SOLUTIONS OF THE DIRAC EQUATION IN A BARDEEN BLACK HOLE GEOMETRY

Ciprian A. Sporea^{a,*}

^a *Faculty of Physics, West University of Timișoara
300223 V. Pârvan Ave 4, Timișoara, Romania*

* ciprian.sporea@e-uvv.ro

Article info	Abstract
<p><i>Received:</i> 16.07.2018 <i>Accepted:</i> 05.11.2018</p> <p>Keywords: analytical solutions, Dirac equation, Bardeen black hole.</p> <p>PACS: 04.62.+v</p>	<p>In this paper we study the Dirac equation in the geometry of a (regular) Bardeen black hole. We will focus on finding new analytical solutions in the vicinity of the black hole horizon. These solutions can be used with the asymptotic solutions (derived in a previous paper) to compute numerical phase shifts that define the scattering amplitudes.</p>

1. Introduction

At the core of modern physics lay down on one hand the General Theory of Relativity, that describes the Universe at large scales and on the other hand Quantum Field Theory, very successful at describing the physical phenomena happening at very small scales. It is well known that quantum effects become insignificant at large scales and that gravity does not play any role in particle physics at today's energy scales available in experiments. There are several proposals for a quantum theory of gravity such as string theory [1], loop quantum gravity [2] or the more modern proposed theories of spin foam gravity [3] and group field theory [4]. However, so far any attempts to successfully test any of these quantum gravity theories have been unsuccessful. Despite of this, some progress was made using a semi-classical approach which consists in the study of quantum fields on a classical curved background. Black holes proved to be the ideal candidates for applying the semi-classical approach. In doing so Hawking [5] showed that black holes can radiate away particles whose energy spectrum resembles the one of a black body.

In this paper we will use the semi-classical approach to study the propagation of spin 1/2 particles in the vicinity of a Bardeen black hole. Thus the goal is to search for

analytical solutions of the Dirac equation in this geometry. There exists in the literature many studies of the Dirac equation in black hole (BH) geometries. For example the Dirac equation in Schwarzschild BH was investigated in Refs. [6, 7, 8, 9, 10, 11], in a Kerr BH [12, 13, 14, 15, 16], in BertottiRobinson BH [17], in Kerr-Taub-NUT BH [18], in KerrNewmanAdS BH [19] or in Reissner-Nordstrom de Sitter BH [20].

The Bardeen black hole [21] is interesting to study because it is a black hole that has the property of being regular i.e it has no singularity present in origin. The Bardeen black hole was generalized in a recent study [22] to include an entire class of Bardeen-like regular black holes that depend on two parameters. Moreover, as showed in ref. [23] the Bardeen black hole solution can be physically interpreted as the gravitational field produced by a nonlinear magnetic monopole. Furthermore, in our recent paper [24] we have found analytical solutions to the Dirac equation in the far field zone of the Bardeen black hole. Thus, by searching for analytical solutions also in the vicinity of the Bardeen black hole one can use these solutions as a starting point for a matching procedure (consisting in jointing the two solutions) that will allow for the computation of at least numerical phase shifts characteristic to scattering phenomena.

The paper is organized as follows: in section 2 we present a review of the Dirac equation in a curved background focusing on a spacetime describing a spherically symmetric black hole. Section 3 starts with a brief description of a Bardeen black hole and ends with the specific form of the Dirac equation in this particular geometry. The main results of this paper are presented in section 4 that consists in solving the Dirac equation in the vicinity of the Bardeen black hole horizon and finding two sets of approximative analytical solutions valid in this part of the spacetime geometry. The conclusions and some final remarks are presented in the last section 5 of the paper.

2. Preliminaries: Dirac equation and spherically symmetric black holes

In a curved spacetime background the Dirac equation, governing the motion of fermion fields, is obtained from the gauge invariant action

$$\mathcal{S} = \int d^4x \sqrt{-g} \left\{ \frac{i}{2} \bar{\psi} \gamma^a D_a \psi - \frac{i}{2} (\overline{D_a \psi}) \gamma^a \psi - m \bar{\psi} \psi \right\} \quad (1)$$

