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ON A SCHWARSZCHILD-LIKE METRIC

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Dedicated to the 85th birthday of Professor Cleopatra Mociuţchi

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Abstract

In this short Note we would like to bring into the attention of people working in General Relativity a Schwarzschild like metric found by Professor Cleopatra Mociuţchi in sixties. It was obtained by the A. Sommerfeld reasoning from his treatise "Elektrodynamik" but using instead of the energy conserving law from the classical Physics, the relativistic energy conserving law.

1. From Newton's Mechanics to General Relativity

To begin with, let's recall the laws of Newton's mechanics :

1. A particle moves with constant velocity if no force acts on it.
2. The acceleration of a particle is proportional to the force acting on it.
3. The forces of action and reaction are equal and opposite.

These laws hold in their simplest form only in inertial frames. The fact is stated as the Galilean principle of relativity :

Galilean PR : The laws of Mechanics have the same form in all inertial frames.

It implies the classical velocity addition law that has many confirmations in classical Mechanics but does not hold for the propagation of light. The speed of light c is the same in all inertial frame. Thus the PR was extended in the form of the next two postulates:

P1 : *Physical laws have the same form in all inertial frames.*

P2 : *The speed of light is fnite and equal in all inertial frames.*

These are basic for Special Relativity (SR). The SR is due to A.Einstein who came at the conclusion that the concepts of space and time are *relative* i.e. dependent on the reference frame of an observer. Mathematically, the postulates P1 and P2 implies that two inertial frames are connected by a Lorentz transformation of coordinates. This fact leads to a new law for the addition of velocities and to two real physical effects: length contraction and time dilatation. The Lorentz transformations show also that is more natural to speak about *spacetime* as a set of points (events) with four coordinates than about space and time separately. The squared differential distance between neighboring events is given by the Lorentz invariant quadratic form.

$$ds^2 = c^2 dt^2 - dy^2 - dz^2 \tag{1.1}$$

However, the non-inertial (accelerated) frames do exist. A. Einstein noticed that the dynamical effects of a gravitational field and an accelerated frame can not be distinguished and he formulated the principle of equivalence (PE).

PE_1: *Every non-inertial frame is locally equivalent to some gravitational field.*

This is more completely restated as

PE_2: *The gravitational field is locally equivalent to the field of inertial forces of a convenient accelerated frame of reference.*

A freely falling reference frame is called a locally inertial frame. Using this term the PE can be also restated as:

PE_3: *At every point in an arbitrary gravitational field we can choose a locally inertial frame in which the laws of Physics take the same form as in SR.*

A. Einstein considered SR as incomplete because of the role played by the inertial frames. He generalized the relativity of inertial motions to the relativity of all motions by formulating the general principle of relativity.

General PR : The form of physical laws is the same in all reference frames.

Mathematically, the general PR can be realized with the help of the principle of general covariance.

General covariance : The form of physical laws does not depend on the choice of coordinates.

This principle says that the physical laws have to be given in tensorial form. For details and historical motivations of the principles reviewed in the above we refer to the first chapter of the book [1].

2. The Einstein equations

Let's relabel the time and the spatial variable in a spacetime M_4 as $x = (x^0 = t; x^1; x^2; x^3)$. Then (1.1) is a particular form of the following quadratic form (the Einstein 's convention on summation is implied)

$$ds^2 = g_{ij} dx^i dx^j, i, j, k... = 0,1,2,3. \qquad (2.1)$$

Here the entries of the matrix ($g_{ij}(x)$) are the local components of a pseudo or semi-Riemannian metric g on M_4. We suppose that the canonical form of the matrix ($g_{ij}(x)$) is diag(+;-;-;-). One says that M_4 is a Lorentz manifold. There exists an unique linear connection ∇ with the local coefficients given by the usual form of the Christofell symbols which has no torsion and makes g covariant constant i.e. $g_{ij;k} = 0$, where ; means the covariant derivative. Then the Ricci tensor is $R_{ik} = R^j_{ik}$ where R^h_{ijk} is the curvature tensor of ∇ and the scalar curvature is R = $g^{ij}R_{ij}$. A. Einstein postulates the basic equation for gravitational processes as it follows (the Einstein equations)

$$EE : R_{ik} - \frac{1}{2} R g_{ik} = k T_{ik} \qquad (2.2)$$

where k is a constant and T_{ik} is the energy -momentum tensor. The tensor T_{ik} refers to the free formations (of the substance having non-zero rest mass, the free electromagnetic field, etc). This should be divergence free since the Einstein tensor (given by the left hand of the Einstein equation (2.2)) is divergence free.

