Fuzzy Supervised Multi-Period Time Series Forecasting

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Abstract: *The goal of this paper is to propose a new method for fuzzy forecasting of time series with supervised learning and k -order fuzzy relationships. In the training phase based on k previous historical periods, a multidimensional matrix of fuzzy dependencies is constructed. During the test stage, the fitted fuzzy model is run for validating the observations and each output value is predicted by using a fuzzy input vector of k previous intervals. The proposed algorithm is verified by a benchmark dataset for fuzzy time series forecasting. The results obtained are similar or better than those of other fuzzy time series prediction methods. Comparative analysis shows the high potential of the new algorithm as an alternative to fuzzy prediction and reveals some opportunities for its further improvement.*

Keywords: *Fuzzy set, fuzzy time series, forecasting, membership function, fuzzy relationships.*

1. Introduction

The purpose of this paper is to suggest, analyze and assess a new supervised machine-learning algorithm for Time Series (TS) prediction with Fuzzy Transition Relationships Matrix (FTRM). Examining a training dataset with a sliding window of size k , the new method calculates a cumulative k -dimensional fuzzy matrix of transition dependencies (training period).

Then, in the second test period, using the new model, for each k consecutive elements from the test dataset, the next output value is obtained. The proposed algorithm is verified on a stock index benchmarking dataset. Comparative analysis of obtained prognostic values demonstrates that the new FTRM model is a feasible and flexible alternative to other Fuzzy Time Series (FTS) prediction models.

Time series analysis and forecasting are of particular importance in economics, engineering, biology, medicine, meteorology and a number of other fields. Regarding the complex and volatile nature of market prices, classical statistical methods, such as classification and regression, are poor choices in building an adequate model for non-linear stock exchanges. A major disadvantage of statistical models is that they cannot reveal hidden patterns of variations between price values.

The term “fuzzy time series” was coined by Song and Chissom [25]. Nowadays it includes various approaches and algorithms for forecasting fuzzy time series. Fuzzy sets approaches have been successfully implemented in various research domains [9-13, 16-24, 27]. Logically, they are successfully implemented in time series modeling [14].

In their seminal works, Song and Chissom [25, 26] build several predictive methods based on fuzzy relations between every two consecutive observations. Replacing complicated max-min composition operations from [26] with simplified arithmetic operations, Chen [2] suggests a more efficient method. Yu [28] adjusted the lengths of intervals with the use of a dynamic approach – distribution and optimization technique.

In the last decade, intelligent methods such as Artificial Neural Networks (ANN), Genetic Algorithms (GA) and Principal Component Analysis (PCA) are finding an increasing application to solve the task of fuzzy forecasting of time series [7, 8, 15]. Chang, Wei and Cheng [1] propose a hybrid Adaptive Network-based Fuzzy Inference System (ANFIS) model that employs AR and volatility to solve stock price forecast problems. Cheng et al. [4] model implements the Ordered Weighted Averaging (OWA) operator to fuse high-order data into the aggregated values of single attributes and an ANFIS procedure for forecasting stock price. Chen and Jian propose a PSA-based algorithm to get the optimal partition of the intervals in the Universe Of Discourse (UOD) of the main factor TAIEX and to discover the optimal partition of intervals in the UOD of the Secondary Factor (SF), where $SF \in \{\text{Dow Jones, NASDAQ, M1B}\}$. Based on the proposed PSA-based algorithm, constructed two-factor second-order fuzzy-trend logical relationship groups, and similarity measures between the subscripts of fuzzy sets, they create a new method for predicting the TAIEX and the NTD/USD exchange rates [3].

Meanwhile, improvements in information technologies and falling prices of integrated circuits made it possible to combine classical probability methods with modern fuzzy data processing algorithms. Research continues and involves more advanced forms of fuzzy numbers such as neutrosophic sets [5, 6]. H. Guan, S. Guan and A. Zhao [5] present new mechanisms for fuzzy neutrosophic logical relation representation and similarity measure to forecast time series fluctuation.

The short review of literature regarding fuzzy time series prediction shows that there is no common model or algorithm that solves this problem. Adoption of fuzzy methods with machine learning will help stock, bonds, derivatives, and forex traders to increase the accuracy and reliability of their decisions. This work develops and validates a new supervised fuzzy model, which performance is equal or exceeds that of previous forecasting methods, due to precisely selected UOD intervals.

The remaining part of this paper is organized as follows. Section 2 briefly introduces the basic concepts of FTS and transition relationships between consecutive time periods. Section 3 presents the peculiarities of the new fuzzy time series forecasting method. Section 4 analyses the empirical results obtained by using different prediction models. The last section concludes the paper.

