Measure of departure from cumulative local symmetry for square contingency tables having ordered categories

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Summary

For square contingency tables with nominal categories, a local symmetry model which indicates the symmetric structure of probabilities for only one pair of symmetric cells is proposed. For ordinal square tables, the present paper proposes (1) another local symmetry model for cumulative probabilities from the upper-right and lower-left corners of the table, and (2) a measure to represent the degree of departure from the proposed model. The measure has the form of a weighted harmonic mean of the diversity index, which includes the Shannon entropy as a special case. Examples are given in which the proposed method is applied to square table data on decayed teeth in Japanese women patients.

Key words: comparison, confidence interval, diversity index, local symmetry, measure, ordered category, weighted harmonic mean

1. Introduction

For an $r \times r$ contingency table with the same row and column classifications, let p_{ij} denote the probability that an observation will fall in the i-th row and j-th column of the table ($i = 1, \ldots, r; j = 1, \ldots, r$). Several symmetry models with respect to the main diagonal are considered. The symmetry (S) model (Bowker, 1948; Bishop *et al.*, 1975, p.282) is defined as

$$p_{ij} = \psi_{ij} \quad \text{where } \psi_{ij} = \psi_{ji} \quad \text{for all } i \neq j.$$

The partial symmetry (PS) model (Saigusa *et al.*, 2016) is defined as

$$p_{ij} = \psi_{ij} \quad \text{where } \psi_{st} = \psi_{ts} \quad \text{for at least one } s \neq t.$$

The local symmetry (LS) model (Saigusa *et al.*, 2019b) is defined as

$$p_{ij} = \psi_{ij} \quad \text{where } \psi_{ab} = \psi_{ba} \quad \text{for only one } a \neq b.$$

The LS model describes that the cell probability that an observation will fall in the a-th row category and the b-th $(> a)$ column category is equal to the probability that the observation falls in the b-th row category and the a-th column category for only one (a, b).

For a square contingency table having ordered categories, some symmetry models for cumulative probabilities from the upper-right and lower-left corners of the table are also considered. Denote the row and column variables by X and Y. The cumulative probability is defined as

$$G_{ij} = \begin{cases} \displaystyle\sum_{s=1}^{i} \sum_{t=j}^{r} p_{st} = \Pr(X \leqslant i, Y \geqslant j) & \text{when } i < j, \\ \displaystyle\sum_{s=i}^{r} \sum_{t=1}^{j} p_{st} = \Pr(X \geqslant i, Y \leqslant j) & \text{when } i > j. \end{cases}$$

Then the S model can be expressed as

$$G_{ij} = \Psi_{ij} \quad \text{where } \Psi_{ij} = \Psi_{ji} \quad \text{for all } i \neq j.$$

The cumulative partial symmetry (CPS) model (Saigusa *et al.*, 2019a) is defined as

$$G_{ij} = \Psi_{ij} \quad \text{where } \Psi_{st} = \Psi_{ts} \quad \text{for at least one } s \neq t.$$

In data analysis, when a model fits the data poorly, we are interested in measuring the degree of departure from the model. For square contingency tables having nominal categories, measures to represent the degrees of departure from the S, PS and LS models have been developed (Tomizawa, 1994; Tomizawa *et al.*, 1998; Saigusa *et al.*, 2016; Saigusa *et al.*, 2019b). These measures are given as forms of weighted arithmetic, geometric and harmonic means of a diversity index (Patil and Taillie, 1982), consisting of cell probabilities. These measures may not be appropriate for ordered tables, in the sense that the values of these measures do not depend on the order of categories. For square contingency tables with ordered categories,

some measures for structures on the cumulative probabilities have been developed to incorporate information on the order of categories. The measures for the S and CPS models are given as weighted arithmetic and geometric means of the diversity index consisting of the cumulative probabilities. The details of some of the measures introduced in the above are presented in the Appendix.

The purpose of this paper is to propose another local symmetry model for the cumulative probabilities and a weighted harmonic mean measure for the proposed model. The rest of this paper is organized as follows. Section 2 considers the new model and measure. Section 3 gives the approximate confidence interval of the proposed measure. Section 4 provides an analysis of Japanese decayed teeth data in terms of another local symmetry by using the proposed measure. Sections 5 and 6 contain discussion and brief concluding remarks.