where $g = \det(g_{\mu\nu})$ and D_a denotes the covariant derivative

$$D_a = \partial_a + \frac{i}{2} S^b_c \omega^c_{ab} \quad (2)$$

The generators $S^{ab} = \frac{i}{4} [\gamma^a, \gamma^b]$ satisfy the standard commutation relations of the spinor representation of the $SL(2, \mathbb{C})$ group [25]. The Dirac matrices γ^a are point-independent and satisfy the relation $\{\gamma^a, \gamma^b\} = 2\eta^{ab}$. The object ω^c_{ab} represents the spin-connection and can be computed using the following equation

$$\omega^c_{ab} = e^\mu_a e^\nu_b (\hat{e}^\lambda_\nu \Gamma^\lambda_{\mu\nu} - \hat{e}^c_{\nu,\mu}) \quad (3)$$

with $\Gamma_{\mu\nu}^\lambda$ the GR Christoffel symbols. The point dependent tetrad fields $e_a(x)$ and $\hat{e}^a(x)$ define local (non-holonomic) frames and co-frames such that

$$\begin{aligned} e_a(x) \cdot e_b(x) &= \eta_{ab}, & \hat{e}^a(x) \cdot \hat{e}^b(x) &= \eta^{ab} \\ \hat{e}_\mu^a e_b^\mu &= \delta_b^a, & \hat{e}_\mu^a e_a^\nu &= \delta_\mu^\nu \end{aligned} \quad (4)$$

where $\eta = \text{diag}(1, -1, -1, -1)$ for the Minkowski metric η_{ab} . Moreover, passing from a local frame to a natural frame is done using the following simple relations

$$e_a = e_a^\mu \partial_\mu, \quad \hat{e}^a = \hat{e}_\mu^a dx^\mu \quad (5)$$

Furthermore, using the tetrad fields the line element ds^2 is expressed as

$$ds^2 = \eta_{ab} \hat{e}_\mu^a dx^\mu \hat{e}_\nu^b dx^\nu = g_{\mu\nu}(x) dx^\mu dx^\nu \quad (6)$$

while the components for the metric tensor read

$$\begin{aligned} g_{\mu\nu} &= \eta_{ab} \hat{e}_\mu^a \hat{e}_\nu^b, & \eta_{ab} &= g_{\mu\nu} e_a^\mu e_b^\nu \\ g^{\mu\nu} &= \eta^{ab} e_a^\mu e_b^\nu, & \eta^{ab} &= g^{\mu\nu} \hat{e}_\mu^a \hat{e}_\nu^b \end{aligned} \quad (7)$$

Returning to the Dirac equation, one can write it into the following explicit form [26]

$$(i\gamma^a e_a^\mu \partial_\mu - m) \psi + \frac{i}{2} \frac{1}{\sqrt{-g}} \partial_\mu (\sqrt{-g} e_a^\mu) \gamma^a \psi - \frac{1}{4} \{\gamma^a, S_c^b\} \omega_{ab}^c \psi = 0 \quad (8)$$

Let us now consider the case of static spherically symmetric black holes, described by the general line element

$$ds^2 = h(r) dt^2 - \frac{dr^2}{h(r)} - r^2 (d\theta^2 + \sin^2 \theta d\phi^2) \quad (9)$$

Our next aim is to see how eq. (8) will look like using the above line element. Because we are working in local frames, we must first define or choose a gauge for the tetrad fields. The most common used gauge in the literature is the diagonal one [27, 28, 29] defined by the 1-forms $\hat{e}^a(x) = \hat{e}_\mu^a dx^\mu$:

$$\hat{e}^0 = \sqrt{h} dt, \quad \hat{e}^1 = \frac{1}{\sqrt{h}} dr, \quad \hat{e}^2 = r d\theta, \quad \hat{e}^3 = r \sin \theta d\phi \quad (10)$$

However, we will use here another gauge first defined in [30] and later developed in [26, 32, 33] that proved to be very useful in studying the Dirac equation in curved backgrounds [7, 26, 31, 32, 33, 34, 35, 36, 37]. This gauge, called the Cartesian gauge, is defined in

spherical coordinates by the following tetrad fields:

$$\begin{aligned}
\hat{e}^0 &= \sqrt{h} dt \\
\hat{e}^1 &= \frac{1}{\sqrt{h}} \sin \theta \cos \phi dr + r \cos \theta \cos \phi d\theta - r \sin \theta \sin \phi d\phi \\
\hat{e}^2 &= \frac{1}{\sqrt{h}} \sin \theta \sin \phi dr + r \cos \theta \sin \phi d\theta + r \sin \theta \cos \phi d\phi \\
\hat{e}^3 &= \frac{1}{\sqrt{h}} \cos \theta dr - r \sin \theta d\theta
\end{aligned} \tag{11}$$