2. Preliminaries

Definition 1 [15]. Let $L = \{l_1, l_2, \dots, l_g\}$ be a fuzzy set in the universe of discourse U . It can be defined by its membership function, $\mu_L: U \rightarrow [0, 1]$, where $\mu_L(u_i)$ denotes the grade of membership of u_i , $U = \{u_1, u_2, \dots, u_i, \dots, u_l\}$.

Definition 2 [15]. Let $F(t) \in \mathbb{R}$, $t = 1, 2, \dots, T$ be a time series, where T is its number of periods. $G(t)$ is defined as a fluctuation time series, where

$$G(t) = F(t) - F(t-1).$$

Each element of $G(t)$ can be represented by a fuzzy set $S(t)$, $t = 2, 3, \dots, T$, as defined in Definition 1. In other words, we “fuzzify” time series $G(t)$ into a Fuzzy-Fluctuation Time Series (FFTS) $S(t)$.

Definition 3 [15]. Let $S(t)$, $t = k+1, k+2, \dots, T$, $k \geq 1$, be a FFTS. If $S(t)$ is determined by $S(t-1)$, $S(t-2)$, $\dots$, $S(t-k)$, then the fuzzy-fluctuation logical relationship is represented by k -th order fuzzy fluctuation:

$$(1) \quad S(t-1), S(t-2), \dots, S(t-k) \rightarrow S(t), t \geq k+2.$$

Based on Type-1 fuzzy set theory, we propose the concept of fuzzy transition relationship, which employs the fuzzy values from previous time periods to reflect the fuzzy fluctuation trends and variations of an k -th order FFTS.

Definition 4. Let

$$\begin{aligned} M_{S(t-k)} &= \{\mu_{S(t-k)1}, \mu_{S(t-k)2}, \dots, \mu_{S(t-k)l}\}, \dots, \\ M_{S(t-2)} &= \{\mu_{S(t-2)1}, \mu_{S(t-2)2}, \dots, \mu_{S(t-2)l}\}, \\ M_{S(t-1)} &= \{\mu_{S(t-1)1}, \mu_{S(t-1)2}, \dots, \mu_{S(t-1)l}\}, \dots, \\ \text{and } M_{S(t)} &= \{\mu_{S(t)1}, \mu_{S(t)2}, \dots, \mu_{S(t)l}\} \end{aligned}$$

denote the FFTS $S(t-k)$, $\dots$, $S(t-2)$, $S(t-1)$ and $S(t)$. In order to uncover the dependence in price change between periods $t-k$, $\dots$, $t-2$, $t-1$, and t , we calculate the mutual relation between $M_{S(t-k)}$, $\dots$, $M_{S(t-2)}$, $M_{S(t-1)}$ and $M_{S(t)}$, summing the products obtained using the following formulas:

$$(2) \quad \Delta_{i\dots op} = M_{S(t)} * \mu_{S(t-k)i} * \dots * \mu_{S(t-2)o} * \mu_{S(t-1)p}, \quad i, \dots, o, p = 1, 2, \dots, l,$$

$$(3) \quad \text{FTRM}_{i\dots op}^S = \text{FTRM}_{i\dots op}^{S-1} + \Delta_{i\dots op},$$

for each k successive periods in the training set. Here the indices $i, \dots, op$ number is k . After normalization of the last k -dimensional matrix of fuzzy Type-1 numbers, we get final FTRM.

3. FFTS-FTRM algorithm

The new FFTS prediction method consists of several main parts: time series data fuzzification, new FTRM model establishment, a model-based FFTS output prediction and finally, defuzzifications of predicted fuzzy time series values into crisp ones. A flowchart of the new algorithm is depicted in Fig. 1.

A detailed step-by-step description of new algorithms can be found below:

Step 1. Enter the time series elements $F(t)$, $t = 2, 3, \dots, T$.

Step 2. Calculate the differences between every two adjacent periods $G(t)$, $t = 2, 3, \dots, T$. After that, determine the number of intervals g and calculate the

boundaries of each interval L . Let $g = 7$ and $L = \{l_1, l_2, \dots, l_7\}$. Then define seven levels of fuzziness as follows: $u_1 = (-\infty, l_2)$, $u_2 = [l_1, l_3)$, $u_3 = [l_2, l_4)$, $u_4 = [l_3, l_5)$, $u_5 = [l_4, l_6)$, $u_6 = [l_5, l_7)$, $u_7 = [l_6, +\infty)$.

