}{\mathbf{Q}}]\mathbf{D}^{i1} + [\overset{1}{\mathbf{Q}}]\mathbf{D}^i = \lambda_i \mathbf{D}^{i1} + \lambda_{i1} \mathbf{D}^i,$$

$$[\overset{\circ}{\mathbf{Q}}]\mathbf{D}^{i2} + [\overset{1}{\mathbf{Q}}]\mathbf{D}^{i1} = \lambda_i \mathbf{D}^{i2} + \lambda_{i1} \mathbf{D}^{i1} + \lambda_{i2} \mathbf{D}^i. \quad (3)$$

Equation (3)₁ meets the conditions of identity. The second equation is expanded with two unknown values: scalar λ_{i1} and vector $\mathbf{D}^{i1}$, whereas the third equation is expanded with the scalar λ_{i2} and vector $\mathbf{D}^{i2}$. The solution can be limited in order to determine the values λ_{i1} and $\mathbf{D}^{i1}$. The equation (3)₂ shows that:

$$\left([\overset{\circ}{\mathbf{Q}}] - \lambda_i [\overset{1}{\mathbf{I}}]\right) \mathbf{D}^{i1} = \left(\lambda_{i1} [\overset{1}{\mathbf{I}}] - [\overset{1}{\mathbf{Q}}]\right) \mathbf{D}^i, \quad (4)$$

where $[\overset{1}{\mathbf{I}}]$ is a unitary matrix.

Equation (4) has the solution when its right side meets specific conditions. In this case the scalar λ_{i1} can be determined [3]. Let us introduce denotation:

$$\left([\overset{\circ}{\mathbf{Q}}] - \lambda_i [\overset{1}{\mathbf{I}}]\right) \mathbf{D}^{i1} = \mathbf{z}. \quad (5)$$

Vector $\mathbf{D}^{i1}$ is presented in the base $\mathbf{D}^i$:

$$\mathbf{D}^{i1} = b_1 \mathbf{D}^1 + b_2 \mathbf{D}^2 + b_3 \mathbf{D}^3. \quad (6)$$

In the same base, we introduce the vector $\mathbf{z}$:

$$\mathbf{z} = c_1 \mathbf{D}^1 + c_2 \mathbf{D}^2 + c_3 \mathbf{D}^3. \quad (7)$$

It can be written that:

$$\left([\overset{\circ}{\mathbf{Q}}] - \lambda_i [\overset{1}{\mathbf{I}}]\right) \sum_{j=1}^3 b_j \mathbf{D}^j = \mathbf{z} = \sum_{j=1}^3 c_j \mathbf{D}^j. \quad (8)$$

Substituting (3)₁ we obtain:

$$\sum_{j=1}^3 b_j (\lambda_j - \lambda_i) \mathbf{D}^j = \sum_{j=1}^3 c_j \mathbf{D}^j. \quad (9)$$

In order to determine c_j we equate values at $\mathbf{D}^j$. This yields:

$$b_j (\lambda_j - \lambda_i) = c_j \text{ for } i=1,2,3, \quad (10)$$

for:

$$\lambda_i = \lambda_1 \text{ and } c_i = c_1 = 0, \quad (11)$$

b_1 is arbitrary, however:

$$b_2 = \frac{c_2}{(\lambda_2 - \lambda_1)}; \quad b_3 = \frac{c_3}{(\lambda_3 - \lambda_1)}. \quad (12)$$

Condition (11)₂ means that the equation (5) has solution only when the vector $\mathbf{z}$ is orthogonal to the vector $\mathbf{D}^1$. Let us calculate the scalar product of vectors $\mathbf{z}$ and $\mathbf{D}^1$:

$$\mathbf{zD}^1 = c_1 \mathbf{D}^1 \mathbf{D}^1 + c_2 \mathbf{D}^2 \mathbf{D}^1 + c_3 \mathbf{D}^3 \mathbf{D}^1. \quad (13)$$

As results from the product:

$$\mathbf{zD}^1 = 0, \quad (14)$$

thus, the vector $\mathbf{z}$ is perpendicular to $\mathbf{D}^1$ (Fig.1).

Fig. Chyba! Dokument neobsahuje žiadny text so zadaným štýlom.1: Distribution of the vector $\mathbf{z}$ and $\mathbf{D}^{i1}$ in the base $\mathbf{D}^i$.