In this gauge the Dirac eq. (8) can be brought to a reduced form

$$i \left\{ \partial_t + \sqrt{h} \gamma^0 (\vec{\gamma} \cdot \vec{\partial}) + \frac{h - \sqrt{h}}{r^2} \gamma^0 (\vec{\gamma} \cdot \vec{x}) [1 + (\vec{x} \cdot \vec{\partial})] \right\} \psi - \sqrt{h} \gamma^0 m \psi = 0 \tag{12}$$

after a transformation of the wave function of the form [26]

$$\psi(x) \rightarrow h(r)^{-1/4} \psi(x) \tag{13}$$

and observing that for a spherically symmetric metric the last term in eq. (8) is vanishing due to the fact that $\{\gamma^a, S^b_c\}$ is completely antisymmetric, while the spin-connection (3) has no such components in the Cartesian gauge. Another advantage of working with the Cartesian gauge is that the free Dirac equation is manifestly covariant under rotations, which in turns imply that the separation of angular variables can be done as in the usual Dirac theory from flat Minkowski spacetime.

One can also bring eq. (12) to a hamiltonian form $H_D \psi = i \partial_t \psi$, where the Dirac Hamiltonian has the following expression [25, 33]

$$H_D = -i \gamma^0 (\vec{\gamma} \cdot \vec{e}_r) \left(h \partial_r + \frac{h}{r} - \frac{\sqrt{h}}{r} K \right) + \sqrt{h} \gamma^0 m \tag{14}$$

and $K = 2\vec{S} \cdot \vec{L} = J^2 - L^2 + 1/4$ is the spin-orbit operator that together with J^2 and J_3 form a complete set obeying the eigenvalue equations [25]

$$\begin{aligned}
J^2 \Phi_{m,\kappa}^\pm &= j(j+1) \Phi_{m,\kappa}^\pm & j &= \frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \dots \\
J_3 \Phi_{m,\kappa}^\pm &= m \Phi_{m,\kappa}^\pm & m &= -j, -j+1, \dots, 0, \dots, +j-1, +j \\
K \Phi_{m,\kappa}^\pm &= -\kappa \Phi_{m,\kappa}^\pm & \kappa &= -(j + \frac{1}{2}), +(j + \frac{1}{2})
\end{aligned} \tag{15}$$

The four-component angular spinors $\Phi_{m,\kappa}^\pm(\theta, \phi)$ are the same as in special relativity, having

expressions in terms of spherical harmonics Y_l^m [38]

$$\Phi_{m,\kappa}^+(\theta, \phi) = \begin{pmatrix} i \Omega_{j\mp 1/2}^m(\theta, \phi) \\ 0 \end{pmatrix}, \quad \Phi_{m,\kappa}^-(\theta, \phi) = \begin{pmatrix} 0 \\ \Omega_{j\pm 1/2}^m(\theta, \phi) \end{pmatrix} \quad (16)$$

$$\Omega_{j-1/2}^m = \frac{1}{\sqrt{2j}} \begin{pmatrix} \sqrt{j+m} Y_{j-1/2}^{m-1/2} \\ \sqrt{j-m} Y_{j-1/2}^{m+1/2} \end{pmatrix} \quad (17)$$

$$\Omega_{j+1/2}^m = \frac{1}{\sqrt{2(j+1)}} \begin{pmatrix} \sqrt{j-m+1} Y_{j+1/2}^{m-1/2} \\ -\sqrt{j+m+1} Y_{j+1/2}^{m+1/2} \end{pmatrix} \quad (18)$$

Searching now for particular positive frequency solutions with energy E of the type

$$\psi(x) = \psi_{E,j,m,\kappa}(t, r, \theta, \phi) = \frac{e^{-iEt}}{r h(r)^{1/4}} \{ f_{E,\kappa}^+(r) \Phi_{m,\kappa}^+(\theta, \phi) + f_{E,\kappa}^-(r) \Phi_{m,\kappa}^-(\theta, \phi) \} \quad (19)$$

that are in fact the common eigenspinors for the $\{H_D, K, J_3\}$ operators with eigenvalues $\{E, \kappa, m_j\}$, eq. (12) reduces to only a radial Dirac equations for the radial wave functions $f_{E,\kappa}^\pm(r)$. The matrix form of this equation reads [7, 35]

$$\begin{pmatrix} m\sqrt{h(r)} & -h(r)\frac{d}{dr} + \frac{\kappa}{r}\sqrt{h(r)} \\ h(r)\frac{d}{dr} + \frac{\kappa}{r}\sqrt{h(r)} & -m\sqrt{h(r)} \end{pmatrix} \begin{pmatrix} f_{E,\kappa}^+(r) \\ f_{E,\kappa}^-(r) \end{pmatrix} = E \begin{pmatrix} f_{E,\kappa}^+(r) \\ f_{E,\kappa}^-(r) \end{pmatrix} \quad (20)$$