Alternatively, the EE can be derived from Hamilton's principle

$$\delta \int (bR + L)\sqrt{-g}\, d\tau = 0 \qquad (2.3)$$

Where $\sqrt{-g}\, d\tau$ is the four dimensional volume element and in the Hilbert- Palatini Lagrange function bR + L, b is a constant, whereas the function L leads to T_{ik}.

Using the Einstein equations one may try to determine g_{ik} assuming that T_{ik} is given or to take a special form of g_{ik} and to determine T_{ik}. The second task is easier. The first is much harder. In absence of the matter (void or empty space) we have $T_{ik} = 0$ and a contraction with g^{ik} in (2.2) leads to a simpler form of the EE : $R_{ik} = 0$. The simplest and the most important exact solution of this last equation is the Schwarzschild metric to be discussed in the next Section. For details we refer to [2].

3. The Schwarzschild metric

The Schwarzschild metric is the first found exact solution of the equation

$$R_{ik} = 0. \qquad (3.1)$$

It was found by Karl Schwarzschild in 1916 and played an important role in confirming the predictions of the GR theory. Now there are many ways to deduce it. It is valid in the empty space surrounding a static body with mass spherical symmetric. Passing to spherical coordinates $(x^0; x^1; x^2; x^3) = (t; r; \theta; \varphi)$, the spherical symmetry and an obvious re-scaling of r reduces ds^2 to the following form (see [2,p.45])

$$ds^2 = Adt^2 - Bdr^2 - r^2(d\theta^2 + \sin^2\theta d\varphi^2), \tag{3.1}$$

where because of the chosen signature we must have A > 0 and B > 0 and the both tend to 1 at large distance from the source ($r \to \infty$). Then one computes the Christofell symbols Γ^i_{jk} which are inserted in a formula for R_{ik} , (5.27 in[2]). The Einstein equations (3.1) give first that the product AB is a constant (equal to 1 as r tends to ∞). Hence B = 1/A. Then (r/B)` = 1, where ` denotes the derivative with respect to *r* and upon integration one gets the Schwarzschild metric (SM)

$$ds^2 = (1-\frac{\lambda}{r})dt^2 - (1-\frac{\lambda}{r})^{-1}dr^2 - r^2(d\theta^2 + \sin^2\theta d\varphi^2),\ \ \lambda = \frac{2GM}{c^2} \tag{3.2}$$

where G is the universal gravitational constant and M is the mass of the central body (star, planet...). We stress that this metric is valid in the empty space outside of the central body.

In his treatise [3], A. Sommerfeld derives the SM directly from the equivalence principle in form PE_3. Here is his reasoning.

Let be a Point *P* lying in the empty space surrounding a central mass M which is distributed with spherical symmetry, its center being a point *O*. Within spherical coordinate, *P* will be determined by the $r, \theta, \varphi.$ Let us consider a local frame of reference in *P*, one of the axes of the frame being OP. A second frame of reference, with the coordinates $\dot{r}, \theta, \varphi.$ slides on the OP along the first, the relative velocity being *v*, that is the velocity a material point of mass *m* has under the action of the gravitational field of the central body. The metric connected to the mass *m* will be the Minkowskian one, that is

$$ds^2 = c^2 d\dot{t}^2 - d\dot{r}^2 - \dot{r}^2(d\theta^2 + \sin^2\theta\varphi^2) \tag{3.4}$$

An observer having a fixed position in P will notice a momentary contraction of the distance as well as a momentary dilatation of the duration, that is

$$dr = \sqrt{1 - v^2/c^2}\, d\dot{r};\ \ dt = d\dot{t}\Big/\sqrt{1 - v^2/c^2} \tag{3.5}$$

where $r = \dot{r}$, as we are talking about the same point *P*. It remains to determine the velocity v=v(r), where the dependence on *r* is due to the fact that we a talking about an accelerated material point, therefore *v* varies permanently during the motion. In order to do this, A. Sommerfeld has used the energy conserving law from the classical Physics:

$$\frac{1}{2}mv^2 + mV(r) = const., \tag{3.6}$$

Where $V(r) = \frac{GM}{r}$ is the gravitational field potential of the central mass. When r→∞ the constant is zero and so

$$\frac{v^2}{c^2} = \frac{\lambda}{r}, \quad \lambda = 2GM/c^2 \tag{3.7}$$

Using this in (3.4) one _nds the SM in the form (3.2).