2. Model and measure

For square contingency tables with ordered categories, we propose the cumulative local symmetry (CLS) model defined by

$$G_{ij} = \Psi_{ij} \quad \text{where} \quad \Psi_{ab} = \Psi_{ba} \quad \text{for only one } a \neq b.$$

The CLS model describes that the probability that an observation will fall in the a-th row category or below and the b-th ($> a$) column category or above is equal to the probability that the observation falls in the b-th row category or above and the a-th column category or below for only one (a, b). Note that the proposed CLS model has a different structure from the LS model.

Assume that $G_{ij} + G_{ji} \neq 0$ for $i < j$ and $G_{kl} \neq G_{lk}$ for $k < l$ except for any one pair (k, l). Then we propose a measure to represent the degree of departure from the CLS model as follows:

$$\gamma_{CLS}^{(\lambda)} = \frac{\prod\limits_{a=1}^{r-1} \prod\limits_{b=a+1}^{r} \gamma_{ab}^{(\lambda)}}{\sum\limits_{i=1}^{r-1} \sum\limits_{j=i+1}^{r} \left[\left(G_{ij}^* + G_{ji}^* \right) \prod\limits_{a=1}^{r-1} \prod\limits_{\substack{b=a+1 \\ (a,b) \neq (i,j)}}^{r} \gamma_{ab}^{(\lambda)} \right]} \quad \text{for} \quad \lambda > -1,$$

where

$$G_{ij}^* = \frac{G_{ij}}{\Delta}, \quad \Delta = \sum_{i=1}^{r-1} \sum_{j=i+1}^{r} (G_{ij} + G_{ji}), \quad G_{ij}^c = \frac{G_{ij}}{G_{ij} + G_{ji}},$$

$$\gamma_{ij}^{(\lambda)} = \begin{cases} 1 - \dfrac{\lambda 2^\lambda}{2^\lambda - 1} H_{ij}^{(\lambda)} & \text{for } \lambda \neq 0, \\ 1 - \dfrac{1}{\log 2} H_{ij}^{(0)} & \text{for } \lambda = 0, \end{cases}$$

$$H_{ij}^{(\lambda)} = \begin{cases} \dfrac{1}{\lambda} \left(1 - \left(G_{ij}^c \right)^{\lambda+1} - \left(G_{ji}^c \right)^{\lambda+1} \right) & \text{for } \lambda \neq 0, \\ -G_{ij}^c \log G_{ij}^c - G_{ji}^c \log G_{ji}^c & \text{for } \lambda = 0. \end{cases}$$

When the CLS model does not hold, we can see

$$\gamma_{CLS}^{(\lambda)} = \frac{1}{\displaystyle\sum_{i=1}^{r-1} \sum_{j=i+1}^{r} \dfrac{G_{ij}^* + G_{ji}^*}{\gamma_{ij}^{(\lambda)}}} \quad \text{for } \lambda > -1.$$

Therefore, the measure $\gamma_{CLS}^{(\lambda)}$ is expressed as the weighted harmonic mean of the submeasure $\gamma_{ij}^{(\lambda)}$ for the degree of asymmetry on $\{G_{ij}^c, G_{ji}^c\}$. For any λ (> -1), we have some properties of the measure $\gamma_{CLS}^{(\lambda)}$ as follows:

  (i) $\gamma_{CLS}^{(\lambda)}$ must lie between 0 and 1,

 (ii) $\gamma_{CLS}^{(\lambda)} = 0$ if and only if the CLS model holds,

(iii) $\gamma_{CLS}^{(\lambda)} = 1$ if and only if the degree of departure from the CLS model is the largest in the sense that $G_{ij} = 0$ (then $G_{ji} > 0$) or $G_{ji} = 0$ (then $G_{ij} > 0$) for all $i < j$, i.e., the cell probabilities in either the upper-right or lower-left triangle of the table are all zeros.

Denote the measures for S and CPS by $\gamma_S^{(\lambda)}$ and $\gamma_{CPS}^{(\lambda)}$, respectively (see Appendix). The value of the measure $\gamma_{CLS}^{(\lambda)}$ applied to a table is equal to or less than the values of the existing measures $\gamma_S^{(\lambda)}$ and $\gamma_{CPS}^{(\lambda)}$ applied to the table. This is a natural relationship, because the constraint of CLS is weaker than those of S and CPS.