Depending on the complexity of the function $h(r)$ that defines a specific black hole geometry, the above radial equation can be solved analytically in some specific parts of the geometry. In the following section we will search for approximative analytical solutions of the radial Dirac equation (20) in the case of a Bardeen black holes geometry. We will focus our attention on the spacetime region in the vicinity of the black hole horizon.

3. Dirac equation in a Bardeen spacetime

The first regular black hole solution was proposed by Bardeen in ref. [21] and it was latter showed in ref. [23] that the Bardeen solution can be interpreted as a self-gravitating field of a magnetic monopole described by a nonlinear electrodynamics. More recently in ref. [22] the authors construct an entire two-parameter family of Bardeen-like solutions.

The field equations of the Einstein-field nonlinear electrodynamics theory are obtained

from the action

$$S = \frac{1}{16\pi} \int d^4x \sqrt{-g} (R - \mathcal{L})$$

$$\mathcal{L} = \frac{12}{a} \left(\frac{\sqrt{aF}}{1 + \sqrt{aF}} \right)^{5/2}, \quad F = F_{\mu\nu}F^{\mu\nu}, \quad a = \frac{Q^3}{M} \quad (21)$$

The resulting field equations are

$$R_\mu^\nu - \frac{1}{2}g_\mu^\nu R - 2 \left(\frac{\partial \mathcal{L}}{\partial F} F_{\mu\lambda} F^{\mu\lambda} - \delta_\mu^\nu \mathcal{L} \right) = 0, \quad \nabla_\mu \left(\frac{\partial \mathcal{L}}{\partial F} F^{\alpha\mu} \right) = 0 \quad (22)$$

Searching for spherically symmetric solutions of type (9) one finds the metric for the Bardeen spacetime, namely [21]

$$ds^2 = \left(1 - \frac{2Mr^2}{\sqrt{(r^2 + Q^2)^3}} \right) dt^2 - \left(1 - \frac{2Mr^2}{\sqrt{(r^2 + Q^2)^3}} \right)^{-1} dr^2 - r^2 (d\theta^2 + \sin^2 \theta d\phi^2) \quad (23)$$

where M is the mass of the black hole and Q the monopole charge. This metric describes a regular Bardeen black hole that has two horizons if $|Q| < 4/3\sqrt{3}$, one horizon if $|Q| = 4/3\sqrt{3}$ giving rise to an extreme Bardeen black hole (as in the Reissner-Nordström solution) and no horizons exists if $|Q| > 4/3\sqrt{3}$. The radii of the horizons have lengthy expressions that result from solving the equation $h(r) = 0$ which is equivalent to following cubic equation in r^2

$$(r^2 + Q^2)^3 - 4M^2 r^4 = 0 \quad (24)$$

Before moving on to solve the Dirac equation for the Bardeen metric, it proves useful to introduce the new Novikov-like variable [39]

$$x = \sqrt{\frac{r}{r_+}} - 1, \quad r = r_+(1 + x^2) \quad (25)$$

valid only when the black hole has at least one horizon, with r_+ the radius of the (outer) black hole horizon. In terms of the new variable the function $h(r)$ can be rewritten as

$$h(x) = 1 - \frac{\gamma (1 + x^2)^2}{[(1 + x^2)^2 + \delta]^{3/2}} \quad (26)$$

where we introduced the notations $\gamma = 2M/r_+$ and $\delta = (Q/r_+)^2$. Inserting (26) into eq. (20) a new radial Dirac equation in the variable x is obtained:

$$\begin{pmatrix} \mathcal{A} & \mathcal{B} \\ \mathcal{C} & \mathcal{D} \end{pmatrix} \begin{pmatrix} f_{E,\kappa}^+(x) \\ f_{E,\kappa}^-(x) \end{pmatrix} = 0 \quad (27)$$