The main point here is the use of the energy conserving law from (3.6). But accordingly to the *PE* in its from *PE₃* it is not only more natural but it is just compulsory the use of the relativistic law of energy conservation as a law from *SR*. It is true that the *SR* includes also the classical laws but only in limits and with some nuances due to different groups of symmetries. This remark belongs to Professor Cleopatra Mociuţchi. Based on it she uses the relativistic law of energy conservation and so she arrived to and studied in the sixties, [4-8], a Schwarzschild like metric as follows.

The relativistic law of energy conservation has the form

$$(m - m_0)c^2 - \frac{GmM}{r} = 0, \quad m_0 = m\sqrt{1 - v^2/c^2} \tag{3.8}$$

It comes out that

$$\sqrt{1 - v^2/c^2} = 1 - \mu/r \tag{3.9}$$

an so (3.5) reduces to what we call the Mociuţchi metric (MM) :

$$ds^2 = c^2(1 - \frac{\mu}{r})^2 dt^2 - (1 - \frac{\mu}{r})^{-2} dr^2 - r^2(d\theta^2 + \sin^2\theta d\varphi^2), \quad \mu = \frac{GM}{c^2} = \lambda/2 \tag{3.10}$$

Here are some properties of the Mociutţchi metric:

1. It returns all the General Relativity tests,
2. It reduces to the SM if (μ/r)2~0,
3. The black hole radius that results from it is half from the Schwarzschild radius,
4. When passing through the radius of the black hole, the variables r and t do not change significance between themselves in the case of MM, contrary to SM.

5. The MM results from (2.3), L is the Lagrange function of the gravitational field of the central mass.

The last enumerated property may be connected with the remark that if one computes the Einstein tensor $R_{ik} - \frac{1}{2} R g_{ik}$ for the MM it comes out that four of its components are non null. More precisely, we have

$$T_{22} = \frac{GM^2}{8\pi r^2}, \quad T_{00} = T_{22}(1-\frac{\mu}{r})^2, \quad T_{11} = -T_{22}(1-\frac{\mu}{r})^{-2}, \quad T_{33} = T_{22}\sin^2\theta$$

In the other words the MM is not an exact solution of the Einstein equation with the energy momentum tensor $T_{ij} = 0$. However, we derived it in the hypothesis of the absence of the matter and fields (in the empty space outside of a star or a planet). The contradiction could be eliminated if we assume that even in this empty space a kind of manifestation of the gravitational field does exist. But we can do this if we slightly modify the PE as follows :

P`E: *The* only dynamical effects *of a gravitational field are locally equivalent to the field of inertial forces of a convenient accelerated frame of reference*.

Moreover, some results on interaction of various fields confirm , [9-11], a more general principle of equivalence :

P"E: *The* dynamical effects of certain fields *are locally equivalent to the field of inertial forces of a convenient accelerated frame of reference.*

The non-vanishing of T_{ij} for the MM implies that now the Einstein equations are derived from a Lagrangian

$$L = bR + L_0 + L_1 + L_2 + \ldots$$

where L_0 is the Lagrangian of a free particle moving in the gravitational, electromagnetic etc. fields whose Lagrangians are L_1, L_2, etc...Such a Lagrangian is similat to that from field theory only the interraction Lagrangian is replaced by bR. Thus one may say that the General Relativity theory is a field theory in which the interaction curves the space. See also [9-12].

Addenda. Prof. G.W. Gibbons from University of Cambridge U.K. write us " The metric you describe in your paper is a special, so-called extreme, case of the well known Reissner-Nordstrom solution of the Einstein-Maxwell equations." The remark is true but the MM was independently found by Prof. Cleopatra Mociuţchi and the coincidence is mathematically formal only. Indeed, in the case of the R-NM, the (0, 0) -component is

$$g_{00} = c^2(1 - 2\frac{GM}{c^2 r} + \frac{GM^2}{c^4 r^2})$$

where M is the mass and e is the charge of the central body. The (0; 0)- component of the MM metric is

$$g_{00}^{M} = c^2(1 - 2\frac{GM}{c^2 r} + \frac{GM^2}{c^4 r^2})$$

and in the particular case $e^2 = GM^2$, that is for a certain value of the electric charge one gets $g_{00}^{M} = g_{00}$ and so the MM reduces to the R-NM. But in the MM only mass enters. The MM refers to a pure gravitational field. So we arrive at the conclusion that the principle of equivalence should be more precisely formulated as follows:

PE: The only dynamical effects of a gravitational field are locally equivalent to the field of inertial forces of a convenient accelerated frame of reference.