<p><i>Training</i></p> <ol style="list-style-type: none"> 1. Input training time series. Transform time series into a fluctuation time series by using first difference operator. 2. Define the universe of discourse for obtained differences by using an approximately equal number of instances in each interval. Calculate the boundaries of each interval. 3. Set the membership function for each FFTS value. 4. Generate FTRM forecasting model with a sliding window of size k.
<p><i>Testing</i></p> <ol style="list-style-type: none"> 5. Predict the test dataset fuzzy output variables by the new model and k previous observations. 6. Defuzzify predicted output values. 7. Evaluate the model's prediction accuracy.

Fig. 1. Flowchart of the supervised fuzzy time series k -order prediction algorithm with transition relationships

Step 3. Convert the differences $G(t)$ into their corresponding triangular fuzzy numbers $S(t)$, $t = 2, 3, \dots, T$, in a given universe of discourse U . The membership degree μ_i of each index value is defined as follows:

$$(4) \quad \mu_1 = \begin{cases} 1, & x < l_1, \\ \frac{l_2 - x}{l_2 - l_1}, & l_1 \leq x < l_2, \\ 0, & l_2 \leq x, \end{cases} \quad \mu_k = \begin{cases} 0, & x < l_{k-1}, \\ \frac{x - l_{k-1}}{l_k - l_{k-1}}, & l_{k-1} \leq x < l_k, \\ \frac{l_{k+1} - x}{l_{k+1} - l_k}, & l_k \leq x < l_{k+1}, \\ 0, & l_{k+1} \leq x, \end{cases} \quad k = 2, \dots, 6,$$

$$\mu_7 = \begin{cases} 0, & x < l_6, \\ \frac{x - l_6}{l_7 - l_6}, & l_6 \leq x < l_7, \\ 1, & l_7 \leq x. \end{cases}$$

Step 4. Let FTRM be a fuzzy relationship matrix between each k successive FFTS elements. In case of $k = 2$, FTRM is a two-dimensional array of fuzzy numbers with l rows and l columns. Let the first dimension of the matrix correspond to the first factor $S(t - 1)$, and the second dimension – to the second factor $S(t - 2)$. Let denote $S(t - 2)$, $S(t - 1)$ and $S(t)$ from FFTS with fuzzy numbers $M_a = \{\mu_{a1}, \mu_{a2}, \dots, \mu_{al}\}$, $M_b = \{\mu_{b1}, \mu_{b2}, \dots, \mu_{bl}\}$, and $M_c = \{\mu_{c1}, \mu_{c2}, \dots, \mu_{cl}\}$. To determine the output change over time periods $t - 2$, $t - 1$ and t , compute the relationship between M_a , M_b and M_c by the next formula:

$$(5) \quad \Delta_{ij} = M_c * \mu_{ai} * \mu_{bj}.$$

Add Δ_{ij} matrix to the current FTRM state according to Equation (3). Repeat Equation (5) and Equation (3) for $t = 4, 5, \dots, T$. Normalize the final FTRM $_{l \times l}$.

Step 5. To forecast next value multiply FTRM and k previous periods values $M_{S(t-k)}, \dots, M_{S(t-2)}, M_{S(t-1)}$ and $M_{S(t)}$. In case of $k = 2$ the equation is:

$$(6) \quad \text{FTRM} = \text{FTRM} * M_a * M_b$$

By using contraction, we convert the result FTRM into a Type-1 fuzzy number:

$$(7) \quad \hat{S}(t) = \sum_{i=1, \dots, o, p=1}^l \text{FTRM}.$$

Step 6. Defuzzify $\hat{S}(t)$ into $G'(t)$, the forecasting value of future fluctuation $G(t)$. Calculate the predicted $F'(t+1)$ value by using the following formulae:

$$(8) \quad F'(t+1) = F(t) + G'(t).$$

Repeat Step 5 and Step 6 for each element in the test dataset.

Step 7. In order to assess prediction performance, calculate the differences between forecasted actual values. Apply one of the widely used indicator in time series models evaluations – Root of the Mean Squared Error (RMSE), defined as follows:

$$(9) \quad \text{RMSE} = \sqrt{\frac{\sum_{t=1}^n (\text{forecast}(t) - \text{actual}(t))^2}{n}}.$$

