CHARACTERISTIC-BASED SPLIT MESHLESS SOLUTION FOR COUETTE FLOW

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Abstract

The paper deals with use of the meshless method for incompressible fluid flow analysis. There are many formulations of the meshless methods. The article presents the Meshless Local Petrov-Galerkin method (MLPG) – local weak formulation of the Navier-Stokes equations. The shape function construction is the crucial part of the meshless numerical analysis in the construction of shape functions. The article presents the radial point interpolation method (RPIM) for the shape functions construction.

Keywords:

CBS algorithm;
Meshless analysis;
Meshless Petrov-Galerkin method;
Navier-Stokes equation.

1. Introduction

Incompressible Navier-Stokes flow in two dimensions is one of the several major problems in fluid mechanics that have been extensively studied both theoretically and numerically. In general, the formulation of incompressible Navier-Stokes equations using primitive variables is often used, but it has limitation in approximating the velocity and pressure. The meshless Local Petrov-Galerkin method (MLPG) is truly meshless method which requires no elements or global background mesh, for either interpolation or integration purposes. The first article applying MLPG method to compute convection-diffusion and incompressible flow was by Lin and Atluri [1]. In their work, two kinds of upwind schemes were constructed to overcome oscillations produced by convection term. They applied the upwind schemes to solve incompressible flow problem based on primitive variables formulation and added the perturbation term to the continuity equation to satisfy Babuška-Brezzi condition. But there still persists the problem of perturbation parameter determination for high Reynolds number problems. The present paper focuses on the MLPG primitive variable method using characteristic-based split method to achieve velocity-pressure decoupling to solve incompressible viscous flow [2].

2. Meshless Shape Function Formulation – Radial Point Interpolation

Using radial basis functions (RBF) is one of the best solutions to construct the meshless shape function. Multi-quadratic RBF function is one of the most popular radial function [3] and is defined as

$$R_i(\mathbf{X}) = (r_i^2 + \varepsilon^2)^q, \quad (1)$$

where r_i is the distance between the desired point ($\mathbf{X}$) and the field node i ($\mathbf{X}_i$) defined simply as 2D Euclidean distance. Constants ε and q in the Eq. 1 depend on the type of problem. The RPIM interpolation augmented with polynomials can be written as

$$u(\mathbf{X}) = \sum_{i=1}^n R_i(\mathbf{X}) a_i + \sum_{j=1}^m p_j(\mathbf{X}) b_j = \mathbf{R}^T(\mathbf{X}) \mathbf{a} + \mathbf{p}^T(\mathbf{X}) \mathbf{b}, \quad (2)$$

where $R_i(\mathbf{X})$ is the radial basis function (RBF), n is the number of RBFs, $p_j(\mathbf{X})$ is polynomial basis function, m is number of polynomial basis function, a_i and b_j are interpolation coefficients. In order to determine a_i and b_j a support domain is formed for the point of interest at $\mathbf{X}$, and n field nodes are included in the support domain. Interpolation coefficients can be determined by enforcing the Eq. 2 to

be satisfied at these n nodes surrounding the point of interest $\mathbf{X}$. This leads to n linear equations, one for each node. The equation system in matrix form can be expressed as

$$\mathbf{U}_s = \mathbf{R}\mathbf{a} + \mathbf{P}_m \mathbf{b}, \quad (3)$$

where $\mathbf{U}_s$ is the vector of displacement function values, $\mathbf{R}$ is the RBF moment matrix and the $\mathbf{P}_m$ is the polynomial moment matrix. Eq. 3 can be subsequently rewritten as

$$u(\mathbf{X}) = \mathbf{R}^T(\mathbf{X})\mathbf{a} + \mathbf{p}^T(\mathbf{X})\mathbf{b} = \begin{bmatrix} \mathbf{R}^T(\mathbf{X}) & \mathbf{p}^T(\mathbf{X}) \end{bmatrix} \begin{bmatrix} \mathbf{a} \\ \mathbf{b} \end{bmatrix} \quad (4)$$

and following set of equations is used to obtain unknown vectors of parameters $\mathbf{a}$, $\mathbf{b}$

$$\tilde{\mathbf{U}}_s = \begin{bmatrix} \mathbf{U}_s \\ \mathbf{0} \end{bmatrix} = \begin{bmatrix} \mathbf{R} & \mathbf{P}_m \\ \mathbf{P}_m^T & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{a} \\ \mathbf{b} \end{bmatrix} = \mathbf{G} \begin{bmatrix} \mathbf{a} \\ \mathbf{b} \end{bmatrix}. \quad (5)$$