3. Approximate confidence interval of measure

In this section, we shall give the approximate standard error of estimate of $\gamma_{CLS}^{(\lambda)}$ and the large-sample confidence interval of $\gamma_{CLS}^{(\lambda)}$. Assume that a multinomial distribution applies to the $r \times r$ table. Let n_{ij} denote the observed frequency in cell (i, j) of the table $(i = 1, \ldots, r; j = 1, \ldots, r)$. The sample version of $\gamma_{CLS}^{(\lambda)}$, denoted by $\hat{\gamma}_{CLS}^{(\lambda)}$, is given by $\gamma_{CLS}^{(\lambda)}$ with $\{p_{ij}\}$ replaced by $\{\hat{p}_{ij}\}$, where $\hat{p}_{ij} = n_{ij}/n$ and $n = \sum\sum n_{ij}$. Using the delta method (Agresti, 2013, p.587), we obtain the following theorem.

Theorem 1. $\sqrt{n}(\hat{\gamma}_{CLS}^{(\lambda)} - \gamma_{CLS}^{(\lambda)})$ *asymptotically* $(n \to \infty)$ *has a normal distribution with mean zero and variance* $\sigma^2[\gamma_{CLS}^{(\lambda)}]$, *where*

$$\sigma^2[\gamma_{CLS}^{(\lambda)}] = \sum_{k=1}^{r}\sum_{\substack{l=1 \\ l \neq k}}^{r} p_{kl}\left(\omega_{kl}^{(\lambda)}\right)^2 - \left(\sum_{k=1}^{r}\sum_{\substack{l=1 \\ l \neq k}}^{r} p_{kl}\,\omega_{kl}^{(\lambda)}\right)^2 \quad for \ \ \lambda > -1,$$

with

$$\omega_{kl}^{(\lambda)} = \frac{\gamma_{CLS}^{(\lambda)}}{\Delta}\left(d_{kl} - \gamma_{CLS}^{(\lambda)}\xi_{kl}^{(\lambda)}\right),$$

$$d_{kl} = \begin{cases} \dfrac{(l-k)(l-k+1)}{2} & for \ \ k < l, \\[3mm] \dfrac{(k-l)(k-l+1)}{2} & for \ \ k > l, \end{cases}$$

$$\xi_{kl}^{(\lambda)} =$$

$$\begin{cases} \displaystyle\sum_{i=k}^{l-1}\sum_{j=i+1}^{l} \frac{1}{\gamma_{ij}^{(\lambda)}}\left\{1 - \frac{2^\lambda(\lambda+1)}{2^\lambda - 1}\frac{G_{ji}^c}{\gamma_{ij}^{(\lambda)}}\left((G_{ij}^c)^\lambda - (G_{ji}^c)^\lambda\right)\right\} & for \ \ \lambda \neq 0; k < l, \\[4mm] \displaystyle\sum_{i=l}^{k-1}\sum_{j=i+1}^{k} \frac{1}{\gamma_{ij}^{(\lambda)}}\left\{1 - \frac{2^\lambda(\lambda+1)}{2^\lambda - 1}\frac{G_{ij}^c}{\gamma_{ij}^{(\lambda)}}\left((G_{ji}^c)^\lambda - (G_{ij}^c)^\lambda\right)\right\} & for \ \ \lambda \neq 0; k > l, \\[4mm] \displaystyle\sum_{i=k}^{l-1}\sum_{j=i+1}^{l} \frac{1}{\gamma_{ij}^{(0)}}\left\{1 - \frac{1}{\log 2}\frac{G_{ji}^c}{\gamma_{ij}^{(0)}}\log\frac{G_{ij}^c}{G_{ji}^c}\right\} & for \ \ \lambda = 0; k < l, \\[4mm] \displaystyle\sum_{i=l}^{k-1}\sum_{j=i+1}^{k} \frac{1}{\gamma_{ij}^{(0)}}\left\{1 - \frac{1}{\log 2}\frac{G_{ij}^c}{\gamma_{ij}^{(0)}}\log\frac{G_{ji}^c}{G_{ij}^c}\right\} & for \ \ \lambda = 0; k > l. \end{cases}$$

The asymptotic distribution of $\sqrt{n}(\hat{\gamma}_{CLS}^{(\lambda)} - \gamma_{CLS}^{(\lambda)})$ is applicable only when $0 < \gamma_{CLS}^{(\lambda)} < 1$.