An example training and test function pseudocode of the new forecasting FTRM method is shown in Figs 2 and 3.

```
function FTRM_training(n_2, n_1, n) {
  var Ma = realToFuzzy(n_2);
  var Mb = realToFuzzy(n_1);
  var Mc = realToFuzzy(n);
  for (var i = 0, l = Ma.length; i < l; i++) {
    if (Ma[i] != 0.0) {
      for (var j = 0, h = Mb.length; j < h; j++) {
        var p = Ma[i] * Mb[j];
        if (p != 0.0) {
          for (var k = 0; k < N; k++) {
            FTRM_Array[i][j][k] += Mc[k] * p;
          };
        }
      }
    }
  }
};
```

Fig. 2. The training function pseudocode

```
function FTRM_test(n_2, n_1) {
  var Ma = realToFuzzy(n_2);
  var Mb = realToFuzzy(n_1);
  var Mc = zeros(N);
  for (var i = 0, l = Ma.length; i < l; i++) {
    if (x[i] != 0.0) {
      for (var j = 0, h = Mb.length; j < h; j++) {
        var p = Ma[i] * Mb[j];
        if (p != 0.0) {
          for (var k = 0; k < N; k++) {
            Mc[k] += FTRM_Array[i][j][k] * p;
          };
        }
      }
    }
  };
  Mc = normalize(Mc);
  return fuzzyToReal(Mc);
};
```

Fig. 3. The test function pseudocode

4. Numerical example

In this section, we compute future TAIEX values step by step with the new FTRM method. We also compare the forecasting results of the new method with the ones of other existing methods [1-5, 7-8, 15, 28].

Given: TAIEX Close prices from 01.01.1999 to 31.10.1999

Find: TAIEX forecasting Close prices from 01.11.1999 to 31.12.1999.

Step 1. We used TAIEX data from 01.01.1999 to 31.10.1999 to discover its price patterns. Transform the TAIEX time series into a fluctuation time series (Web Appendix Table A1).

(http://web.uni-plovdiv.bg/galili/FTRM_TAIEX_1999/Web_Appendix.pdf)

Step 2. In order to define the universe of discourse, we calculate the number of intervals, so all of them contain approximately the same number of observations and obtain that $g = 7$. Then we compute the intervals' boundaries: $l_1 = -110.56$, $l_2 = -50.06$, $l_3 = -18.91$, $l_4 = -0.61$, $l_5 = 29.24$, $l_6 = 74.09$, $l_7 = 136.01$. The corresponding μ_i values are depicted in Fig. 4.

Fig. 4. The membership functions $\mu_i, i = 1, \dots, 7$

Step 3. Fuzzify each fluctuation time series value, for example

$$F(3) = 6404.31 \text{ and } F(4) = 6421.75, G(4) = F(4) - F(3) = 17.44, \\ S(4) = \{0.00, 0.00, 0.00, 0.40, 0.60, 0.00, 0.00\}.$$

In this way, all the fuzzified fluctuation dataset were shown in Web Appendix Table A1.

Step 4. According to the new algorithm description, we generate the FTRM and in our experiment $k = 2$. From Step 3:

$$S(4) = \{0.00, 0.00, 0.00, 0.60, 0.40, 0.00, 0.00\}, \text{ and} \\ S(5) = \{0.00, 0.00, 0.77, 0.23, 0.00, 0.00, 0.00\}.$$

Then we calculate the product $S(4) * S(5)$ and obtain a sparse matrix with four non-zero elements: 0.308, 0.092, 0.462 and 0.138 in i, j positions (4, 3); (4, 4); (5, 3); and (5, 4) respectively. After $S(4) * S(5)$ multiplying with $S(6)$ the non-zero FTRM's elements are as follows:

$$\begin{aligned} \text{In position (4, 3)} &= \{0.000, 0.240, 0.068, 0.000, 0.000, 0.000, 0.000\}, \\ \text{in position (4, 4)} &= \{0.000, 0.360, 0.030, 0.000, 0.000, 0.000, 0.000\}, \\ \text{in position (5, 3)} &= \{0.000, 0.360, 0.102, 0.000, 0.000, 0.000, 0.000\}, \\ \text{in position (5, 4)} &= \{0.000, 0.108, 0.030, 0.000, 0.000, 0.000, 0.000\}. \end{aligned}$$

All results after 221 – 2 training steps are shown in Web Appendix Table A2.

