

Improved Cross Entropy Measures of Single Valued Neutrosophic Sets and Interval Neutrosophic Sets and Their Multicriteria Decision Making Methods

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Abstract: *Due to some drawbacks of the cross entropy between Single Valued Neutrosophic Sets (SVNSs) in dealing with decision-making problems, the existing single valued neutrosophic cross entropy indicates an asymmetrical phenomenon or may produce an undefined (unmeaningful) phenomenon in some situations. In order to overcome these disadvantages, this paper proposes an improved cross entropy measure of SVNSs and investigates its properties, and then extends it to a cross entropy measure between interval neutrosophic sets (INSs). Furthermore, the cross entropy measures are applied to multicriteria decision making problems with single valued neutrosophic information and interval neutrosophic information. In decision making methods, through the weighted cross entropy measure between each alternative and the the ideal alternative, one can obtain the ranking order of all alternatives and the best one. The decision-making methods using the proposed cross entropy measures can efficiently deal with decision making problems with incomplete, indeterminate and inconsistent information which exist usually in real situations. Finally, two illustrative examples are provided to demonstrate the application and efficiency of the developed decision making approaches under single valued neutrosophic and interval neutrosophic environments.*

Keywords: *Neutrosophic set, single valued neutrosophic set, interval neutrosophic set, cross entropy; multicriteria decision making.*

1. Introduction

Cross entropy is an important tool to judge the relation between two objects. Therefore, it can be widely applied to data analysis and classification, decision making, pattern recognition and so on. Therefore, many researchers have proposed various cross entropy measures. Zadeh [1, 2] firstly introduced fuzzy entropy

concept. Shannon [3] developed a cross entropy approach as information theory. Kullback and Leibler [4] proposed a measure of the “cross entropy distance” between two probability distributions. After that, Lin [5] introduced a modified cross entropy measure. Shang and Jiang [6] presented a fuzzy cross entropy measure and a symmetric discrimination information measure between fuzzy sets. As a generalization of De Luca and Termini [7] nonprobabilistic entropy, Vlachos and Sergiadis [8] have proposed an intuitionistic fuzzy cross entropy measure and applied it to pattern recognition, medical diagnosis and image segmentation. Then, Zhang and Jiang [9] defined a vague cross entropy measure by analogy with the cross entropy of probability distributions and applied it to pattern recognition and medical diagnosis and then Ye [10] further applied the cross entropy of vague sets to the fault diagnosis problem of turbine. Also, Ye [11] has applied the intuitionistic fuzzy cross entropy to multicriteria fuzzy decision-making problems. As a generalization of the vague cross-entropy [9], Ye [12] proposed an interval-valued intuitionistic fuzzy cross-entropy measure and applied it to multicriteria decision-making problems. Since a Single Valued Neutrosophic Set (SVNS) is an extension of an intuitionistic fuzzy set, Ye [13] extended the intuitionistic fuzzy cross entropy to SVNSs and proposed a single valued neutrosophic cross entropy measure, and then applied it to multicriteria decision-making problems with single valued neutrosophic information. However, the single valued neutrosophic cross entropy defined by Ye [13] has some drawbacks in some situations. For instance, it indicates an asymmetric phenomenon of the cross entropy measure of SVNSs or may produce an undefined (unmeaningful) phenomenon in some situations (details are given in the next section). In order to overcome these disadvantages, the paper proposes an improved cross entropy measure of SVNSs and investigates its properties, and then extends it to a cross entropy measure between Interval Neutrosophic Sets (INSs). Furthermore, the cross entropy measures of SVNSs and INSs are applied to multicriteria decision-making problems with single valued neutrosophic information and interval neutrosophic information.

The rest of the paper is organized as follows. Section 2 briefly describes some concepts of SVNSs, the cross entropy between SVNSs and its drawbacks in some cases. In Section 3 we propose an improved cross entropy measure between SVNSs and investigate its properties. Section 4 extends the improved cross entropy measure of SVNSs to the cross entropy measure of INSs. Section 5 applies the cross entropy measures of SVNSs and INSs to multicriteria decision making problems with single valued neutrosophic information and interval neutrosophic information. In Section 6 two illustrative examples are provided to demonstrate the application and efficiency of the developed decision making approaches under single valued neutrosophic and interval neutrosophic environments. Section 7 gives conclusions and perspectives of future work.

2. Single valued neutrosophic cross entropy

The neutrosophic set proposed firstly by Smarandache [14] generalizes an intuitionistic fuzzy set and an interval-valued intuitionistic fuzzy set from

philosophical point of view. It can represent uncertainty, imprecise, incomplete and inconsistent information, which the intuitionistic fuzzy set and the interval-valued intuitionistic fuzzy set cannot express. From scientific or engineering point of view, the neutrosophic set will be difficult to be applied in real science and engineering areas [15]. Therefore, as an instance of the neutrosophic set, Wang et al. [15] proposed a SVNS concept and set-theoretic operators for real scientific and engineering applications. SVNS is a generalization of an intuitionistic fuzzy set and gives us an additional possibility to represent uncertainty, imprecise, incomplete, and inconsistent information, which exists in real world, and then it would be more suitable to be applied in indeterminate information and inconsistent information measures. In the following discussion, we introduce the definition of SVNS [15].