with

$$\begin{aligned}
\mathcal{A} &= \mu \sqrt{1 - \frac{\gamma(1+x^2)^2}{[(1+x^2)^2 + \delta]^{3/2}}} - \varepsilon \\
\mathcal{B} &= - \left(1 - \frac{\gamma(1+x^2)^2}{[(1+x^2)^2 + \delta]^{3/2}} \right) \frac{1}{2x} \frac{d}{dx} + \frac{\kappa}{1+x^2} \sqrt{1 - \frac{\gamma(1+x^2)^2}{[(1+x^2)^2 + \delta]^{3/2}}} \\
\mathcal{C} &= \left(1 - \frac{\gamma(1+x^2)^2}{[(1+x^2)^2 + \delta]^{3/2}} \right) \frac{1}{2x} \frac{d}{dx} + \frac{\kappa}{1+x^2} \sqrt{1 - \frac{\gamma(1+x^2)^2}{[(1+x^2)^2 + \delta]^{3/2}}} \\
\mathcal{D} &= -\mu \sqrt{1 - \frac{\gamma(1+x^2)^2}{[(1+x^2)^2 + \delta]^{3/2}}} - \varepsilon \\
\mu &= r_+ m, \quad \varepsilon = r_+ E
\end{aligned} \tag{28}$$

4. Analytical solutions of the Dirac equation

In this section we will show how one can obtain approximative analytical solutions of the radial Dirac equation (20). We will focus our attention on finding solutions near the black hole horizon. The solutions in the asymptotic zone of the Bardeen black hole giving the scattering modes were discussed in ref. [24] where an analysis of fermion scattering by Bardeen black holes was also performed.

The variable $x \rightarrow 0$ as one approaches the black hole horizon $r \rightarrow r_+$. Thus by making a Taylor expansion with respect to x of the operators $\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}$ and using the fact that $\gamma^2 = (1 + \delta)^3$, the following expressions will be obtained

$$\begin{aligned}
\mathcal{A} &\rightarrow \mu x \sqrt{\frac{1-2\delta}{1+\delta}} - \varepsilon + O(x^2) \\
\mathcal{B} &\rightarrow -\frac{(1-2\delta)}{2(1+\delta)} x \frac{d}{dx} + \kappa x \sqrt{\frac{1-2\delta}{1+\delta}} + O(x^2) \\
\mathcal{C} &\rightarrow \frac{(1-2\delta)}{2(1+\delta)} x \frac{d}{dx} + \kappa x \sqrt{\frac{1-2\delta}{1+\delta}} + O(x^2) \\
\mathcal{D} &\rightarrow -\mu x \sqrt{\frac{1-2\delta}{1+\delta}} - \varepsilon + O(x^2)
\end{aligned} \tag{29}$$

If in the above expansions one discards the terms of order $O(x^2)$ and higher, eq. (27) reduces to

$$\begin{cases} (a\mu x - \varepsilon) f_{E,\kappa}^+(x) + x \left(\kappa a - \frac{a^2}{2} \frac{d}{dx} \right) f_{E,\kappa}^-(x) = 0 \\ x \left(\kappa a + \frac{a^2}{2} \frac{d}{dx} \right) f_{E,\kappa}^+(x) - (a\mu x + \varepsilon) f_{E,\kappa}^-(x) = 0 \end{cases}, \quad a = \sqrt{\frac{1-2\delta}{1+\delta}} \tag{30}$$

Figure 1: Comparison of the approximative analytical solutions (31)-(32), red dash-dotted curves, with the numerical solutions of the full equations (28), solid blue curves. In the left panel the function $f_{E,\kappa}^+(x)$ is plotted and in the right panel the function $f_{E,\kappa}^-(x)$.

This system of coupled differential equations has an analytical solution that reads

$$f_{E,\kappa}^+(x) = e^{-\lambda x} \left[C_1 x^{i\beta} HeunC \left(\alpha, 2i\beta, -2, 0, \sigma, -\frac{\mu a}{\varepsilon} x \right) + C_2 x^{-i\beta} HeunC \left(\alpha, -2i\beta, -2, 0, \sigma, -\frac{\mu a}{\varepsilon} x \right) \right] \quad (31)$$