Substitution of the relationships (12) into (6) yields the single parameter equation that is presented by the following construction of the vector $\mathbf{D}^{i1}$:

$$\mathbf{D}^{i1} = b_1 \mathbf{D}^1 + \frac{c_2}{(\lambda_2 - \lambda_1)} \mathbf{D}^2 + \frac{c_3}{(\lambda_3 - \lambda_1)} \mathbf{D}^3. \quad (15)$$

Further, substitution of the obtained vector $\mathbf{D}^{i1}$ to the denotation (5) yields:

$$\begin{aligned} & \left([\mathbf{Q}] - \lambda_i [\mathbf{I}] \right) b_1 \mathbf{D}^1 + \frac{c_2}{(\lambda_2 - \lambda_1)} \left([\mathbf{Q}] - \lambda_i [\mathbf{I}] \right) \mathbf{D}^2 + \\ & + \frac{c_3}{(\lambda_3 - \lambda_1)} \left([\mathbf{Q}] - \lambda_i [\mathbf{I}] \right) \mathbf{D}^3 = 0, \end{aligned} \quad (16)$$

whereas for $i=1$, regardless of the value b_1 , the equation is obtained:

$$b_1 \left([\mathbf{Q}] - \lambda_1 [\mathbf{I}] \right) \mathbf{D}^1 = 0. \quad (17)$$

Vector $\mathbf{D}^{i1}$ can be normalized:

$$\mathbf{p} = \frac{\mathbf{D}^{i1}}{|\mathbf{D}^{i1}|} = \frac{b_1}{\sqrt{b_1^2 + \left(\frac{c_2}{\lambda_2 - \lambda_1} \right)^2 + \left(\frac{c_3}{\lambda_3 - \lambda_1} \right)^2}} \mathbf{D}^1 + \dots \quad (18)$$

The second and third components of the vector $\mathbf{D}^{i1}$ are non-zero; b_1 can adopt any values including zero.

The condition of orthogonality (14) resulting from scalar product of vectors $\mathbf{z}$ and $\mathbf{D}^1$ can be rewritten based on the equation (4):

$$\left\{ \left(\lambda_{il} [\mathbf{I}] - [\mathbf{Q}] \right) \mathbf{D}^i \right\} \mathbf{D}^i = 0. \quad (19)$$

This leads to a form which is more convenient for calculation:

$$\left(\lambda_{il} [\mathbf{I}] \mathbf{D}^i \right) \mathbf{D}^i = \left([\mathbf{Q}] \mathbf{D}^i \right) \mathbf{D}^i, \quad (20)$$

of the first correction for eigenvalue. The scalar λ_{il} is finally given by:

$$\lambda_{il} = \left([\mathbf{Q}] \mathbf{D}^i \right) \mathbf{D}^i. \quad (21)$$

2. Calculations for the homogeneous state of strain

According to [4] we analyse homogeneous static strain of a compressible body. In the specific coincident Cartesian coordinate systems $\{x^i\}$ and $\{X^\alpha\}$, strain can be given by the equations:

$$x^1 = \lambda_1 X^1; \quad x^2 = \lambda_2 X^2; \quad x^3 = \lambda_3 X^3, \quad (22)$$

where $\lambda_1, \lambda_2, \lambda_3 = \text{const.}$

Let us assume that the surface of discontinuity propagates towards in the direction which is perpendicular to the axis X^3 . The unit vector which is normal to the surface Σ_R (see Fig. 2) in the material system $\{X^\alpha\}$ has the components:

$$\mathbf{N} = [\cos \alpha, \sin \alpha, 0]. \quad (23)$$

Fig. **Chyba! Dokument neobsahuje žádný text so zadaným štýlom.**2: Extension in the direction of the axis X^1 for the surface of discontinuity Σ_R in the material system at any angle.

Assuming that $\varepsilon = 8,5 \cdot 10^{-4}$ and taking into consideration the basic matrix $[\mathbf{Q}]$ and the matrix $[\mathbf{Q}]^1$, treated as a disturbance, we obtain:

$$[\mathbf{Q}] = [\overset{\circ}{\mathbf{Q}}] + \varepsilon [\overset{1}{\mathbf{Q}}]. \quad (24)$$

The basic matrix of the acoustic tensor Q_{ik} can be calculated for the non-deformed medium. We adopt that $\lambda_1 = \lambda_2 = \lambda_3 = 1$. Therefore, the strain tensor invariants are $I_1 = I_2 = 3$. For the calculations, we adopt the elastic Murnaghan material according to [4, 5]. The calculations performed in [4] lead to the conclusion that the basic matrix $[\overset{\circ}{\mathbf{Q}}]$ has the following components:

$$[\overset{\circ}{\mathbf{Q}}] = \begin{bmatrix} \mu + (\lambda + \mu) \cos^2 \alpha & (\lambda + \mu) \sin \alpha \cos \alpha & 0 \\ (\lambda + \mu) \sin \alpha \cos \alpha & \mu + (1 + \mu) \sin^2 \alpha & 0 \\ 0 & 0 & \mu \end{bmatrix}, \quad (25)$$

where α is the angle of inclination of the vector which is normal to the surface of discontinuity parallel to the axis X^3 of the Cartesian coordination system (Fig. 2).