Definition 1 [15]. Let X be a space of points (objects) with generic elements in X denoted by x . A SVNS A in X is characterized by a truth-membership function $T_A(x)$, an indeterminacy-membership function $I_A(x)$, and a falsity-membership function $F_A(x)$, where $T_A(x), I_A(x), F_A(x) \in [0, 1]$ for each point x in X . Therefore, a SVNS A can be expressed as

$$A = \{ \langle x, T_A(x), I_A(x), F_A(x) \rangle \mid x \in X \},$$

whereas, the sums of $T_A(x)$, $I_A(x)$ and $F_A(x)$ satisfy the condition

$$0 \leq T_A(x) + I_A(x) + F_A(x) \leq 3.$$

The following inclusion, equality, complement for SVNSs A and B are defined, respectively, as follows [15]:

- (1) $A \subseteq B$ if and only if $T_A(x) \leq T_B(x)$, $I_A(x) \geq I_B(x)$, $F_A(x) \geq F_B(x)$ for any x in X ,
- (2) $A = B$ if and only if $A \subseteq B$ and $B \subseteq A$,
- (3) $A^c = \{ \langle x, F_A(x), 1 - I_A(x), T_A(x) \rangle \mid x \in X \}$.

As the extension of the intuitionistic fuzzy cross entropy, Ye [13] has introduced a cross-entropy measure between SVNSs, which provides a better and more efficient mathematical framework in handling uncertain and inconsistent information.

For two SVNSs A and B , Ye [13] defined the following single valued neutrosophic cross entropy measure between A from B .

Definition 2 [13]. Let $A = \{ \langle x_i, T_A(x_i), I_A(x_i), F_A(x_i) \rangle \mid x_i \in X \}$ and $B = \{ \langle x_i, T_B(x_i), I_B(x_i), F_B(x_i) \rangle \mid x_i \in X \}$ be two SVNSs A and B in a universe of discourse $X = \{x_1, x_2, \dots, x_n\}$. Then, a single valued neutrosophic cross-entropy measure between A from B is defined as

$$(1) \quad E(A, B) = \sum_{i=1}^n \left[T_A(x_i) \log_2 \frac{T_A(x_i)}{(T_A(x_i) + T_B(x_i))/2} + (1 - T_A(x_i)) \log_2 \frac{1 - T_A(x_i)}{1 - (T_A(x_i) + T_B(x_i))/2} \right] + \\ + \sum_{i=1}^n \left[I_A(x_i) \log_2 \frac{I_A(x_i)}{(I_A(x_i) + I_B(x_i))/2} + (1 - I_A(x_i)) \log_2 \frac{1 - I_A(x_i)}{1 - (I_A(x_i) + I_B(x_i))/2} \right] + \\ + \sum_{i=1}^n \left[F_A(x_i) \log_2 \frac{F_A(x_i)}{(F_A(x_i) + F_B(x_i))/2} + (1 - F_A(x_i)) \log_2 \frac{1 - F_A(x_i)}{1 - (F_A(x_i) + F_B(x_i))/2} \right]$$

which also indicates the discrimination degree of A from B . Then, it satisfies the following properties [13]:

- 1) $E(A, B) \geq 0$,
- 2) $E(A, B) = 0$ if and only if $T_A(x_i) = T_B(x_i)$, $I_A(x_i) = I_B(x_i)$, and $F_A(x_i) = F_B(x_i)$, $x_i \in X$,
- 3) $E(A^c, B^c) = E(A, B)$.

However, one can find some drawbacks of Equation (1) as follows:

1) For two SVNNSs A and B , if one of the three functions $T_A(x_i)$, $I_A(x_i)$, $F_A(x_i)$ is equal to 0 or one of the three functions $T_B(x_i)$, $I_B(x_i)$ and $F_B(x_i)$ is equal to 1 for any x_i in X ($i = 1, 2, \dots, n$), Equation (1) is undefined or unmeaningful. In this case, one cannot utilize them to calculate the cross entropy between A from B .

2) For two SVNNSs A and B , there is $E(A, B) \neq E(B, A)$. This means that $E(A, B)$ is not symmetric.

As for the asymmetric problem of $E(A, B)$, Ye [13] has modified it into a symmetric discrimination information measure for SVNNSs:

$$(2) \quad D(A, B) = E(A, B) + E(B, A).$$

The bigger the difference between A and B is, the larger $D(A, B)$ is.

As for the disadvantages of Equation (1), we shall propose another form of single valued neutrosophic cross entropy in the following section to overcome the disadvantages.

3. Improved single valued neutrosophic cross entropy

This section defines an improved cross entropy measure of SVNNSs to overcome the aforementioned disadvantages in the existing single-valued neutrosophic cross-entropy measure [13].

Definition 3. For any two SVNNSs A and B in a universe of discourse $X = \{x_1, x_2, \dots, x_n\}$, the cross entropy between SVNNSs A from B is defined as follows:

$$(3) \quad N(A, B) = \sum_{i=1}^n \left[\sqrt{\frac{T_A^2(x_i) + T_B^2(x_i)}{2}} - \left(\frac{\sqrt{T_A(x_i)} + \sqrt{T_B(x_i)}}{2} \right)^2 + \right. \\ \left. + \sqrt{\frac{I_A^2(x_i) + I_B^2(x_i)}{2}} - \left(\frac{\sqrt{I_A(x_i)} + \sqrt{I_B(x_i)}}{2} \right)^2 + \right. \\ \left. + \sqrt{\frac{[1 - I_A(x_i)]^2 + [1 - I_B(x_i)]^2}{2}} - \left(\frac{\sqrt{1 - I_A(x_i)} + \sqrt{1 - I_B(x_i)}}{2} \right)^2 + \right. \\ \left. + \sqrt{\frac{F_A^2(x_i) + F_B^2(x_i)}{2}} - \left(\frac{\sqrt{F_A(x_i)} + \sqrt{F_B(x_i)}}{2} \right)^2 \right].$$

Thus, we can derive the following Theorem 1 from the cross entropy measure of SVNNSs A and B .