Using Eq. 5 we can obtain

$$u(\mathbf{X}) = \begin{bmatrix} \mathbf{R}^T(\mathbf{X}) & \mathbf{p}^T(\mathbf{X}) \end{bmatrix} \mathbf{G}^{-1} \tilde{\mathbf{U}}_s = \tilde{\Phi}^T(\mathbf{X}) \tilde{\mathbf{U}}_s \quad (6)$$

and finally the RPIM shape functions corresponding to the nodal velocity vector $\Phi(\mathbf{X})$ are obtained as

$$\Phi^T(\mathbf{X}) = [\phi_1(\mathbf{X}) \quad \phi_2(\mathbf{X}) \quad \dots \quad \phi_n(\mathbf{X})]. \quad (7)$$

Then the velocity field approximation can be written using the RPIM shape functions and nodal displacements

$$u(\mathbf{X}) = \Phi^T(\mathbf{X}) \mathbf{U}_s = \sum_{i=1}^n \phi_i u_i \quad (8)$$

and the l -th derivatives of velocity field $u(\mathbf{X})$ are easily obtained as

$$\Phi^{(l)}(\mathbf{X}) = \begin{bmatrix} \frac{\partial^l \mathbf{R}^T(\mathbf{X})}{\partial \mathbf{X}^l} & \frac{\partial^l \mathbf{p}^T(\mathbf{X})}{\partial \mathbf{X}^l} \end{bmatrix} \mathbf{G}^{-1}. \quad (9)$$

Note that $\mathbf{G}^{-1}$ usually exists for arbitrarily scattered nodes. Therefore, there is no singularity problem in the RPIM as a small number of nodes are used in the local support domain [4].

3. Governing Equations and Characteristic-Based split algorithm

The governing equations for unsteady incompressible viscous fluid flow are the Navier-Stokes equations with the continuity equation in the convection term [2, 5]. This equation can be written as:

$$\frac{\partial u}{\partial t} = \frac{1}{\text{Re}} \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right) - u \frac{\partial u}{\partial x} - v \frac{\partial u}{\partial y} - \frac{\partial p}{\partial x} + f_x, \quad (10)$$

$$\frac{\partial v}{\partial t} = \frac{1}{\text{Re}} \left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} \right) - u \frac{\partial v}{\partial x} - v \frac{\partial v}{\partial y} - \frac{\partial p}{\partial y} + f_y, \quad (11)$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \quad (12)$$

where u and v are the velocities in x and y direction respectively, p is the pressure, f_x and f_y are the body forces, Re is Reynolds number. Eq. 10 and Eq. 11 are the momentum equations and Eq. 12 is the continuity equation. A characteristic-based split algorithm is used to solve this problem (see [2, 3]). As the first step, the pressure term from Eq. 10 and Eq. 11 is dropped. Assuming the Lagrangian fluid dynamics approach as the fluid particle moves along the characteristic wave, the convection term in Eq. 10 and Eq. 11 disappear

$$\frac{\partial u}{\partial t} = \frac{1}{Re} \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right), \quad (13)$$

$$\frac{\partial v}{\partial t} = \frac{1}{Re} \left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} \right). \quad (14)$$

The time derivative of the velocity vector in Eq.13 and Eq.14 can be replaced with a difference approximation and following relation is obtained (Fig.1.)