Further, let us calculate the matrix $[\overset{1}{\mathbf{Q}}]$, considered as a disturbance. We adopt that:

$$\varepsilon_1 = \varepsilon \varepsilon_{11}; \quad \varepsilon_2 = \varepsilon \varepsilon_{22}; \quad \varepsilon_3 = \varepsilon \varepsilon_{33}. \quad (26)$$

In order to determine the components of the matrix $[\overset{1}{\mathbf{Q}}]$ it is necessary to calculate the partial derivatives of the elastic potential with respect to the invariants λ_1, λ_2 and λ_3 and partial derivatives of the acoustic tensor $[Q_{ik}]$. After the transformations, we obtain the final form of the matrix $[\overset{1}{\mathbf{Q}}]$:

$$[\overset{1}{\mathbf{Q}}] = \begin{bmatrix} \left[\begin{array}{c} \phi_1(\varepsilon_1 + \varepsilon_2) + \phi_4 \varepsilon_3 + \\ + \left(\phi_2 \varepsilon_1 + \phi_3 \varepsilon_2 + \right) \cos^2 \alpha \end{array} \right] & \left[\begin{array}{c} \phi_6(\varepsilon_1 + \varepsilon_2) + \\ + \phi_5 \varepsilon_3 \end{array} \right] \sin \alpha \cos \alpha & 0 \\ \left[\begin{array}{c} \phi_6(\varepsilon_1 + \varepsilon_2) + \\ + \phi_5 \varepsilon_3 \end{array} \right] \sin \alpha \cos \alpha & \left[\begin{array}{c} \phi_1(\varepsilon_1 + \varepsilon_2) + \phi_4 \varepsilon_3 + \\ + \left(\phi_3 \varepsilon_1 + \phi_2 \varepsilon_2 + \right) \sin^2 \alpha \end{array} \right] & 0 \\ 0 & 0 & \left[\begin{array}{c} \phi_1(\varepsilon_1 + \varepsilon_2 + \varepsilon_3) + \\ - \phi_7 \left(\varepsilon_1 \sin^2 \alpha + \right. \\ \left. + \varepsilon_2 \cos^2 \alpha \right) \end{array} \right] \end{bmatrix} \quad (27)$$

Substitution of (25) and (27) to the equation (24) yields the matrix $[\mathbf{Q}]$:

$$[\mathbf{Q}] = \begin{bmatrix} \overset{\circ}{Q}_{11} & \overset{\circ}{Q}_{12} & 0 \\ \overset{\circ}{Q}_{21} & \overset{\circ}{Q}_{22} & 0 \\ 0 & 0 & \overset{\circ}{Q}_{33} \end{bmatrix} + \varepsilon \begin{bmatrix} \overset{1}{Q}_{11} & \overset{1}{Q}_{12} & 0 \\ \overset{1}{Q}_{21} & \overset{1}{Q}_{22} & 0 \\ 0 & 0 & \overset{1}{Q}_{33} \end{bmatrix}. \quad (28)$$

Let us calculate eigenvectors and eigenvalues:

- for $U_1^2 \rho_R = \lambda + 2\mu$, from the condition:

$$\begin{bmatrix} \overset{\circ}{Q}_{11} - (\lambda + 2\mu) & \overset{\circ}{Q}_{12} & 0 \\ \overset{\circ}{Q}_{21} & \overset{\circ}{Q}_{22} - (\lambda + 2\mu) & 0 \\ 0 & 0 & \overset{\circ}{Q}_{33} - (\lambda + 2\mu) \end{bmatrix} \begin{bmatrix} D_1 \\ D_2 \\ D_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \quad (29)$$

this yields vector $\mathbf{D}^1$ with coordinates:

$$\mathbf{D}^1 = [\cos \alpha, \sin \alpha, 0], \quad (30)$$

- for $U_2^2 \rho_R = \mu$, from the condition:

$$\begin{bmatrix} \overset{\circ}{Q}_{11} - \mu & \overset{\circ}{Q}_{12} & 0 \\ \overset{\circ}{Q}_{21} & \overset{\circ}{Q}_{22} - \mu & 0 \\ 0 & 0 & \overset{\circ}{Q}_{33} - \mu \end{bmatrix} \begin{bmatrix} D_1 \\ D_2 \\ D_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \quad (31)$$

this yields vector $\mathbf{D}^2$ with coordinates:

$$\mathbf{D}^2 = [-\sin \alpha, \cos \alpha, 0], \quad (32)$$

- for $U_3^2 \rho_R = \mu$, from the condition:

$$\begin{bmatrix} \overset{\circ}{Q}_{11} - \mu & \overset{\circ}{Q}_{12} & 0 \\ \overset{\circ}{Q}_{21} & \overset{\circ}{Q}_{22} - \mu & 0 \\ 0 & 0 & \overset{\circ}{Q}_{33} - \mu \end{bmatrix} \begin{bmatrix} D_1 \\ D_2 \\ D_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \quad (33)$$

this yields vector $\mathbf{D}^3$ with coordinates:

$$\mathbf{D}^3 = [0, 0, 1]. \quad (34)$$

Below are presented the eigenvectors of the basic matrix $[\overset{\circ}{\mathbf{Q}}]$:

$$\begin{aligned} U_1^2 \rho_R = \lambda + 2\mu & \quad \mathbf{D}^1 = [\cos \alpha, \sin \alpha, 0], \\ U_2^2 \rho_R = \mu & \quad \mathbf{D}^2 = [-\sin \alpha, \cos \alpha, 0], \\ U_3^2 \rho_R = \mu & \quad \mathbf{D}^3 = [0, 0, 1]. \end{aligned} \quad (35)$$

It should be noted that the eigenvectors can also adopt the values with the opposite sign ($-\mathbf{D}^i$ for $i=1,2,3$).

According to (21), in order to determine the eigenvalue λ_{i1} it is convenient to first calculate the product for the matrix $[\overset{1}{\mathbf{Q}}]$ and the unit vector $\mathbf{D}^i$, for $i=1,2,3$. This yields:

- for $\lambda_{11} = \left([\overset{1}{\mathbf{Q}}] \mathbf{D}^1 \right) \mathbf{D}^1$,

$$[\overset{1}{\mathbf{Q}}] \mathbf{D}^1 = \begin{bmatrix} \overset{1}{Q}_{11} \cos \alpha + \overset{1}{Q}_{12} \sin \alpha \\ \overset{1}{Q}_{21} \cos \alpha + \overset{1}{Q}_{22} \sin \alpha \\ 0 \end{bmatrix}, \quad (36)$$

$$\lambda_{11} = \overset{1}{Q}_{11} \cos^2 \alpha + \overset{1}{Q}_{12} \sin \alpha \cos \alpha + \overset{1}{Q}_{21} \sin \alpha \cos \alpha + \overset{1}{Q}_{22} \sin^2 \alpha, \quad (37)$$

- for $\lambda_{21} = \left([\overset{1}{\mathbf{Q}}] \mathbf{D}^2 \right) \mathbf{D}^2$,

$$[\overset{1}{\mathbf{Q}}] \mathbf{D}^2 = \begin{bmatrix} -\overset{1}{Q}_{11} \sin \alpha + \overset{1}{Q}_{12} \cos \alpha \\ -\overset{1}{Q}_{21} \sin \alpha + \overset{1}{Q}_{22} \cos \alpha \\ 0 \end{bmatrix}, \quad (38)$$

$$\lambda_{21} = \overset{1}{Q}_{11} \sin^2 \alpha - \overset{1}{Q}_{12} \sin \alpha \cos \alpha - \overset{1}{Q}_{21} \sin \alpha \cos \alpha + \overset{1}{Q}_{22} \cos^2 \alpha, \quad (39)$$