Step 5. The forecast value of the TAIEX on November 1, 1999 was computed as follows:

$$\begin{aligned} S_{29.10.1999} &= \{0.00, 0.00, 0.00, 0.15, 0.85, 0.00, 0.00\}, \\ S_{30.10.1999} &= \{0.00, 0.00, 0.00, 0.00, 0.00, 0.00, 1.00\}. \end{aligned}$$

Then we calculate the product $S_{29.10.1999} * S_{30.10.1999}$ and obtain a sparse matrix with two non-zero elements: 0.15 and 0.85 in i, j positions (4, 7) and (5, 7), respectively. From FTRM forecasting model we get:

$$\begin{aligned} \text{In position (4, 7)} &= \{0.00, 0.72, 0.91, 0.19, 0.12, 0.14, 0.28\}, \\ \text{in position (5, 7)} &= \{0.14, 0.88, 1.27, 1.14, 0.98, 1.34, 0.22\}. \end{aligned}$$

After FTRM multiplying with $S_{29.10.1999} * S_{30.10.1999}$ we obtain next two fuzzy numbers:

In position (4, 7) = {0.000, 0.108, 0.1365, 0.0285, 0.018, 0.021, 0.042},

in position (5, 7) = {0.119, 0.748, 1.0795, 0.969, 0.833, 1.139, 0.187}.

Step 6. The fluctuation, obtained by summing of fluctuation fuzzy numbers and normalization is as follows:

$$\hat{S}(t) = \{-2.424, -7.894, -4.236, -0.112, 4.584, 15.832, 5.738\},$$

where t is 30.10.1999. The defuzzified value is $G'(t) = 11.488$. Then $F'(t + 1) = 7854.85 + 11.488 = 7866.338$. The forecasting results for training dataset are shown in Table 1.

Table 1. Forecasting results from 1 November 1999 to 30 December 1999

Date (DD.MM.YYYY)	Actual	Forecast	(Forecast– Actual) ²	Date (DD.MM.YYYY)	Actual	Forecast	(Forecast– Actual) ²
01.11.1999	7814.89	7866.34	2646.72	01.12.1999	7766.2	7772.67	41.85
02.11.1999	7721.59	7822.08	10097.46	02.12.1999	7806.26	7771.17	1231.22
03.11.1999	7580.09	7718.86	19257.13	03.12.1999	7933.17	7834.84	9669.59
04.11.1999	7469.23	7604.20	18217.55	04.12.1999	7964.49	7960.29	17.66
05.11.1999	7488.26	7504.69	269.80	06.12.1999	7894.46	7929.95	1259.81
06.11.1999	7376.56	7489.95	12857.61	07.12.1999	7827.05	7852.58	651.55
08.11.1999	7401.49	7338.37	3984.40	08.12.1999	7811.02	7834.43	548.04
09.11.1999	7362.69	7401.97	1542.84	09.12.1999	7738.84	7811.4	5264.53
10.11.1999	7401.81	7344.67	3265.06	10.12.1999	7733.77	7757.44	560.43
11.11.1999	7532.22	7423.22	11880.53	13.12.1999	7883.61	7733.52	22527.63
15.11.1999	7545.03	7557.83	163.92	14.12.1999	7850.14	7899.40	2426.88
16.11.1999	7606.2	7525.85	6456.39	15.12.1999	7859.89	7856.90	8.96
17.11.1999	7645.78	7607.41	1471.96	16.12.1999	7739.76	7858.77	14163.12
18.11.1999	7718.06	7668.75	2431.88	17.12.1999	7723.22	7703.90	373.09
19.11.1999	7770.81	7754.67	260.36	18.12.1999	7797.87	7729.82	4631.02
20.11.1999	7900.34	7798.9	10290.11	20.12.1999	7782.94	7814.76	1012.51
22.11.1999	8052.31	7942.01	12165.51	21.12.1999	7934.26	7797.64	18666.26
23.11.1999	8046.19	8059.81	185.47	22.12.1999	8002.76	7967.69	1230
24.11.1999	7921.85	8047.44	15771.89	23.12.1999	8083.49	8007.71	5742.98
25.11.1999	7904.53	7910.23	32.49	24.12.1999	8219.45	8126.09	8715.74
26.11.1999	7595.44	7911.11	99644.64	27.12.1999	8415.07	8278.32	18701.53
29.11.1999	7823.9	7619.51	41776.96	28.12.1999	8448.84	8416.74	1030.57
30.11.1999	7720.87	7794.3	5391.97	RMSE			94.11

To compare the performance of different FTS methods for prediction we calculate RMSE for forecasting the TAIEX 1999 (Table 2).

Table 2. Accuracy comparison for TAIEX 1999 prediction

Method	RSME
C h e n [2]	120
Y u [28]	145
H u a r n g and Y u [8]	109
H s i e h, H s i a o and Y e h [7]	86
C h a n g, W e i and C h e n g [1]	100
C h e n g et al. [4]	103
C h e n and J i a n [3]	99
J i a, Z h a o and G u a n [15]	99.12
H. G u a n, S. G u a n and A. Z h a o [5]	99.03
FTRM method	94.11

Though one of others methods, Hsieh, H s i a o and Y e h [7], has a better result, however, it relies on a computationally intensive programming algorithm (wavelet transformation, regression-correlation selection, neural network training, and artificial bee colony optimization). Our method is relatively simple, but flexible and feasible.