Let $\hat{\sigma}^2[\gamma_{CLS}^{(\lambda)}]$ denote $\sigma^2[\gamma_{CLS}^{(\lambda)}]$ with $\{p_{ij}\}$ replaced by $\{\hat{p}_{ij}\}$. According to Theorem 1, $\hat{\sigma}[\gamma_{CLS}^{(\lambda)}]/\sqrt{n}$ is an estimated approximate standard error of $\hat{\gamma}_{CLS}^{(\lambda)}$, and also $\hat{\gamma}_{CLS}^{(\lambda)} \pm z_{\alpha/2}\, \hat{\sigma}[\gamma_{CLS}^{(\lambda)}]/\sqrt{n}$ is an approximate $100(1-\alpha)$ percent confidence interval of $\gamma_{CLS}^{(\lambda)}$, where $z_{\alpha/2}$ is the upper $(\alpha/2)$-th quantile of the standard normal distribution.

4. Examples

Table 1, taken from Tomizawa *et al.* (2006), contains data on decayed teeth in 363 women patients aged from 18 to 39 visiting a dental clinic in Sapporo City, Japan, from 2001 to 2005. Table 1(a) is classified by numbers of decayed teeth in the left side of the patient's mouth and those in the right side. Note that each of these patients has at least one decayed tooth. Table 1(b) is reclassified by the numbers of decayed teeth in the lower side of the patient's mouth and those in the upper side.

Table 1. Decayed teeth data for 363 women patients aged 18–39 visiting a dental clinic in Sapporo City, Japan, from 2001 to 2005: (a) classified by numbers of decayed teeth in the left and right sides, and (b) classified by those in the upper and lower sides (Tomizawa *et al.*, 2006)

(a) For left and right decayed teeth

Left	Right			Total
	1-4	5-8	9+	
1-4	103	45	1	149
5-8	35	84	33	152
9+	3	17	42	62
Total	141	146	76	363

(b) For lower and upper decayed teeth

Lower	Upper			Total
	1-4	5-8	9+	
1-4	97	62	15	174
5-8	20	63	75	158
9+	2	6	23	31
Total	119	131	113	363

Table 2. Estimates of $\gamma_{CLS}^{(\lambda)}$, estimated approximate standard errors of $\hat{\gamma}_{CLS}^{(\lambda)}$ and approximate 95% confidence intervals of $\gamma_{CLS}^{(\lambda)}$, applied to the data in each of Tables 1(a) and 1(b)

(a) For Table 1(a)

λ	Estimated measure $\hat{\gamma}_{CLS}^{(\lambda)}$	Standard error	Confidence interval
-0.5	0.006	0.013	$(-0.019, 0.031)$
0	0.010	0.021	$(-0.032, 0.052)$
0.5	0.013	0.027	$(-0.040, 0.065)$
1	0.014	0.030	$(-0.044, 0.072)$
1.5	0.014	0.030	$(-0.045, 0.074)$
2	0.014	0.030	$(-0.044, 0.072)$

(b) For Table 1(b)

λ	Estimated measure $\hat{\gamma}_{CLS}^{(\lambda)}$	Standard error	Confidence interval
-0.5	0.222	0.059	$(0.106, 0.337)$
0	0.345	0.083	$(0.181, 0.508)$
0.5	0.409	0.093	$(0.226, 0.591)$
1	0.437	0.096	$(0.249, 0.625)$
1.5	0.444	0.096	$(0.255, 0.632)$
2	0.437	0.096	$(0.249, 0.625)$

Table 2 gives the estimated values of the measure $\gamma_{CLS}^{(\lambda)}$ applied to each of the data in Tables 1(a) and 1(b). It also gives the estimated approximate standard errors and the approximate 95% confidence intervals of the measure. It is seen from Table 2 that the confidence interval of $\gamma_{CLS}^{(\lambda)}$ applied to the data in Table 1(a) includes zero for any λ. This means that there might be a CLS structure for numbers of decayed teeth in the left and right sides. On the other hand, it is inferred that there is not a CLS structure for numbers of decayed teeth in the upper and lower sides, because the confidence interval of $\gamma_{CLS}^{(\lambda)}$ applied to the data in Table 1(b) does not include zero for any λ. The confidence interval of the measure $\gamma_{CLS}^{(\lambda)}$ is also useful for comparing the degrees of departure from CLS for different tables. According to Table 2, the degree of departure from CLS for numbers of decayed teeth in the lower and upper sides may be larger than for decayed teeth in the left and right sides.