$$\frac{u^{n+1}|_{x_k} - u^n|_{x_k - \Delta x_k}}{\Delta t} = \frac{1}{Re} \left(\frac{\partial^2 u^n}{\partial x^2} + \frac{\partial^2 u^n}{\partial y^2} \right) \Big|_{x_k - \Delta x_k}, \quad (15)$$

$$\frac{v^{n+1}|_{x_k} - v^n|_{x_k - \Delta x_k}}{\Delta t} = \frac{1}{Re} \left(\frac{\partial^2 v^n}{\partial x^2} + \frac{\partial^2 v^n}{\partial y^2} \right) \Big|_{x_k - \Delta x_k}, \quad (16)$$

where $x_k - \Delta x_k$ and x_k are the spatial coordinates along the characteristic at the n and $n+1$ time step.

Fig. 1: Characteristic in space-time domain.

Using the Taylor expansion in space, neglecting high order terms, Eq.15 and Eq.16 becomes

$$\begin{aligned} \tilde{u}^{n+1} = & u^n + \Delta t \left[\frac{1}{Re} \left(\frac{\partial^2 u^n}{\partial x^2} + \frac{\partial^2 u^n}{\partial y^2} \right) - u^n \frac{\partial u^n}{\partial x} - v^n \frac{\partial u^n}{\partial y} \right] \\ & + \frac{\Delta t^2}{2} \left[u^n \left(\frac{\partial u^n}{\partial x} \frac{\partial^2 u^n}{\partial x^2} + \frac{\partial v^n}{\partial x} \frac{\partial^2 u^n}{\partial x \partial y} \right) + v^n \left(\frac{\partial u^n}{\partial y} \frac{\partial^2 u^n}{\partial x \partial y} + \frac{\partial v^n}{\partial y} \frac{\partial^2 u^n}{\partial y^2} \right) \right] \end{aligned} \quad (17)$$

$$\begin{aligned} \tilde{v}^{n+1} = v^n + \Delta t \left[\frac{1}{\text{Re}} \left(\frac{\partial^2 v^n}{\partial x^2} + \frac{\partial^2 v^n}{\partial y^2} \right) - u^n \frac{\partial v^n}{\partial x} - v^n \frac{\partial v^n}{\partial y} \right] \\ + \frac{\Delta t^2}{2} \left[u^n \left(\frac{\partial u^n}{\partial x} \frac{\partial^2 v^n}{\partial x^2} + \frac{\partial v^n}{\partial x} \frac{\partial^2 v^n}{\partial x \partial y} \right) + v^n \left(\frac{\partial u^n}{\partial y} \frac{\partial^2 v^n}{\partial x \partial y} + \frac{\partial v^n}{\partial y} \frac{\partial^2 v^n}{\partial y^2} \right) \right], \end{aligned} \quad (18)$$

where $\tilde{u}$ and $\tilde{v}$ are the intermediate velocities and the last term in Eq.17 and Eq.18 in the square brackets acts as the upwinding or stabilizing term. The intermediate velocities do not satisfy the continuity equation (Eq. 12). Therefore, the intermediate velocities need to be corrected. To do so, first the pressure field is computed and then the corrected velocities are found using following equations

$$\begin{aligned} u^{n+1} = u^n + \Delta t \left[\frac{1}{\text{Re}} \left(\frac{\partial^2 u^n}{\partial x^2} + \frac{\partial^2 u^n}{\partial y^2} \right) - u^n \frac{\partial u^n}{\partial x} - v^n \frac{\partial u^n}{\partial y} \right] + \\ \frac{\Delta t^2}{2} \left[u^n \left(\frac{\partial u^n}{\partial x} \frac{\partial^2 v^n}{\partial x^2} + \frac{\partial v^n}{\partial x} \frac{\partial^2 v^n}{\partial x \partial y} \right) + v^n \left(\frac{\partial u^n}{\partial y} \frac{\partial^2 v^n}{\partial x \partial y} + \frac{\partial v^n}{\partial y} \frac{\partial^2 v^n}{\partial y^2} \right) \right] - \Delta t \frac{\partial p^{n+1}}{\partial x} \end{aligned} \quad (19)$$