5. Conclusions

In this work, a model-based and adaptive fuzzy time series algorithm for k -th order forecasting is presented. The machine learning technique enables the efficient training of a model by using hidden statistical dependencies. The robustness of the new algorithm was proven by univariate TAIEX prediction with window size equal to two. Our experiments have shown that the new FTRM method achieves better performance than existing fuzzy prediction models with only one exception.

This study shows that the problem of price index prediction can be solved successfully utilizing the concept of fuzzy time series and a fuzzy matrix of transition relations. The paper demonstrates the application of a type-1 model with a nine-point scale (no influence, very low, low, medium, high, very high, absolutely high) for the TAIEX index dataset. According to the experiments conducted, the number of intervals and their boundaries directly affect prognosis quality. Best results are obtained when all intervals contain the same or approximately equal number of instances. The accuracy of the proposed FTPM predictive method is comparable or superior to that of the existing FTS models;

Our plans about future research include experiment with other input transformation operations (square root, cube root, ln, log) and multivariate input data, check the accuracy of the new model by predicting the prices of other stock indices, and establish the relevance and performance of the model for other types of data (production planning, quality control, marketing, finance).

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Appendix A

Table A1. Historical training data and fuzzified fluctuation data of TAIEX 1999

Date (DD. MM. YYYY)	TAIEX	Fluc- tua- tion	Fuz- zi- fied	Date (DD. MM. YYYY)	TAIEX	Fluc- tua- tion	Fuz- zi- fied	Date (DD. MM. YYYY)	TAIEX	Fluc- tua- tion	Fuz- zi- fied
05.01.1999	6152.43			01.02.1999	5862.79	-135.53	2	08.03.1999	6431.96	10.23	4
06.01.1999	6199.91	47.48	5	02.02.1999	5749.64	-113.15	3	09.03.1999	6493.43	61.47	5
07.01.1999	6404.31	204.4	6	03.02.1999	5743.86	-5.78	4	10.03.1999	6486.61	-6.82	4
08.01.1999	6421.75	17.44	4	04.02.1999	5514.89	-228.97	1	11.03.1999	6436.8	-49.81	3
11.01.1999	6406.99	-14.76	4	05.02.1999	5474.79	-40.1	4	12.03.1999	6462.73	25.93	4
12.01.1999	6363.89	-43.1	3	06.02.1999	5710.18	235.39	7	15.03.1999	6598.32	135.59	6
13.01.1999	6319.34	-44.55	3	08.02.1999	5822.98	112.8	5	16.03.1999	6672.23	73.91	5
14.01.1999	6241.32	-78.02	3	09.02.1999	5723.73	-99.25	3	17.03.1999	6757.07	84.84	5
15.01.1999	6454.6	213.28	7	10.02.1999	5798	74.27	5	18.03.1999	6895.01	137.94	6
16.01.1999	6483.3	28.7	4	20.02.1999	6072.33	274.33	7	19.03.1999	6997.29	102.28	5
18.01.1999	6377.25	-106.05	3	22.02.1999	6313.63	241.3	7	20.03.1999	6993.38	-3.91	4
19.01.1999	6343.36	-33.89	4	23.02.1999	6180.94	-132.69	2	22.03.1999	7043.23	49.85	5
20.01.1999	6310.71	-32.65	4	24.02.1999	6238.87	57.93	5	23.03.1999	6945.48	-97.75	3
21.01.1999	6332.2	21.49	4	25.02.1999	6275.53	36.66	4	24.03.1999	6889.42	-56.06	3
22.01.1999	6228.95	-103.25	3	26.02.1999	6318.52	42.99	5	25.03.1999	6941.38	51.96	5
25.01.1999	6033.21	-195.74	2	01.03.1999	6312.25	-6.27	4	26.03.1999	7033.25	91.87	5
26.01.1999	6115.64	82.43	5	02.03.1999	6263.54	-48.71	3	29.03.1999	6901.68	-131.57	2
27.01.1999	6138.87	23.23	4	03.03.1999	6403.14	139.6	6	30.03.1999	6898.66	-3.02	4
28.01.1999	6063.41	-75.46	3	04.03.1999	6393.74	-9.4	4	31.03.1999	6881.72	-16.94	4
29.01.1999	5984	-79.41	3	05.03.1999	6383.09	-10.65	4	01.04.1999	7018.68	136.96	6
30.01.1999	5998.32	14.32	4	06.03.1999	6421.73	38.64	4	02.04.1999	7232.51	213.83	7