- for $\lambda_{31} = \left([\overset{1}{\mathbf{Q}}] \mathbf{D}^3 \right) \mathbf{D}^3$,

$$[\overset{1}{\mathbf{Q}}] \mathbf{D}^3 = \begin{bmatrix} 0 \\ 0 \\ \overset{1}{Q}_{33} \end{bmatrix}, \quad (40)$$

$$\lambda_{31} = \overset{1}{Q}_{33}. \quad (41)$$

Comparison of the eigenvalues λ_{i1} of the matrix $[\overset{1}{\mathbf{Q}}]$ for $i=1,2,3$:

$$\lambda_{11} = \overset{1}{Q}_{11} \cos^2 \alpha + \overset{1}{Q}_{22} \sin^2 \alpha + \left(\overset{1}{Q}_{12} + \overset{1}{Q}_{21} \right) \sin \alpha \cos \alpha,$$

$$\lambda_{21} = \overset{1}{Q}_{11} \sin^2 \alpha + \overset{1}{Q}_{22} \cos^2 \alpha - \left(\overset{1}{Q}_{12} + \overset{1}{Q}_{21} \right) \sin \alpha \cos \alpha,$$

$$\lambda_{31} = \overset{1}{Q}_{33}. \quad (42)$$

Based on the obtained eigenvalues λ_{i1} (42), the values c_i and b_i can be calculated for $i=1,2,3$.

According to the equations (4) and (8), it can be written that:

$$\left(\lambda_{i1} [\mathbf{I}] - [\overset{1}{\mathbf{Q}}] \right) \mathbf{D}^i = \mathbf{z} = \sum_{j=1}^3 c_j \mathbf{D}^j, \quad (43)$$

- after expansion for λ_{11}

$$\mathbf{z} = \begin{bmatrix} \left(\lambda_{11} - \overset{1}{Q}_{11} \right) \cos \alpha + \overset{1}{Q}_{12} \sin \alpha \\ \overset{1}{Q}_{21} \cos \alpha + \left(\lambda_{11} - \overset{1}{Q}_{22} \right) \sin \alpha \\ 0 \end{bmatrix}, \quad (44)$$

c_j can be determined from the equation (43):

$$c_j = \left\{ \left(\lambda_{ji} [\mathbf{I}] - [\mathbf{Q}] \right) \mathbf{D}^i \right\} \mathbf{D}^j = \mathbf{z} \mathbf{D}^j. \quad (45)$$

Components c_j , for $j=1,2,3$ are, respectively:

$$\begin{aligned} c_1 &= \mathbf{z} \mathbf{D}^1 = 0, \\ c_2 &= \mathbf{z} \mathbf{D}^2 = \dot{Q}_{12} \cos 2\alpha + \frac{1}{2} \left(\dot{Q}_{11} - \dot{Q}_{22} \right) \sin 2\alpha, \\ c_3 &= \mathbf{z} \mathbf{D}^3 = 0. \end{aligned} \quad (46)$$

Having the values c_1, c_2, c_3 and according to (10), the values b_1, b_2, b_3 can be determined. They are, respectively:

$$\begin{aligned} b_1 &= b_3 = 0, \\ b_2 &= -\frac{1}{\lambda + \mu} \left[\dot{Q}_{12} \cos 2\alpha + \frac{1}{2} \left(\dot{Q}_{11} - \dot{Q}_{22} \right) \sin 2\alpha \right], \end{aligned} \quad (47)$$

- for λ_{21} according to (44), we obtain:

$$\mathbf{z} = \begin{bmatrix} -\left(\lambda_{21} - \dot{Q}_{11} \right) \sin \alpha + \dot{Q}_{12} \cos \alpha \\ -\dot{Q}_{21} \sin \alpha + \left(\lambda_{21} - \dot{Q}_{22} \right) \cos \alpha \\ 0 \end{bmatrix}. \quad (48)$$

According to the equation (45), the components c_1, c_2, c_3 are, respectively:

$$\begin{aligned} c_1 &= \dot{Q}_{12} \cos 2\alpha + \frac{1}{2} \left(\dot{Q}_{11} - \dot{Q}_{22} \right) \sin 2\alpha, \\ c_2 &= c_3 = 0. \end{aligned} \quad (49)$$

Values b_1, b_2, b_3 can be determined as in (47) based on the equation (10). They are, respectively:

$$\begin{aligned} b_1 &= \frac{1}{\lambda + \mu} \left[\dot{Q}_{12} \cos 2\alpha + \frac{1}{2} \left(\dot{Q}_{11} - \dot{Q}_{22} \right) \sin 2\alpha \right], \\ b_2 &= b_3 = 0, \end{aligned} \quad (50)$$

- for λ_{31} according to (44), we obtain:

$$\mathbf{z} = \begin{bmatrix} 0 \\ 0 \\ \lambda_{31} - \dot{Q}_{33} \end{bmatrix}. \quad (51)$$

According to the equation (45), the components can be obtained:

$$c_1 = c_2 = c_3 = 0. \quad (52)$$

Therefore, based on the equation (10), we find that:

$$b_1 = b_2 = b_3 = 0. \quad (53)$$

For the first corrections of the eigenvalues λ_{i1} and based on the condition (6), it is possible to determine the corrections of the vector $\mathbf{D}^{i1}$.