Of course, this strange mathematical coincidence could have and other explanations. To find some from the physical community was the aim of our paper.

Prof. Y. Itin proposed to our attention his paper [13]. Studying it we add more explanation as follows. The paper by Prof. Y. Itin is a study of the metrics that, in the framework of the GR constructed by A. Einstein, return all the GR tests in the observational limits.The MM metric presented in our paper suggests a possible generalization of the Einsteinian theory in which to admit that the gravitational field has itself a non vanishing energy-momentum tensor included in the right hand of the Einstein equations, equivalently, its Einsteinian tensor is non -null. Thus it seems that the curvature of space appears as a dynamical effect of the gravitational potential (the metric tensor field), but the gravitational field is much more than dynamical effects and a manifestation of this much more appears also in the second hand of the Einstein equation. The GR tests are also returned, but the movement of a free particle is no longer on geodesics.

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In memory of Prof. Dr. Ioan Gottlieb

Ioan Gottlieb, Profesor Emeritus and researcher in Theoretical Physics at the Alexandru Ioan Cuza University of Iasi, Romania, has passed away in Iaşi , Romania on September 2, 2011. He had been active in Physics and the University administration for six decades.

Professor Ioan Gottlieb was born on January 21, 1929 in Baia-Mare, Romania.During World War II he was detainee at the infamous nazi concentration camps. He was liberated at the end of war (1945). Upon graduating from the Faculty of Mathematics and Physics, Department of Mathematics of the Babes-Bolyai University of Cluj in 1951, he takes on research work in

Theoretical Physics, in which field he would later develop his doctoral degree dissertation, held at the Alexandru Ioan Cuza University of Iasi in 1962, under the supervision of Professor Teofil T. Vescan. The same year he marries Mrs. Cleopatra Mociuţchi, also a university Professor at the Faculty of Physics of Iaşi, who would become his lifelong partner and professional collaborator.

Professor Gottlieb begins in 1953 teaching at the Aexandru Ioan Cuza University of Iaşi, where he evolves up the hierarchy to Professor (1972). Initiator and leader of the Iaşi modern and contemporary school of Theoretical Physics Head of Department between 1964-1971, and 1990-1999, doctoral degree supervisor since 1971, Professor Gottlieb proves exceptional didactic skills and great character. He formed 21 PhD. students in Physics, some of which became PhD. supervisors themselves, some academic leaders and top researchers. Throughout his 60-year academic career, Professor Gottlieb taught and published in important branches of modern Physics: Quantum Mechanics, Electrodynamics and Theory of Relativity, Thermodynamics, Statistical Physics, Field Quantum Theory, Astronomy, Mathematical Methods for Theoretical Physics, and Philosophical issues in Physics. He published in Romania and abroad 18 books and over 150 scientific papers.

A true European man of culture, Professor Gottlieb was a member of several Academies and Scientific Societies: the European Academy for Sciences and Arts, American Mathematical Society, International Society of General Relativity and Gravitation, Committee on Space Research (COSPAR), the Society of Mathematics and Physics, The Society of Physics and Chemistry, the Mathematical Society Janos Bolyai of Hungary, and of course the Romanian Society of Physics. He was a founding member of the Romanian Society of General Relativity and Gravitation, becoming also its first President. As an acknowledgment of his distinguished scientific and academic merits, he was granted the title of Professor Emeritus by the University Aexandru Ioan Cuza of Iaşi back in 2004.

He will always be missed, with his lasting optimistic smile, his wise and experienced advice. He will be at the hearts of the all those who had known him, as a living example of a dedicated researcher, deeply committed educator, a true friend and a wonderful person!

Professor Gottlieb thought and wrote on various issues on Physics until nearly his last breath.In 2011 he published a final version of his vision on the Universe, in his last book titled A Fractal Model of the Universe (Ars Longa Publishing House, Iaşi, Romania). His last research paper, co-written with M.Anastasiei, was posted as arXiv : 1104.4023 and was polished by its authors in the summer of 2011. In the above is printed the version with the final contributions of Professor Ioan Gottlieb.