Table A1 (continued)

Date (DD. MM. YYYY)	TAIEX	Fluc- tuation	Fuz- zi- fied	Date (DD. MM. YYYY)	TAIEX	Fluc- tuation	Fuz- zi- fied	Date (DD. MM. YYYY)	TAIEX	Fluc- tuation	Fuz- zi- fied
03.04.1999	7182.2	-50.31	3	06.05.1999	7560.05	-12.11	4	05.06.1999	7639.3	48.86	5
06.04.1999	7163.99	-18.21	4	07.05.1999	7469.33	-90.72	3	07.06.1999	7802.69	163.39	6
07.04.1999	7135.89	-28.1	4	10.05.1999	7484.37	15.04	4	08.06.1999	7892.13	89.44	5
08.04.1999	7273.41	137.52	6	11.05.1999	7474.45	-9.92	4	09.06.1999	7957.71	65.58	5
09.04.1999	7265.7	-7.71	4	12.05.1999	7448.41	-26.04	4	10.06.1999	7996.76	39.05	4
12.04.1999	7242.4	-23.3	4	13.05.1999	7416.2	-32.21	4	11.06.1999	7979.4	-17.36	4
13.04.1999	7337.85	95.45	5	14.05.1999	7592.53	176.33	6	14.06.1999	7973.58	-5.82	4
14.04.1999	7398.65	60.8	5	15.05.1999	7576.64	-15.89	4	15.06.1999	7960	-13.58	4
15.04.1999	7498.17	99.52	5	17.05.1999	7599.76	23.12	4	16.06.1999	8059.02	99.02	5
16.04.1999	7466.82	-31.35	4	18.05.1999	7585.51	-14.25	4	17.06.1999	8274.36	215.34	7
17.04.1999	7581.5	114.68	5	19.05.1999	7614.6	29.09	4	21.06.1999	8413.48	139.12	6
19.04.1999	7623.18	41.68	4	20.05.1999	7608.88	-5.72	4	22.06.1999	8608.91	195.43	6
20.04.1999	7627.74	4.56	4	21.05.1999	7606.69	-2.19	4	23.06.1999	8492.32	-116.59	3
21.04.1999	7474.16	-153.58	2	24.05.1999	7588.23	-18.46	4	24.06.1999	8589.31	96.99	5
22.04.1999	7494.6	20.44	4	25.05.1999	7417.03	-171.2	2	25.06.1999	8265.96	-323.35	1
23.04.1999	7612.8	118.2	5	26.05.1999	7426.63	9.6	4	28.06.1999	8281.45	15.49	4
26.04.1999	7629.09	16.29	4	27.05.1999	7469.01	42.38	5	29.06.1999	8514.27	232.82	7
27.04.1999	7550.13	-78.96	3	28.05.1999	7387.37	-81.64	3	30.06.1999	8467.37	-46.9	3
28.04.1999	7496.61	-53.52	3	29.05.1999	7419.7	32.33	4	02.07.1999	8572.09	104.72	5
29.04.1999	7289.62	-206.99	2	31.05.1999	7316.57	-103.13	3	03.07.1999	8563.55	-8.54	4
30.04.1999	7371.17	81.55	5	01.06.1999	7397.62	81.05	5	05.07.1999	8593.35	29.8	4
03.05.1999	7383.26	12.09	4	02.06.1999	7488.03	90.41	5	06.07.1999	8454.49	-138.86	2
04.05.1999	7588.04	204.78	6	03.06.1999	7572.91	84.88	5	07.07.1999	8470.07	15.58	4
05.05.1999	7572.16	-15.88	4	04.06.1999	7590.44	17.53	4	08.07.1999	8592.43	122.36	5
09.07.1999	8550.27	-42.16	3	09.08.1999	7028.01	-21.73	4	08.09.1999	7973.3	27.54	4
12.07.1999	8463.9	-86.37	3	10.08.1999	7269.6	241.59	7	09.09.1999	8025.02	51.72	5
13.07.1999	8204.5	-259.4	1	11.08.1999	7228.68	-40.92	4	10.09.1999	8161.46	136.44	6
14.07.1999	7888.66	-315.84	1	12.08.1999	7330.24	101.56	5	13.09.1999	8178.69	17.23	4
15.07.1999	7918.04	29.38	4	13.08.1999	7626.05	295.81	7	14.09.1999	8092.02	-86.67	3
16.07.1999	7411.58	-506.46	1	16.08.1999	8018.47	392.42	7	15.09.1999	7971.04	-120.98	3
17.07.1999	7366.23	-45.35	3	17.08.1999	8083.43	64.96	5	16.09.1999	7968.9	-2.14	4
19.07.1999	7386.89	20.66	4	18.08.1999	7993.71	-89.72	3	17.09.1999	7916.92	-51.98	3
20.07.1999	7806.85	419.96	7	19.08.1999	7964.67	-29.04	4	18.09.1999	8016.93	100.01	5
21.07.1999	7786.65	-20.2	4	20.08.1999	8117.42	152.75	6	20.09.1999	7972.14	-44.79	3