and

$$\begin{aligned} v^{n+1} = v^n + \Delta t \left[\frac{1}{\text{Re}} \left(\frac{\partial^2 v^n}{\partial x^2} + \frac{\partial^2 v^n}{\partial y^2} \right) - u^n \frac{\partial v^n}{\partial x} - v^n \frac{\partial v^n}{\partial y} \right] + \\ \frac{\Delta t^2}{2} \left[u^n \left(\frac{\partial u^n}{\partial x} \frac{\partial^2 v^n}{\partial x^2} + \frac{\partial v^n}{\partial x} \frac{\partial^2 v^n}{\partial x \partial y} \right) + v^n \left(\frac{\partial u^n}{\partial y} \frac{\partial^2 v^n}{\partial x \partial y} + \frac{\partial v^n}{\partial y} \frac{\partial^2 v^n}{\partial y^2} \right) \right] - \Delta t \frac{\partial p^{n+1}}{\partial y} \end{aligned} \quad (20)$$

The pressure term has not been removed and is evaluated at the time $n+1$ and is not treated explicitly. If pressure at the time $n+1$ is known, the correct velocities are determined using equations Eq.19 and Eq.20. When we compare Eq. 17, 18 and Eq. 19, 20 we get

$$u^{n+1} = \tilde{u}^{n+1} - \Delta t \frac{\partial p^{n+1}}{\partial x}, \quad (21)$$

$$v^{n+1} = \tilde{v}^{n+1} - \Delta t \frac{\partial p^{n+1}}{\partial y}. \quad (22)$$

The velocities u^{n+1} and v^{n+1} must satisfy the continuity equation which implies

$$\frac{\partial^2 p^{n+1}}{\partial x^2} + \frac{\partial^2 p^{n+1}}{\partial y^2} = \frac{1}{\Delta t} \left(\frac{\partial \tilde{u}^{n+1}}{\partial x} + \frac{\partial \tilde{v}^{n+1}}{\partial y} \right). \quad (23)$$

Eq. 23 is the Poisson's equation with non-zero source term [2]. The pressure Eq. 23 is solved using MLPG over problem domain with boundary conditions $p^n|_{\Gamma_u} = \bar{p}^n$ and $\partial p^n / \partial n = \bar{q}^n$.

4. The MLPG Method and the Local Weak Formulation

The meshless Local Petrov-Galerkin method (MLPG) is truly meshless method which requires no elements or global background mesh, for either interpolation or integration purposes. In MLPG the problem domain is represented by a set of arbitrarily distributed nodes. The weighted residual method is used to create the discrete system equation by integrating the governing equation over local quadrature domains. The quadrature domain can be arbitrary in theory, but very simple regularly shaped domain, such as rectangles for 2D problems are often used for ease of implementation [6].

A generalized local weak form of the pressure Poisson Eq. 21 defined over local sub-domain Ω_s can be written as

$$\int_{\Omega_s} \left[\left(\frac{\partial^2 p^{n+1}}{\partial x^2} + \frac{\partial^2 p^{n+1}}{\partial y^2} \right) - \frac{1}{\Delta t} \left(\frac{\partial \tilde{u}^{n+1}}{\partial x} + \frac{\partial \tilde{v}^{n+1}}{\partial y} \right) \right] w d\Omega = 0, \quad (24)$$

where p is the trial function, w is the test function defined as

$$w(r_i) = \begin{cases} 1 - 6r_i^2 + 8r_i^3 - 3r_i^4 & r_i \leq 1 \\ 0 & r_i > 1 \end{cases}; \quad r_i = \frac{\|x - x_i\|}{d_s}, \quad (25)$$

where d_s is the size of the local quadrature domain, so it is evident that weighting function value is zero on its boundary. Using the divergence theorem the Eq. 22 has changed to