Theorem 1. For any two SVNNSs A and B in a universe of discourse $X = \{x_1, x_2, \dots, x_n\}$, the cross entropy measure $N(A, B)$ satisfies the following properties:

- 1) $N(A, B) \geq 0$,
 - 2) $N(A, B) = 0$ if and only if $T_A(x_i) = T_B(x_i)$, $I_A(x_i) = I_B(x_i)$, and $F_A(x_i) = F_B(x_i)$, $x_i \in X$,
 - 3) $N(A^c, B^c) = N(A, B)$,
 - 4) $N(A, B) = N(B, A)$.
- Proof:*

1) Since there exists the inequality $\sqrt{\frac{x^2 + y^2}{2}} \geq \left(\frac{\sqrt{x} + \sqrt{y}}{2}\right)^2$ for all real numbers x and y , one can yield $N(A, B) \geq 0$.

2) The inequality $\sqrt{\frac{x^2 + y^2}{2}} \geq \left(\frac{\sqrt{x} + \sqrt{y}}{2}\right)^2$ becomes the equality

$\sqrt{\frac{x^2 + y^2}{2}} = \left(\frac{\sqrt{x} + \sqrt{y}}{2}\right)^2$ if and only if $x = y$, thus there is $N(A, B) = 0$ if and only if

$A = B$, i.e., $T_A(x_i) = T_B(x_i)$, $I_A(x_i) = I_B(x_i)$, and $F_A(x_i) = F_B(x_i)$, $x_i \in X$.

3) It is obvious that $N(A^c, B^c) = N(A, B)$ is true.

4) It is obvious that $N(A, B) = N(B, A)$ is true.

Therefore, these proofs are completed. ■

It is clear that the proposed single valued neutrosophic cross entropy measure can overcome the drawbacks of the one proposed by Ye [13].

Let us consider the following example to demonstrate the efficiency of the proposed cross entropy measure.

Example 1. Let A and B be two SVNNSs in the universe of discourse X , which are given by $A = \{\langle x, 1, 0, 0 \rangle | x \in X\}$ and $B = \{\langle x, 0.5, 0.4, 0.2 \rangle | x \in X\}$.

Obviously, Equation (1) is undefined in this case. Then, one can obtain $N(A, B) = N(B, A) = 0.3736$ by using Equation (3). Thus, the proposed cross entropy can overcome the disadvantages of Equation (1). Hence, Equation (3) is more reasonable and more efficient than Equation (1).

When the differences of importance are considered in the elements in the universe, one needs to take the weight of the element x_i , $i = 1, 2, \dots, n$, into account. In the following, we introduce a weighted cross entropy measure of SVNNSs.

Let w_i be the weight for each element x_i , $i = 1, 2, \dots, n$, $w_i \in [0, 1]$ and $\sum_{i=1}^n w_i = 1$, then the weighted cross entropy measure between SVNNSs A from B can be defined by

$$\begin{aligned}
(4) \quad N_w(A, B) = & \sum_{i=1}^n w_i \left[\sqrt{\frac{T_A^2(x_i) + T_B^2(x_i)}{2}} - \left(\frac{\sqrt{T_A(x_i)} + \sqrt{T_B(x_i)}}{2} \right)^2 \right. \\
& + \sqrt{\frac{I_A^2(x_i) + I_B^2(x_i)}{2}} - \left(\frac{\sqrt{I_A(x_i)} + \sqrt{I_B(x_i)}}{2} \right)^2 + \\
& + \sqrt{\frac{[1 - I_A(x_i)]^2 + [1 - I_B(x_i)]^2}{2}} - \left(\frac{\sqrt{1 - I_A(x_i)} + \sqrt{1 - I_B(x_i)}}{2} \right)^2 + \\
& \left. + \sqrt{\frac{F_A^2(x_i) + F_B^2(x_i)}{2}} - \left(\frac{\sqrt{F_A(x_i)} + \sqrt{F_B(x_i)}}{2} \right)^2 \right].
\end{aligned}$$

It is obvious that $N_w(A, B)$ also satisfy the four properties of Theorem 1.

Theorem 2. Let w_i be the weight for each element x_i , $i = 1, 2, \dots, n$, $w_i \in [0, 1]$, and $\sum_{i=1}^n w_i = 1$, then the weighted cross entropy measure $N_w(A, B)$ also satisfies the following properties:

- 1) $N_w(A, B) \geq 0$,
- 2) $N_w(A, B) = 0$ if and only if $T_A(x_i) = T_B(x_i)$, $I_A(x_i) = I_B(x_i)$, and $F_A(x_i) = F_B(x_i)$, $x_i \in X$,
- 3) $N_w(A^c, B^c) = N_w(A, B)$,
- 4) $N_w(A, B) = N_w(B, A)$.

Since the process to prove these properties is similar to that in Theorem 1, it is not repeated here.

4. Cross entropy measure of INSs

For real scientific and engineering applications, Wang et al. [16] have also introduced the concept of an INS, which is also a subclass of the neutrosophic set, and give the definition of an INS.

Definition 4 [16]. Let X be a space of points (objects) with generic elements in X denoted by x . An INS A in X is characterized by a truth-membership function $T_A(x)$, an indeterminacy-membership function $I_A(x)$, and a falsity-membership function $F_A(x)$, where $T_A(x) = [\inf T_A(x), \sup T_A(x)] \subseteq [0, 1]$, $I_A(x) = [\inf I_A(x), \sup I_A(x)] \subseteq [0, 1]$, and $F_A(x) = [\inf F_A(x), \sup F_A(x)] \subseteq [0, 1]$ for each point x in X . Then, an INS A can be expressed as

$$\begin{aligned}
A &= \left\{ \langle x, T_A(x), I_A(x), F_A(x) \rangle \mid x \in X \right\} = \\
&= \left\{ \langle x, [\inf T_A(x), \sup T_A(x)], [\inf I_A(x), \sup I_A(x)], [\inf F_A(x), \sup F_A(x)] \rangle \mid x \in X \right\}.
\end{aligned}$$

Obviously, the sum of $T_A(x)$, $I_A(x)$ and $F_A(x)$ satisfies the condition $0 \leq \sup T_A(x) + \sup I_A(x) + \sup F_A(x) \leq 3$.