$$\begin{aligned} f_{E,\kappa}^-(x) = & \frac{e^{-\lambda x}}{a\mu x + \varepsilon} \left[C_1 x^{i\beta} \left(a\kappa x + i\varepsilon - ax\sqrt{\kappa^2 + \mu^2} \right) HeunC \left(\alpha, 2i\beta, -2, 0, \sigma, -\frac{\mu a}{\varepsilon} x \right) \right. \\ & + C_2 x^{-i\beta} \left(a\kappa x - i\varepsilon - ax\sqrt{\kappa^2 + \mu^2} \right) HeunC \left(\alpha, -2i\beta, -2, 0, \sigma, -\frac{\mu a}{\varepsilon} x \right) \Big] \\ & - \frac{a^3 \mu e^{-\lambda x}}{2\varepsilon(a\mu x + \varepsilon)} \left[C_1 x^{1+i\beta} HeunCPrime \left(\alpha, 2i\beta, -2, 0, \sigma, -\frac{\mu a}{\varepsilon} x \right) \right. \\ & + C_2 x^{1-i\beta} HeunCPrime \left(\alpha, -2i\beta, -2, 0, \sigma, -\frac{\mu a}{\varepsilon} x \right) \Big] \end{aligned} \quad (32)$$

where the constants C_1 and C_2 are arbitrary and $HeunC$ stand for the Heun confluent function and it's derivative $HeunCPrime$ [42]. These solutions constitutes the main result of this paper. The parameters used in eqs. (31)-(32) are given by the expressions:

$$\lambda = \frac{2\sqrt{\kappa^2 + \mu^2}}{a}, \quad \alpha = \frac{4\varepsilon\sqrt{\kappa^2 + \mu^2}}{a^2\mu}, \quad \beta = \frac{2\varepsilon}{a^2}, \quad \sigma = 1 + \frac{2\varepsilon\kappa}{a^2\mu} \quad (33)$$

Let us now analyse the validity of the above approximative analytical solutions. Solving numerically (using Maple or Mathematica) the complete system of eqs. (27-28) one obtains the full blue curves in Figs. 1-2. The dash-dotted red curves are obtained using the analytical solutions (31) and we can see that they approximate quite well the full numerical solution in the vicinity of the black hole horizon located at $x = 0$. In obtaining the numerical solutions the initial conditions $f^+(0.0001) = 0$, $f^-(0.0001) = 1$ were used together with the values $M = 1$, $Q = 0.1$, $m = 0.01$, $E = 1$, $\kappa = 1$ for Fig. 1 and $M = 2$, $Q = 0.5$, $m = 0.1$, $E = 2$, $\kappa = 3$ for Fig. 2. Imposing initial conditions in $x = 0$ was not possible because there are some problems when Maple/Mathematica tries to

Figure 2: Comparison of the approximative analytical solutions (31)-(32), red dash-dotted curves, with the numerical solutions of the full equations (28), solid blue curves. In the left panel the function $f_{E,\kappa}^+(x)$ is plotted and in the right panel the function $f_{E,\kappa}^-(x)$.

evaluate numerically the first derivative of the functions $f^\pm(x)$ when solving numerically the differential equations (28). This is probably due to the fact that the solutions shown in Figs. 1-2 are not regular at the origin ($x = 0$), since the derivative can be seen to diverge in that point.

5. Final remarks

In this paper we showed how to find (approximative) analytical solutions of the Dirac equation on a curved background, focusing on the geometry of a regular Bardeen black hole. In our approach we used the Cartesian gauge to write the Dirac equation and as was showed in this gauge the problem reduces to only a radial equation that needed to be solved.

The Cartesian gauge was also used in other studies to find complete analytical solutions of the Dirac equation in de Sitter spacetime [31, 40, 41] and approximative solutions in several black hole geometries [7, 24, 34, 36]. Thus one can safely argue that the Cartesian gauge is very important in studying the Dirac equation in curved spacetimes geometries.

Acknowledgements

This work was supported by a grant of Ministry of Research and Innovation, CNCS - UEFISCDI, project number PN-III-P1-1.1-PD-2016-0842, within PNCDI III.

References

- [1] J. Polchinski, *String Theory, Vol. 2*, Cambridge Univ. Press (2005).
- [2] A. Ashtekar, J. Pullin, *Loop Quantum Gravity: the first 30 years*, World Scientific (2017).
- [3] A. Perez, Living Rev. Relativ. (2013) 16:3.
- [4] L. Freidel, Int. J. Theor. Phys. (2005) 44: 1769.