Table 3 gives the estimated values of the measures $\gamma_{S}^{(\lambda)}$ and $\gamma_{CPS}^{(\lambda)}$ applied to each of the data in Tables 1(a) and 1(b). It can be seen that, for any λ,

the estimated value of $\gamma_{CLS}^{(\lambda)}$ applied to each of Tables 1(a) and 1(b) is less than those of $\gamma_S^{(\lambda)}$ and $\gamma_{CPS}^{(\lambda)}$.

Table 3. Estimates of the measures $\gamma_{CLS}^{(\lambda)}$, $\gamma_{CPS}^{(\lambda)}$ and $\gamma_S^{(\lambda)}$, applied to the data in each of Tables 1(a) and 1(b)

(a) For Table 1(a)				(b) For Table 1(b)			
λ	$\hat{\gamma}_{CLS}^{(\lambda)}$	$\hat{\gamma}_{CPS}^{(\lambda)}$	$\hat{\gamma}_{S}^{(\lambda)}$	λ	$\hat{\gamma}_{CLS}^{(\lambda)}$	$\hat{\gamma}_{CPS}^{(\lambda)}$	$\hat{\gamma}_{S}^{(\lambda)}$
-0.5	0.006	0.009	0.017	-0.5	0.222	0.250	0.281
0	0.010	0.015	0.028	0	0.345	0.380	0.418
0.5	0.013	0.019	0.035	0.5	0.409	0.445	0.483
1	0.014	0.021	0.038	1	0.437	0.473	0.510
1.5	0.014	0.022	0.039	1.5	0.444	0.479	0.516
2	0.014	0.021	0.038	2	0.437	0.473	0.510

5. Discussion

Consider the 4×4 artificial cell and cumulative probability tables given in Tables 4 and 5. The cumulative probabilities given in Tables 5(a)–(f) are calculated from the cell probabilities in Tables 4(a)–(f), respectively.

Table 4. Artificial cell probability tables

(a)			
0.055	0.038	0.010	0.002
0.002	0.050	0.100	0.008
0.040	0.400	0.050	0.030
0.008	0.032	0.120	0.055

(b)			
0.040	0.088	0.010	0.002
0.002	0.040	0.100	0.008
0.040	0.400	0.040	0.030
0.008	0.032	0.120	0.040

(c)			
0.030	0.138	0.010	0.002
0.002	0.025	0.100	0.008
0.040	0.400	0.025	0.030
0.008	0.032	0.120	0.030

(d)			
0.015	0.188	0.010	0.002
0.002	0.015	0.100	0.008
0.040	0.400	0.015	0.030
0.008	0.032	0.120	0.015

(e)			
0.003	0.238	0.010	0.002
0.002	0.002	0.100	0.008
0.040	0.400	0.002	0.030
0.008	0.032	0.120	0.003

(f)			
0.100	0.000	0.000	0.000
0.002	0.099	0.000	0.000
0.040	0.400	0.099	0.000
0.008	0.032	0.120	0.100

There is a CLS structure in Table 4(a) with $G_{12} = G_{21} = 0.05$. For each of Tables 5(b)–(e), the value of G_{kl} equals the value of G_{kl} in Table 5(a) for $k \neq l$ except for $(k, l) = (1, 2)$. The ratio G_{12}/G_{21} is 1.0 for Table 4(a), 2.0 for Table 4(b), 3.0 for Table 4(c), 4.0 for Table 4(d), and 5.0 for Table 4(e). Table 5(f) shows that the cumulative probabilities in the upper-right triangle cells are all zeros for Table 4(f).

Table 5. Artificial cumulative probability tables: the cumulative probabilities in Tables 5(a)–(f) are calculated from Tables 4(a)–(f), respectively

(a)			
	0.050	0.012	0.002
0.050		0.120	0.010
0.048	0.480		0.040
0.008	0.040	0.160	

(b)			
	0.100	0.012	0.002
0.050		0.120	0.010
0.048	0.480		0.040
0.008	0.040	0.160	

(c)			
	0.150	0.012	0.002
0.050		0.120	0.010
0.048	0.480		0.040
0.008	0.040	0.160	

(d)			
	0.200	0.012	0.002
0.050		0.120	0.010
0.048	0.480		0.040
0.008	0.040	0.160	

(e)			
	0.250	0.012	0.002
0.050		0.120	0.010
0.048	0.480		0.040
0.008	0.040	0.160	