For the eigenvectors, the components of the vectors $\mathbf{D}^{i1}$ will be denoted with $(\mathbf{D}^{i1})_m$, where $m = 1, 2, 3$. This yields:

- for λ_{11} ,

$$\begin{aligned} (\mathbf{D}^{11})_1 &= (\mathbf{D}^{11})_3 = 0, \\ (\mathbf{D}^{11})_2 &= -\frac{1}{\lambda + \mu} \left[\bar{Q}_{12} \cos 2\alpha + \frac{1}{2} \left(\bar{Q}_{11} - \bar{Q}_{22} \right) \sin 2\alpha \right], \end{aligned} \quad (54)$$

- for λ_{21} ,

$$\begin{aligned} (\mathbf{D}^{12})_1 &= \frac{1}{\lambda + \mu} \left[\bar{Q}_{12} \cos 2\alpha + \frac{1}{2} \left(\bar{Q}_{11} - \bar{Q}_{22} \right) \sin 2\alpha \right], \\ (\mathbf{D}^{12})_2 &= (\mathbf{D}^{12})_3 = 0, \end{aligned} \quad (55)$$

- for λ_{31} ,

$$(\mathbf{D}^{13})_1 = (\mathbf{D}^{13})_2 = (\mathbf{D}^{13})_3 = 0. \quad (56)$$

Substitution of the matrix components $[\bar{\mathbf{Q}}]$ (27) to (43) yields:

$$\begin{aligned} \lambda_{11} &= (\phi_2 \varepsilon_1 + \phi_3 \varepsilon_2) \cos^4 \alpha + (\phi_3 \varepsilon_1 + \phi_2 \varepsilon_2) \sin^4 \alpha + (\phi_4 + \phi_5) \varepsilon_3 + \\ &+ (\phi_1 + \phi_6 2 \sin^2 \alpha \cos^2 \alpha) (\varepsilon_1 + \varepsilon_2), \\ (\mathbf{D}^{11})_1 &= (\mathbf{D}^{11})_3 = 0, \\ (\mathbf{D}^{11})_2 &= -\frac{\sin 2\alpha}{2(\lambda + \mu)} \left[[\phi_6 (\varepsilon_1 + \varepsilon_2) + 2\phi_5 \varepsilon_3] \cos 2\alpha + \right. \\ &\left. + (\phi_2 \varepsilon_1 + \phi_3 \varepsilon_2) \cos^2 \alpha - (\phi_3 \varepsilon_1 + \phi_2 \varepsilon_2) \sin^2 \alpha \right], \end{aligned} \quad (57)$$

$$\begin{aligned} \lambda_{21} &= [\phi_1 + (\phi_2 + \phi_3 - 2\phi_6) \sin^2 \alpha \cos^2 \alpha] (\varepsilon_1 + \varepsilon_2) + \phi_4 \varepsilon_3, \\ (\mathbf{D}^{12})_1 &= \frac{\sin 2\alpha}{2(\lambda + \mu)} \left[[\phi_6 (\varepsilon_1 + \varepsilon_2) + 2\phi_5 \varepsilon_3] \cos 2\alpha + \right. \\ &\left. + (\phi_2 \varepsilon_1 + \phi_3 \varepsilon_2) \cos^2 \alpha - (\phi_3 \varepsilon_1 + \phi_2 \varepsilon_2) \sin^2 \alpha \right], \\ (\mathbf{D}^{12})_2 &= (\mathbf{D}^{12})_3 = 0, \end{aligned} \quad (58)$$

$$\begin{aligned} \lambda_{31} &= \phi_1 (\varepsilon_1 + \varepsilon_2 + \varepsilon_3) - \phi_7 (\varepsilon_1 \sin^2 \alpha + \varepsilon_2 \cos^2 \alpha), \\ (\mathbf{D}^{13})_1 &= (\mathbf{D}^{13})_2 = (\mathbf{D}^{13})_3 = 0. \end{aligned} \quad (59)$$

3. Conclusions

The first approximations of the eigenvectors and eigenvalues obtained using the perturbation method for the case of homogeneous static strain discussed in the study allow for determination of the velocity of propagation and unit vectors of wave amplitude, also for other non-linear materials. Using the calculations presented in this study, it is possible to verify the correctness of numerical solutions.

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