Especially, when the upper and lower ends of the interval values of $T_A(x)$, $I_A(x)$ and $F_A(x)$ in an INS A are equal, the INS A reduces to SVN A . However, both SVN A s and INS A s are the subclasses of neutrosophic sets.

The following expressions for INSs A and B are defined as follows [16]:

1) The complement A^c for an INS A is denoted as $T_A^c(x) = F_A(x) = [\inf F_A(x), \sup F_A(x)]$, $I_A^c(x) = [1 - \sup I_A(x), 1 - \inf I_A(x)]$, and $F_A^c(x) = T_A(x) = [\inf T_A(x), \sup T_A(x)]$ for any x in X .

2) $A \subseteq B$ if and only if $\inf T_A(x) \leq \inf T_B(x)$, $\sup T_A(x) \leq \sup T_B(x)$, $\inf I_A(x) \geq \inf I_B(x)$, $\sup I_A(x) \geq \sup I_B(x)$, $\inf F_A(x) \geq \inf F_B(x)$, and $\sup F_A(x) \geq \sup F_B(x)$ for any x in X .

3) $A = B$ if and only if $A \subseteq B$ and $B \subseteq A$.

Since the existing literature does not deal with the cross entropy of INSs, we can extend the improved cross entropy measure of SVNNS to the cross entropy measure of INSs.

Definition 5. For any two INSs A and B in a universe of discourse $X = \{x_1, x_2, \dots, x_n\}$, the cross entropy between INSs A from B is defined as follows:

$$(5) \quad M(A, B) = \frac{1}{2} \left\{ \sum_{i=1}^n \left[\sqrt{\frac{(\inf T_A(x_i))^2 + (\inf T_B(x_i))^2}{2}} - \left(\frac{\sqrt{\inf T_A(x_i)} + \sqrt{\inf T_B(x_i)}}{2} \right)^2 \right. \right. \\ + \sqrt{\frac{(\inf I_A(x_i))^2 + (\inf I_B(x_i))^2}{2}} - \left(\frac{\sqrt{\inf I_A(x_i)} + \sqrt{\inf I_B(x_i)}}{2} \right)^2 \\ + \sqrt{\frac{[1 - \inf I_A(x_i)]^2 + [1 - \inf I_B(x_i)]^2}{2}} - \left(\frac{\sqrt{1 - \inf I_A(x_i)} + \sqrt{1 - \inf I_B(x_i)}}{2} \right)^2 \\ \left. \left. + \sqrt{\frac{(\inf F_A(x_i))^2 + (\inf F_B(x_i))^2}{2}} - \left(\frac{\sqrt{\inf F_A(x_i)} + \sqrt{\inf F_B(x_i)}}{2} \right)^2 \right] \right. \\ + \sum_{i=1}^n \left[\sqrt{\frac{(\sup T_A(x_i))^2 + (\sup T_B(x_i))^2}{2}} - \left(\frac{\sqrt{\sup T_A(x_i)} + \sqrt{\sup T_B(x_i)}}{2} \right)^2 \right. \\ + \sqrt{\frac{(\sup I_A(x_i))^2 + (\sup I_B(x_i))^2}{2}} - \left(\frac{\sqrt{\sup I_A(x_i)} + \sqrt{\sup I_B(x_i)}}{2} \right)^2 \\ + \sqrt{\frac{[1 - \sup I_A(x_i)]^2 + [1 - \sup I_B(x_i)]^2}{2}} - \left(\frac{\sqrt{1 - \sup I_A(x_i)} + \sqrt{1 - \sup I_B(x_i)}}{2} \right)^2 \\ \left. \left. + \sqrt{\frac{(\sup F_A(x_i))^2 + (\sup F_B(x_i))^2}{2}} - \left(\frac{\sqrt{\sup F_A(x_i)} + \sqrt{\sup F_B(x_i)}}{2} \right)^2 \right] \right\}.$$

Similarly, $M(A, B)$ also satisfies the four properties of Theorem 1.

Theorem 3. For any two INSs A and B in a universe of discourse $X = \{x_1, x_2, \dots, x_n\}$, the cross entropy $M(A, B)$ satisfies the following properties:

- 1) $M(A, B) \geq 0$,
- 2) $M(A, B) = 0$ if and only if $T_A(x_i) = T_B(x_i)$, $I_A(x_i) = I_B(x_i)$, and $F_A(x_i) = F_B(x_i)$, $x_i \in X$,
- 3) $M(A^c, B^c) = M(A, B)$,
- 4) $M(A, B) = M(B, A)$.

Since the proof process of these properties is similar to that in Theorem 1, it is omitted here.

Taking the weight of the element $x_i, i = 1, 2, \dots, n$, into account, we can also introduce a weighted cross entropy measure between INSS.