(f)			
	0.000	0.000	0.000
0.050		0.000	0.000
0.048	0.480		0.000
0.008	0.040	0.160	

Table 6 gives the values of the measure $\gamma_{CLS}^{(\lambda)}$ applied to each of Tables 4(a)–(f). We see from Table 6 that (1) the value of $\gamma_{CLS}^{(\lambda)}$ for Table 4(a) is 0, (2) for any fixed λ, the value of $\gamma_{CLS}^{(\lambda)}$ increases as the ratio G_{12}/G_{21} increases, and (3) the value of $\gamma_{CLS}^{(\lambda)}$ for Table 4(f) is 1. Therefore, the measure $\gamma_{CLS}^{(\lambda)}$ would be natural to represent the degree of departure from the CLS model.

Consider the artificial cell probability table in Table 7(a), and the modified table interchanging 2 and 3 in categories in Table 7(b). Denote the measure for LS by $\Phi_{LS}^{(\lambda)}$ (see Appendix). Tables 7(c)–(d) give the values of the measures $\Phi_{LS}^{(\lambda)}$ and $\gamma_{CLS}^{(\lambda)}$ applied to each of Tables 7(a)–(b), respectively.

Table 6. Values of $\gamma_{CLS}^{(\lambda)}$ for Tables 4(a)–(f)

Applied	λ		
tables	0	0.5	1.5
Table 4(a)	0.000	0.000	0.000
Table 4(b)	0.208	0.253	0.279
Table 4(c)	0.256	0.309	0.340
Table 4(d)	0.278	0.334	0.366
Table 4(e)	0.293	0.351	0.384
Table 4(f)	1.000	1.000	1.000

According to Tables 7(c)–(d), for any λ, we can see that the value of $\gamma_{CLS}^{(\lambda)}$ for Table 7(a) is different from Table 7(b), while the value of $\Phi_{LS}^{(\lambda)}$ for Table 7(a) is the same as Table 7(b). This is because the measure $\gamma_{CLS}^{(\lambda)}$ depends on the order of categories, unlike $\Phi_{LS}^{(\lambda)}$.

Table 7. (a) Artificial cell probability table and (b) modified table interchanging 2 and 3 in categories, and (c)–(d) the values of $\Phi_{LS}^{(\lambda)}$ and $\gamma_{CLS}^{(\lambda)}$ applied to the data in each of Tables 7(a)–(b)

(a)					(b)			
0.036	0.045	0.027	0.031		0.036	0.027	0.045	0.031
0.015	0.149	0.061	0.023		0.007	0.156	0.094	0.034
0.007	0.094	0.156	0.034		0.015	0.061	0.149	0.023
0.018	0.095	0.098	0.111		0.018	0.098	0.095	0.111

(c) For Table 7(a)				(d) For Table 7(b)		
λ	$\Phi_{LS}^{(\lambda)}$	$\gamma_{CLS}^{(\lambda)}$		λ	$\Phi_{LS}^{(\lambda)}$	$\gamma_{CLS}^{(\lambda)}$
0	0.075	0.058		0	0.075	0.002
0.5	0.094	0.073		0.5	0.094	0.002
1	0.103	0.080		1	0.103	0.003

6. Concluding remarks

For the square contingency table with ordered categories, we have considered a CLS model, which indicates the symmetry of cumulative probabilities for only one pair of symmetric cells. We have also proposed a measure to represent the degree of departure from the CLS model. The measure $\gamma_{CLS}^{(\lambda)}$ is useful for comparing the degrees of departure from CLS in different square tables (see Examples and Discussion).

Acknowledgements

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Appendix

Measures to represent the degree of departure from the S and PS models have been proposed (Tomizawa, 1994; Tomizawa *et al.*, 1998; Saigusa *et al.*, 2016), although the details are omitted here. Assuming that $p_{ij} + p_{ji} \neq 0$ for $i < j$ and $p_{kl} \neq p_{lk}$ for any $k < l$ except for any one pair (k, l), Saigusa

et al. (2019b) proposed a measure to represent the degree of departure from the LS model as follows:

$$\Phi_{LS}^{(\lambda)} = \frac{\displaystyle\prod_{a=1}^{r-1}\prod_{b=a+1}^{r}\Phi_{ab}^{(\lambda)}}{\displaystyle\sum_{i=1}^{r-1}\sum_{j=i+1}^{r}\left[\left(p_{ij}^{*}+p_{ji}^{*}\right)\prod_{a=1}^{r-1}\prod_{\substack{b=a+1\\(a,b)\neq(i,j)}}^{r}\Phi_{ab}^{(\lambda)}\right]}\quad\text{for}\quad\lambda>-1,$$