- [5] S. W. Hawking, *Comm. Math. Phys.* **43** (1975) 199; *Nature* **248** (1974) 30-31.
- [6] B. Mukhopadhyay, S.K. Chakrabarti, *Class. Quantum Grav.* **16**, 3165 (1999).
- [7] I.I. Cotaescu, *Mod. Phys. Lett. A* **22** (2007) 2493.
- [8] W. G. Unruh, *Phys. Rev. D* **14**, 3251 (1976).
- [9] A. Al-Badawi, M.Q. Owaidat, *Gen Relativ Gravit* **49**:110 (2017).
- [10] A. Zecca, *Nuovo Cimento* **30**, 13091315 (1998).
- [11] J. Jing, *Phys. Rev. D* **70** 065004 (2004); *Phys. Rev. D* **71** 124006 (2005).
- [12] D. Page, *Phys. Rev. D* **14**, 1509 (1976).
- [13] E. G. Kalnins and W. Miller, *J. Math. Phys.* **33** 286 (1992).
- [14] F. Belgiorno, S.L. Cacciatori, *J. Phys. A* **42**, 135207 (2009)
- [15] H. Schmid, *Math. Nachr.* **2004**, 274275, 117129.
- [16] I. Sakalli and M. Halilsoy, *Phys. Rev. D* **69**, 124012 (2004).
- [17] G. -Ortigoza, *Gen. Relativ. Gravit.* **5**, 395 (2001).
- [18] H. Cebeci, N. Ozdemir, *Class. Quantum Grav.* **30**, 175005 (2013).
- [19] F. Belgiorno, S.L. Cacciatori, *J. Math. Phys.* **51**, 033517 (2010).
- [20] Yan Lyu and Song Cui, *Phys. Scr.* **79** 025001 (2009).
- [21] J.M. Bardeen, in: *Conference Proceedings of GR5, Tbilisi, USSR, 1968*, p. 174.
- [22] Z.-Y. Fan and X. Wang, *Phys. Rev. D* **94**, 124027 (2016).
- [23] E. Ayon-Beato and A. Garcia, *Phys. Lett B* **493**, 149 (2000).
- [24] C.A. Sporea. Fermion scattering by a class of Bardeen black holes. arXiv:1806.11462
- [25] B. Thaller, *The Dirac Equation*, Springer Verlag, Berlin Heidelberg, 1992.
- [26] I.I. Cotaescu, *Mod. Phys. Lett. A* **13** (1998) 2923-2936.
- [27] D. R. Brill and J. A. Wheeler, *Rev. Mod. Phys.* **29** (1957) 465.
- [28] D. R. Brill and J. A. Cohen, *J. Math. Phys.* **7** (1966) 238.
- [29] D. G. Boulware, *Phys. Rev. D* **12** (1975) 350.
- [30] V. M. Villalba, *Mod. Phys. Lett. A* **8** (1993) 2351-2364.
- [31] I.I. Cotaescu, *Phys. Rev. D* **60** (1999) 124006.
- [32] I.I. Cotaescu, *J. Phys. A: Math. Gen.* **33** (2000) 1977.
- [33] I.I. Cotaescu, *Rom. J. Phys.* **52** (2007) 895-940.
- [34] C.A. Sporea, *Mod. Phys. Lett. A* **30**, 1550145 (2015).
- [35] I.I. Cotaescu, C. Crucean and C.A. Sporea, *Eur. Phys. J. C* (2016) **76**:102.
- [36] I.I. Cotaescu, C. Crucean and C.A. Sporea, *Eur. Phys. J. C* (2016) **76**:413.
- [37] C.A. Sporea, *Chinese Physics C*, Vol. **41**, No. 12 (2017) 123101.
- [38] V. B. Berestetski, E. M. Lifshitz and L. P. Pitaevski. *Quantum Electrodynamics*. Pergamon Press, Oxford, 1982.

- [39] I.D. Novikov, Doctoral disertation, Sthernberg Astronomical Institute (1963).
- [40] I.I. Cotaescu, C. Crucean, *Mod. Phys. Lett. A* **22** (2008) 3707-3720.
- [41] I.I. Cotaescu, R. Racocanu, C. Crucean, *Mod. Phys. Lett. A* **21** (2006) 1313-1318.
- [42] F. W. J. Olver, D. W. Lozier, R. F. Boisvert and C. W. Clark, *NIST Handbook of Mathematical Functions* (Cambridge University Press, 2010).