Let w_i be the weight for each element $x_i, i = 1, 2, \dots, n, w_i \in [0, 1]$ and $\sum_{i=1}^n w_i = 1$, then we have the weighted cross entropy measure between INSS A and B :

$$\begin{aligned}
(6) \quad M_w(A, B) = & \frac{1}{2} \left\{ \sum_{i=1}^n w_i \left[\sqrt{\frac{(\inf T_A(x_i))^2 + (\inf T_B(x_i))^2}{2}} - \left(\frac{\sqrt{\inf T_A(x_i)} + \sqrt{\inf T_B(x_i)}}{2} \right)^2 \right. \right. \\
& + \sqrt{\frac{(\inf I_A(x_i))^2 + (\inf I_B(x_i))^2}{2}} - \left(\frac{\sqrt{\inf I_A(x_i)} + \sqrt{\inf I_B(x_i)}}{2} \right)^2 + \\
& + \sqrt{\frac{[1 - \inf I_A(x_i)]^2 + [1 - \inf I_B(x_i)]^2}{2}} - \left(\frac{\sqrt{1 - \inf I_A(x_i)} + \sqrt{1 - \inf I_B(x_i)}}{2} \right)^2 + \\
& \left. \left. + \sqrt{\frac{(\inf F_A(x_i))^2 + (\inf F_B(x_i))^2}{2}} - \left(\frac{\sqrt{\inf F_A(x_i)} + \sqrt{\inf F_B(x_i)}}{2} \right)^2 \right] + \right. \\
& + \sum_{i=1}^n w_i \left[\sqrt{\frac{(\sup T_A(x_i))^2 + (\sup T_B(x_i))^2}{2}} - \left(\frac{\sqrt{\sup T_A(x_i)} + \sqrt{\sup T_B(x_i)}}{2} \right)^2 \right. \\
& + \sqrt{\frac{(\sup I_A(x_i))^2 + (\sup I_B(x_i))^2}{2}} - \left(\frac{\sqrt{\sup I_A(x_i)} + \sqrt{\sup I_B(x_i)}}{2} \right)^2 + \\
& + \sqrt{\frac{[1 - \sup I_A(x_i)]^2 + [1 - \sup I_B(x_i)]^2}{2}} - \left(\frac{\sqrt{1 - \sup I_A(x_i)} + \sqrt{1 - \sup I_B(x_i)}}{2} \right)^2 + \\
& \left. \left. + \sqrt{\frac{(\sup F_A(x_i))^2 + (\sup F_B(x_i))^2}{2}} - \left(\frac{\sqrt{\sup F_A(x_i)} + \sqrt{\sup F_B(x_i)}}{2} \right)^2 \right] \right\}.
\end{aligned}$$

Especially, when $T_A(x_i) = \inf T_A(x_i) = \sup T_A(x_i)$, $I_A(x_i) = \inf I_A(x_i) = \sup I_A(x_i)$, and $F_A(x_i) = \inf F_A(x_i) = \sup F_A(x_i)$ in the INS A and $T_B(x_i) = \inf T_B(x_i) = \sup T_B(x_i)$, $I_B(x_i) = \inf I_B(x_i) = \sup I_B(x_i)$, $F_B(x_i) = \inf F_B(x_i) = \sup F_B(x_i)$ in the INS B for any $x_i, i = 1, 2, \dots, n$, in X , the INSS A and B reduce to the SVNSS A and B , while (5) and (6) reduce, respectively, to (3) and (4).

It is obvious that $M_w(A, B)$ also satisfy the four properties of Theorem 2.

Theorem 4. For any two INSS A and B in a universe of discourse $X = \{x_1, x_2, \dots, x_n\}$, the cross entropy $M_w(A, B)$ also satisfies the following properties:

- 1) $M_w(A, B) \geq 0$,
- 2) $M_w(A, B) = 0$ if and only if $T_A(x_i) = T_B(x_i)$, $I_A(x_i) = I_B(x_i)$, and $F_A(x_i) = F_B(x_i)$, $x_i \in X$,
- 3) $M_w(A^c, B^c) = M_w(A, B)$,
- 4) $M_w(A, B) = M_w(B, A)$.

Since the process to prove these properties is similar to that in Theorem 1, it is not repeated here.

5. Multicriteria decision-making methods using the cross entropy measures

A multicriteria decision making problem, where the alternatives can be evaluated according to multiple criteria, is the process of finding the best alternative among all of the feasible alternatives. In general, the multicriteria decision-making problem usually includes uncertainty, imprecise, incomplete and inconsistent information, which exists in real world. Then SVNNS and INNS can represent and handle the information. In this section multicriteria decision making methods, using the proposed cross entropy measures are developed under single valued neutrosophic and interval neutrosophic environments.

For a multicriteria decision-making problem, usually, there is a set of m alternatives $A = \{A_1, A_2, \dots, A_m\}$, which are to be evaluated based on a set of n criteria $C = \{C_1, C_2, \dots, C_n\}$. Assume that the weight of the criterion C_j , $j = 1, 2, \dots, n$, entered by the decision-maker, is w_j , $w_j \in [0, 1]$ and $\sum_{j=1}^n w_j = 1$.

Following, two multicriteria decision-making methods using the cross entropy measures of SVNNS and INNS are developed under single valued neutrosophic and interval neutrosophic environments.

5.1. Decision-making method using the cross entropy measure of SVNNS

In the multicriteria decision-making problem with single valued neutrosophic information, the characteristics of an alternative A_i , $i = 1, 2, \dots, m$, on a criterion C_j , $j = 1, 2, \dots, n$, can be represented by the form of a SVNNS:

$$A_i = \{\langle C_j, T_{A_i}(C_j), I_{A_i}(C_j), F_{A_i}(C_j) \rangle \mid C_j \in C, j = 1, 2, \dots, n\},$$

where $T_{A_i}(C_j), I_{A_i}(C_j), F_{A_i}(C_j) \in [0, 1]$ and $0 \leq T_{A_i}(C_j) + I_{A_i}(C_j) + F_{A_i}(C_j) \leq 3$ for $C_j \in C, j = 1, 2, \dots, n$, and $i = 1, 2, \dots, m$.