where

$$p_{ij}^{*}=\frac{p_{ij}}{\delta},\quad\delta=\sum_{i=1}^{r}\sum_{\substack{j=1\\j\neq i}}^{r}p_{ij},\quad p_{ij}^{c}=\frac{p_{ij}}{p_{ij}+p_{ji}},$$

$$\Phi_{ij}^{(\lambda)}=\begin{cases}1-\dfrac{\lambda 2^{\lambda}}{2^{\lambda}-1}I_{ij}^{(\lambda)} & \text{for}\quad\lambda\neq 0,\\[2mm]1-\dfrac{1}{\log 2}I_{ij}^{(0)} & \text{for}\quad\lambda=0,\end{cases}$$

$$I_{ij}^{(\lambda)}=\begin{cases}\dfrac{1}{\lambda}\left(1-\left(p_{ij}^{c}\right)^{\lambda+1}-\left(p_{ji}^{c}\right)^{\lambda+1}\right) & \text{for}\quad\lambda\neq 0,\\[2mm]-p_{ij}^{c}\log p_{ij}^{c}-p_{ji}^{c}\log p_{ji}^{c} & \text{for}\quad\lambda=0.\end{cases}$$

Note that $I_{ij}^{(\lambda)}$ is the diversity index of degree λ, including the Shannon entropy when $\lambda=0$ (Patil and Taillie, 1982). When the LS model does not hold, we observe that

$$\Phi_{LS}^{(\lambda)}=\frac{1}{\displaystyle\sum_{i=1}^{r-1}\sum_{j=i+1}^{r}\dfrac{p_{ij}^{*}+p_{ji}^{*}}{\Phi_{ij}^{(\lambda)}}}\quad\text{for}\quad\lambda>-1.$$

Therefore, $\Phi_{LS}^{(\lambda)}$ is expressed as the weighted harmonic mean of the submeasure $\Phi_{ij}^{(\lambda)}$.

Assume that $G_{ij}+G_{ji}\neq 0$ for $i<j$. Tomizawa *et al.* (2001) proposed a measure for the S model as follows:

$$\gamma_{S}^{(\lambda)}=\sum_{i=1}^{r-1}\sum_{j=i+1}^{r}(G_{ij}^{*}+G_{ji}^{*})\gamma_{ij}^{(\lambda)}\quad\text{for}\quad\lambda>-1,$$

where

$$G_{ij}^* = \frac{G_{ij}}{\Delta}, \quad \Delta = \sum_{i=1}^{r-1}\sum_{j=i+1}^{r}(G_{ij}+G_{ji}), \quad G_{ij}^c = \frac{G_{ij}}{G_{ij}+G_{ji}},$$

$$\gamma_{ij}^{(\lambda)} = \begin{cases} 1 - \dfrac{\lambda 2^\lambda}{2^\lambda - 1}H_{ij}^{(\lambda)} & \text{for } \lambda \neq 0, \\ 1 - \dfrac{1}{\log 2}H_{ij}^{(0)} & \text{for } \lambda = 0, \end{cases}$$

$$H_{ij}^{(\lambda)} = \begin{cases} \dfrac{1}{\lambda}\left(1 - \left(G_{ij}^c\right)^{\lambda+1} - \left(G_{ji}^c\right)^{\lambda+1}\right) & \text{for } \lambda \neq 0, \\ -G_{ij}^c \log G_{ij}^c - G_{ji}^c \log G_{ji}^c & \text{for } \lambda = 0. \end{cases}$$

Saigusa *et al.* (2019a) proposed a measure for the CPS model as follows:

$$\gamma_{CPS}^{(\lambda)} = \prod_{i=1}^{r-1}\prod_{j=i+1}^{r}[\gamma_{ij}^{(\lambda)}]^{(G_{ij}^*+G_{ji}^*)} \quad \text{for } \lambda > -1.$$

It can be seen that the measures $\gamma_S^{(\lambda)}$ and $\gamma_{CPS}^{(\lambda)}$ are weighted arithmetic and geometric means of the submeasure $\gamma_{ij}^{(\lambda)}$, respectively.