$$\begin{aligned} & \int_{\Omega_s} \left(\frac{\partial p^{n+1}}{\partial x} \frac{\partial w}{\partial x} + \frac{\partial p^{n+1}}{\partial y} \frac{\partial w}{\partial y} \right) d\Omega - \int_{\Gamma_{su}} \frac{\partial p^{n+1}}{\partial n} w d\Gamma = \\ & \int_{\Gamma_{sq}} \bar{q} w d\Gamma - \frac{1}{\Delta t} \int_{\partial\Omega_s} (\tilde{u}^{n+1} n_1 w + \tilde{v}^{n+1} n_2 w) d\Gamma + \frac{1}{\Delta t} \int_{\Omega_s} \left(\tilde{u}^{n+1} \frac{\partial w}{\partial x} + \tilde{v}^{n+1} \frac{\partial w}{\partial y} \right) d\Omega \end{aligned} \quad (26)$$

Because unknown nodal values of the pressure p are constants which can be moved out of the integral the equation, Eq. 24 can be changed to discrete system of linear equations, where global "stiffness" matrix is defined as

$$K_{ij} = \int_{\Omega_s^i} \left(\frac{\partial \phi_j}{\partial x} \frac{\partial w_i}{\partial x} + \frac{\partial \phi_j}{\partial y} \frac{\partial w_i}{\partial y} \right) d\Omega - \int_{\Gamma_{su}^i} \frac{\partial \phi_j}{\partial n} w_i d\Gamma \quad (27)$$

and the right-hand side "load" vector is

$$\begin{aligned} f_i = & \int_{\Gamma_{sq}} \bar{q} w d\Gamma - \frac{1}{\Delta t} \int_{\Gamma_{su}^i} (\tilde{u}^{n+1} n_1 w_i + \tilde{v}^{n+1} n_2 w_i) d\Gamma \\ & - \frac{1}{\Delta t} \int_{\Gamma_{sq}^i} (\tilde{u}^{n+1} n_1 w_i + \tilde{v}^{n+1} n_2 w_i) d\Gamma + \frac{1}{\Delta t} \int_{\Omega_s^i} \left(\tilde{u}^{n+1} \frac{\partial w_i}{\partial x} + \tilde{v}^{n+1} \frac{\partial w_i}{\partial y} \right) d\Omega \end{aligned} \quad (28)$$

5. Numerical Example

In this section the MLPG solution of Navier-Stokes equations developed in the previous sections is validated by solving the Couette flow example as a benchmark problem of fluid flow simulation [7]. The problem domain (see Fig. 2) is formed by regular nodal model with 41x81 nodes and nodal spacing of $n_s = 0.025$.

The plain Couette flow is defined as viscous flow between two parallel plates with zero external pressure gradient. The bottom plate is stationary and the upper plate is moving with constant velocity and the distance between plates $h = 1$ (see Fig. 2). The exact solution of the unsteady Couette flow is often reported as

$$u(y, t) = y + \frac{2}{\pi} \sum_{k=1}^{\infty} \frac{(-1)^k}{k} \sin(k\pi y) e^{-k^2 \pi^2 t \nu}, \quad (29)$$

where ν is the kinematic viscosity of fluid [8]. Rectangular computational domain (see Fig. 2) has been used to solve Couette flow. Because the laminar Couette flow is valid only for very low Reynolds'

number, our computations have been carried out with $Re = 1$. The time step has been constant $\Delta t = 0.0001$ [3].

Fig. 2: Network and boundary conditions for Couette flow.

Fig. 3: Comparison of results for Couette flow.

Horizontal velocity distribution at various time intervals has been compared with the corresponding exact solution according Eq. 29 and presented in Fig. 3. Table 1 shows solution errors according the L2 norm for five time intervals.

6. Conclusions

In this article, a numerical algorithm using the Meshless Local Petrov-Galerkin (MLPG) method for the incompressible Navier-Stokes equations is demonstrated. To deal with convection term, the characteristic-based split method was adopted and the set of recurrent equations was derived for time stepping procedure. The ability of the MLPG code to solve fluid dynamics problems was presented by solving Couette flow problem with reasonable accuracy when compared to exact solution.

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