For convenience, the values of the three functions $T_{A_i}(C_j), I_{A_i}(C_j), F_{A_i}(C_j)$ in an SVNNS A_i are denoted by a Single Valued Neutrosophic Value (SVNV) $d_{ij} = \langle T_{ij}, I_{ij}, F_{ij} \rangle$, $i = 1, 2, \dots, m$, and $j = 1, 2, \dots, n$, which is usually derived from the evaluation of an alternative A_i with respect to a criterion C_j by the expert or decision maker. Thus, we can elicit a single valued neutrosophic decision matrix $D = (d_{ij})_{m \times n}$.

For the ranking order of the alternatives in the decision-making problem, each ideal SVNV is defined as $d_j^* = \langle 1, 0, 0 \rangle$, $j = 1, 2, \dots, n$, in the ideal alternative $A^* = \{d_1^*, d_2^*, \dots, d_n^*\}$. Then, based on Equation (4), the cross entropy between an alternative A_i , $i = 1, 2, \dots, m$, and the ideal alternative A^* can be expressed by

$$\begin{aligned}
(7) \quad N_w(A_i, A^*) = & \sum_{j=1}^n w_j \left[\sqrt{\frac{T_{ij}^2 + (T_j^*)^2}{2}} - \left(\frac{\sqrt{T_{ij}} + \sqrt{T_j^*}}{2} \right)^2 \right. \\
& + \sqrt{\frac{I_{ij}^2 + (I_j^*)^2}{2}} - \left(\frac{\sqrt{I_{ij}} + \sqrt{I_j^*}}{2} \right)^2 + \\
& + \sqrt{\frac{[1 - I_{ij}]^2 + [1 - I_j^*]^2}{2}} - \left(\frac{\sqrt{1 - I_{ij}} + \sqrt{1 - I_j^*}}{2} \right)^2 + \\
& \left. + \sqrt{\frac{F_{ij}^2 + (F_j^*)^2}{2}} - \left(\frac{\sqrt{F_{ij}} + \sqrt{F_j^*}}{2} \right)^2 \right].
\end{aligned}$$

Then the smaller value of the cross entropy $N_w(A_i, A^*)$, $i = 1, 2, \dots, m$, indicates that an alternative is closer to the ideal alternative. Therefore, the ranking order of all the alternatives can be determined according to the increasing order of the cross entropy measure values. The alternative with the smallest cross entropy measure value is the best one.

5.2. Decision-making method using the cross entropy measure of INSS

In the multicriteria decision-making problem with interval neutrosophic information, the characteristics of an alternative A_i , $i = 1, 2, \dots, m$, on a criterion C_j , $j = 1, 2, \dots, n$, can be expressed by the following INS form:

$$\begin{aligned}
A_i = & \{ \langle C_j, T_{A_i}(C_j), I_{A_i}(C_j), F_{A_i}(C_j) \rangle \mid C_j \in C, j = 1, 2, \dots, n \} = \\
= & \left\{ C_j, \left[\inf T_{A_i}(C_j), \sup T_{A_i}(C_j) \right], \left[\inf I_{A_i}(C_j), \sup I_{A_i}(C_j) \right], \right. \\
& \left. \left[\inf F_{A_i}(C_j), \sup F_{A_i}(C_j) \right] \mid C_j \in C, j = 1, 2, \dots, n \right\}
\end{aligned}$$

where $T_{A_i}(C_j)$, $I_{A_i}(C_j)$, $F_{A_i}(C_j) \subseteq [0, 1]$ and $0 \leq \sup T_{A_i}(C_j) + \sup I_{A_i}(C_j) + \sup F_{A_i}(C_j) \leq 3$ for $C_j \in C, j = 1, 2, \dots, n$, and $i = 1, 2, \dots, m$.

For convenience, the interval values of the three functions $T_{A_i}(C_j) = [\inf T_{A_i}(C_j), \sup T_{A_i}(C_j)]$, $I_{A_i}(C_j) = [\inf I_{A_i}(C_j), \sup I_{A_i}(C_j)]$, $F_{A_i}(C_j) = [\inf F_{A_i}(C_j), \sup F_{A_i}(C_j)]$ in an INS A_i are denoted by an interval neutrosophic value (INV) $r_{ij} = \left\langle [T_{ij}^L, T_{ij}^U], [I_{ij}^L, I_{ij}^U], [F_{ij}^L, F_{ij}^U] \right\rangle$, $i = 1, 2, \dots, m$, and $j = 1, 2, \dots, n$, which is usually derived from the evaluation of alternative A_i with respect to a criterion C_j by an expert or decision maker. Thus, one can obtain an interval neutrosophic decision matrix $R = (r_{ij})_{m \times n}$.

In the multicriteria decision-making method, an ideal INV can be defined as $r_j^* = \langle [T_j^{L^*}, T_j^{U^*}], [I_j^{L^*}, I_j^{U^*}], [F_j^{L^*}, F_j^{U^*}] \rangle = \langle [1, 1], [0, 0], [0, 0] \rangle$, $j = 1, 2, \dots, n$, in the ideal alternative $A^* = \{r_1^*, r_2^*, \dots, r_n^*\}$.