Table A1 (continued)

Date		Fluc-	Fuz-	Date		Fluc-	Fuz-	Date		Fluc-	Fuz-
(DD. MM. YYYY)	TAIEX	tua- tion	zi- fied	(DD. MM. YYYY)	TAIEX	tua- tion	zi- fied	(DD. MM. YYYY)	TAIEX	tua- tion	zi- fied
22.07.1999	7678.67	-107.98	3	21.08.1999	8153.57	36.15	4	27.09.1999	7759.93	-212.21	1
23.07.1999	7724.52	45.85	5	23.08.1999	8119.98	-33.59	4	28.09.1999	7577.85	-182.08	2
26.07.1999	7595.71	-128.81	2	24.08.1999	7984.39	-135.59	2	29.09.1999	7615.45	37.6	4
27.07.1999	7367.97	-227.74	1	25.08.1999	8127.09	142.7	6	30.09.1999	7598.79	-16.66	4
28.07.1999	7484.5	116.53	5	26.08.1999	8097.57	-29.52	4	01.10.1999	7694.99	96.2	5
29.07.1999	7359.37	-125.13	3	27.08.1999	8053.97	-43.6	3	02.10.1999	7659.55	-35.44	4
30.07.1999	7413.11	53.74	5	30.08.1999	8071.36	17.39	4	04.10.1999	7685.48	25.93	4
31.07.1999	7326.75	-86.36	3	31.08.1999	8157.73	86.37	5	05.10.1999	7557.01	-128.47	2
02.08.1999	7195.94	-130.81	2	01.09.1999	8273.33	115.6	5	06.10.1999	7501.63	-55.38	3
03.08.1999	7175.19	-20.75	4	02.09.1999	8226.15	-47.18	3	07.10.1999	7612	110.37	5
04.08.1999	7110.8	-64.39	3	03.09.1999	8073.97	-152.18	2	08.10.1999	7552.98	-59.02	3
05.08.1999	6959.73	-151.07	2	04.09.1999	8065.11	-8.86	4	11.10.1999	7607.11	54.13	5
06.08.1999	6823.52	-136.21	2	06.09.1999	8130.28	65.17	5	12.10.1999	7835.37	228.26	7
07.08.1999	7049.74	226.22	7	07.09.1999	7945.76	-184.52	2	13.10.1999	7836.94	1.57	4
14.10.1999	7879.91	42.97	5	20.10.1999	7666.64	-26.32	4	27.10.1999	7701.22	0.93	4
15.10.1999	7819.09	-60.82	3	21.10.1999	7654.9	-11.74	4	28.10.1999	7681.85	-19.37	4
16.10.1999	7829.39	10.3	4	22.10.1999	7559.63	-95.27	3	29.10.1999	7706.67	24.82	4
18.10.1999	7745.26	-84.13	3	25.10.1999	7680.87	121.24	5	30.10.1999	7854.85	148.18	6
19.10.1999	7692.96	-52.3	3	26.10.1999	7700.29	19.42	4				

Table A2. The transition relationship matrix TRM

Fuzzy set	1	2	3	4	5	6	7
11	0.14	0.14	0.14	0.43	0.14	0.00	0.00
12	0.06	0.06	0.12	0.47	0.18	0.06	0.06
13	0.05	0.12	0.20	0.32	0.27	0.02	0.02
14	0.02	0.07	0.20	0.38	0.20	0.09	0.05
15	0.02	0.07	0.24	0.31	0.18	0.09	0.09
16	0.00	0.00	0.07	0.57	0.21	0.07	0.07
17	0.00	0.08	0.15	0.38	0.15	0.08	0.15

Table A3. The normalized transition relationship matrix FTRMⁿ

<p>259</p>	<p>217</p>	<p>42</p>
<p>258</p>	<p>217</p>	<p>42</p>
First difference		
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