Hence, by applying Equation (6) the weighted cross entropy between an alternative A_i , $i = 1, 2, \dots, m$, and the ideal alternative A^* is given by the following formula:

$$(8) \quad M_w(A_i, A^*) = \frac{1}{2} \left\{ \sum_{j=1}^n w_j \left[\sqrt{\frac{(T_{ij}^L)^2 + (T_j^{L^*})^2}{2}} - \left(\frac{\sqrt{T_{ij}^L} + \sqrt{T_j^{L^*}}}{2} \right)^2 + \sqrt{\frac{(I_{ij}^L)^2 + (I_j^{L^*})^2}{2}} - \left(\frac{\sqrt{I_{ij}^L} + \sqrt{I_j^{L^*}}}{2} \right)^2 \right] + \right. \\ \left. + \sqrt{\frac{[1-I_{ij}^L]^2 + [1-I_j^{L^*}]^2}{2}} - \left(\frac{\sqrt{1-I_{ij}^L} + \sqrt{1-I_j^{L^*}}}{2} \right)^2 + \sqrt{\frac{(F_{ij}^L)^2 + (F_j^{L^*})^2}{2}} - \left(\frac{\sqrt{F_{ij}^L} + \sqrt{F_j^{L^*}}}{2} \right)^2 \right] + \\ + \sum_{j=1}^n w_j \left[\sqrt{\frac{(T_{ij}^U)^2 + (T_j^{U^*})^2}{2}} - \left(\frac{\sqrt{T_{ij}^U} + \sqrt{T_j^{U^*}}}{2} \right)^2 + \sqrt{\frac{(I_{ij}^U)^2 + (I_j^{U^*})^2}{2}} - \left(\frac{\sqrt{I_{ij}^U} + \sqrt{I_j^{U^*}}}{2} \right)^2 \right] + \\ \left. + \sqrt{\frac{[1-I_{ij}^U]^2 + [1-I_j^{U^*}]^2}{2}} - \left(\frac{\sqrt{1-I_{ij}^U} + \sqrt{1-I_j^{U^*}}}{2} \right)^2 + \sqrt{\frac{(F_{ij}^U)^2 + (F_j^{U^*})^2}{2}} - \left(\frac{\sqrt{F_{ij}^U} + \sqrt{F_j^{U^*}}}{2} \right)^2 \right] \right\}.$$

Thus, the smaller value of the cross entropy $M_w(A_i, A^*)$, $i = 1, 2, \dots, m$, indicates that an alternative is closer to the ideal alternative. Therefore, the ranking order of all the alternatives can be determined according to the increasing order of the cross entropy measure values. The alternative with the smallest cross entropy measure value is the best one.

6. Illustrative examples

This section provides two illustrative examples for multicriteria decision making problems to demonstrate the application and efficiency of the proposed decision making methods under single valued neutrosophic and interval neutrosophic environments.

6.1. Single valued neutrosophic decision-making problem on selecting global suppliers

Let us consider that the multicriteria decision-making problem adopted from Ye [13] is concerned with a manufacturing company, which wants to select the best global supplier according to the core competencies of suppliers. Now, suppose that there are four suppliers $\{A_1, A_2, A_3, A_4\}$ whose core competencies are evaluated by means of four criteria: (1) C_1 is the level of technology innovation; (2) C_2 is the control ability of the flow; (3) C_3 is the ability of management; (4) C_4 is the level of service. Then, the weight vector for the four criteria is $w = (0.3, 0.25, 0.25, 0.2)^T$. The decision matrix of the suppliers is made up according to the four evaluating criteria. Therefore, the single valued neutrosophic decision matrix of the candidates [13] can be obtained as follows:

$$D = \begin{pmatrix} \langle 0.5, 0.1, 0.3 \rangle & \langle 0.5, 0.1, 0.4 \rangle & \langle 0.7, 0.1, 0.2 \rangle & \langle 0.3, 0.2, 0.1 \rangle \\ \langle 0.4, 0.2, 0.3 \rangle & \langle 0.3, 0.2, 0.4 \rangle & \langle 0.9, 0.0, 0.1 \rangle & \langle 0.5, 0.3, 0.2 \rangle \\ \langle 0.4, 0.3, 0.1 \rangle & \langle 0.5, 0.1, 0.3 \rangle & \langle 0.5, 0.0, 0.4 \rangle & \langle 0.6, 0.2, 0.2 \rangle \\ \langle 0.6, 0.1, 0.2 \rangle & \langle 0.2, 0.2, 0.5 \rangle & \langle 0.4, 0.3, 0.2 \rangle & \langle 0.7, 0.2, 0.1 \rangle \end{pmatrix}.$$

The proposed single valued neutrosophic decision-making method is applied to select the most desirable supplier.

By using Equation (7), one can obtain the values of the cross entropy $N_w(A_i, A^*)$, $i = 1, 2, 3, 4$:

$N_w(A_1, A^*) = 0.2439$, $N_w(A_2, A^*) = 0.2792$, $N_w(A_3, A^*) = 0.2580$ and $N_w(A_4, A^*) = 0.3034$.

According to the cross entropy measure values, the order of the four suppliers is A_1, A_3, A_2 , and A_4 . Hence, the best supplier is A_1 .

For convenient comparison, we show the decision making results and the ones in [13] in Table 1.

Table 1. Results of single valued neutrosophic decision-making methods

Measure/Ranking	Proposed method	Ye's method [13]
Measure result	$N_w(A_1, A^*) = 0.2439$	$D(A_1, A^*) = 1.1101$
	$N_w(A_2, A^*) = 0.2792$	$D(A_2, A^*) = 1.1801$
	$N_w(A_3, A^*) = 0.2580$	$D(A_3, A^*) = 0.9962$
	$N_w(A_4, A^*) = 0.3034$	$D(A_4, A^*) = 1.2406$
Ranking order	A_1, A_3, A_2, A_4	A_3, A_1, A_2, A_4

From the results of Table 1, one can see that the ranking order of the four alternatives and the best choice are different from the results of Ye's method [13] and indicate the difference between A_1 and A_3 . Obviously, the different measure methods may yield different results. However, since the improved cross-entropy measure in this paper can overcome the shortcomings of the existing cross-entropy measure [13], the proposed method is more reasonable and simpler than the existing method [13].

6.2. Interval neutrosophic decision-making problem of investment alternatives

Let us consider the multicriteria decision making problem discussed in [17]. There is an investment company, which wants to invest a sum of money in the best option. To invest the money, there is a panel with four possible alternatives: (1) A_1 is a car company; (2) A_2 is a food company; (3) A_3 is a computer company; (4) A_4 is an arms company. The investment company must take a decision according to the three criteria: (1) C_1 is the risk; (2) C_2 is the growth; (3) C_3 is the environmental impact. The weight vector of the three criteria is given by $w = (0.35, 0.25, 0.4)^T$ [17].

When the four possible alternatives are to be evaluated by the expert under the above three criteria in the form of INVs, one can obtain the following interval neutrosophic decision matrix R [17]:

$$R = \begin{pmatrix} \langle [0.4, 0.5], [0.2, 0.3], [0.3, 0.4] \rangle & \langle [0.4, 0.6], [0.1, 0.3], [0.2, 0.4] \rangle & \langle [0.4, 0.5], [0.7, 0.8], [0.7, 0.9] \rangle \\ \langle [0.6, 0.7], [0.1, 0.2], [0.2, 0.3] \rangle & \langle [0.6, 0.7], [0.1, 0.2], [0.2, 0.3] \rangle & \langle [0.8, 0.9], [0.5, 0.7], [0.3, 0.6] \rangle \\ \langle [0.3, 0.6], [0.2, 0.3], [0.3, 0.4] \rangle & \langle [0.5, 0.6], [0.2, 0.3], [0.3, 0.4] \rangle & \langle [0.7, 0.9], [0.6, 0.8], [0.4, 0.5] \rangle \\ \langle [0.7, 0.8], [0.0, 0.1], [0.1, 0.2] \rangle & \langle [0.6, 0.7], [0.1, 0.2], [0.1, 0.3] \rangle & \langle [0.8, 0.9], [0.6, 0.7], [0.6, 0.7] \rangle \end{pmatrix}.$$

Then, the developed interval neutrosophic decision-making approach is utilized to obtain the most desirable alternative(s).

By using Equation (8), one can obtain the values of the cross entropy $M_w(A_i, A^*)$, $i = 1, 2, 3, 4$:

$$M_w(A_1, A^*) = 0.5872, M_w(A_2, A^*) = 0.3643, M_w(A_3, A^*) = 0.4895, \text{ and} \\ M_w(A_4, A^*) = 0.3724.$$

Thus, the ranking order of the four alternatives is A_2, A_4, A_3 and A_1 . Therefore, the alternative A_2 is the best choice among the four alternatives.

For convenient comparison, we also show the decision making results and the ones in [17] in Table 2. It is worth noting that since the alternative with the largest similarity measure value in Ye's methods (1) and (2) [17] indicates the best one, the similarity measure values in [17] should be ranked in a decreasing order.

Table 2. Results of interval neutrosophic decision-making methods

Measure/Ranking	Proposed method	Ye's method (1) [17]	Ye's method (2) [17]
Measure result	$M_w(A_1, A^*) = 0.5872$	$S_1(A_1, A^*) = 0.7667$	$S_2(A_1, A^*) = 0.7370$
	$M_w(A_2, A^*) = 0.3643$	$S_1(A_2, A^*) = 0.9542$	$S_2(A_2, A^*) = 0.9323$
	$M_w(A_3, A^*) = 0.4895$	$S_1(A_3, A^*) = 0.8625$	$S_2(A_3, A^*) = 0.8344$
	$M_w(A_4, A^*) = 0.3724$	$S_1(A_4, A^*) = 0.9600$	$S_2(A_4, A^*) = 0.9034$
Ranking order	A_2, A_4, A_3, A_1	A_4, A_2, A_3, A_1	A_2, A_4, A_3, A_1

From the results of Table 2, one can see that the ranking order of the four alternatives and the best choice are in agreement with the results of Ye's method (2) [17] and indicate the efficiency of the proposed method. Then, the ranking orders between the proposed method and Ye's method (1) [17] and between Ye's methods [17] only indicate the difference between A_2 and A_4 in different measure methods.

For the above two examples, one can see that the multicriteria decision making methods using the cross entropy measures proposed in this paper are more suitable for real scientific and engineering applications because the proposed cross entropy measures of SVNNSs and INNSs cannot produce undefined phenomena.

7. Conclusion

To overcome the drawbacks of the cross entropy measure defined in [13], this paper has developed an improved cross entropy measure of SVNNSs and investigated its properties, and then extended it to the cross entropy measure of INNSs. Furthermore, the proposed cross entropy measures of SVNNSs and INNSs have been applied to multicriteria decision-making problems with single valued neutrosophic information and interval neutrosophic information. In the two decision-making methods, through the cross entropy measure between each alternative and the the ideal alternative, one can obtain the ranking order of all alternatives and the best one. Finally, two illustrative examples demonstrate the application and efficiency of the decision making approaches in a single valued neutrosophic setting and interval neutrosophic setting. The proposed decision making methods can efficiently deal with decision making problems with single valued neutrosophic information and interval neutrosophic information since the proposed cross entropy can overcome the drawbacks of the cross entropy defined in [13]. In future, we shall apply the

cross entropy measures of SVN_Ss and IN_Ss to other domains, such as pattern recognition, clustering analysis, and medical diagnosis.

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