Stability of Picard operators under operator perturbations

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Dedicated to Professor Mihail Megan on the occasion of his 70th birthday

Abstract. In this paper we study the following problems:

I. Let (M, d) be a complete metric space and $f, g : M \rightarrow M$ be two operators. We suppose that:

- (a) f is a Picard operator with its unique fixed point x_f^* ;
- (b) there exists $\eta > 0$ such that $d(f(x), g(x)) \leq \eta$, for every $x \in M$.

The problem consists in estimating $d(g^n(x), x_f^*)$, for $x \in M$ and $n \in \mathbb{N}^*$.

II. Let B be a Banach space and $f, g : B \rightarrow B$ be two operators. We suppose that f is a Picard operator. The problem is to find sufficient conditions which guarantee that $f + g$ is a Picard operator.

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1 Introduction

There exist various aspects of the stability problem for the solutions of differential and integral equations (see [1], [5], [18], [21], [26], [31], [37], ...), of operatorial equations ([7], [2], [4], [6], [8], [12], [17], [18], [23], [25], [27], [28],

[29], [30], [33], [34], [36], ...) and of functional-operatorial equations ([3], [10], [11], [14], [15], [16], [20], [22], [24], [32], [33], ...).

In this paper, we will study the following two abstract problems:

Problem I. (see Problem 1.4.2 in [30]) Let (M, d) be a complete metric space and $f, g : M \rightarrow M$ be two operators. We suppose that:

- (a) f is a Picard operator with its unique fixed point x_f^* ;
- (b) there exists $\eta > 0$ such that $d(f(x), g(x)) \leq \eta$, for every $x \in M$.

The problem consists in estimating $d(g^n(x), x_f^*)$, for $x \in M$ and $n \in \mathbb{N}^*$.

Problem II. (see Problem 55 in [34]) Let B be a Banach space and $f, g : B \rightarrow B$ be two operators. We suppose that f is a Picard operator. The problem is to find sufficient conditions which guarantee that $f + g$ is a Picard operator.

We will present first some basic notions and results which are essential for the main part of this paper.

Let X be a nonempty set and $f : X \rightarrow X$ be an operator. We consider the fixed point equation

$$x = f(x), \quad x \in X. \quad (1.1)$$

We denote by F_f the fixed point set of f , i.e., $F_f := \{x \in X \mid f(x) = x\}$. If (M, d) is a metric space, then, by definition, f is a Picard operator if $F_f = \{x^*\}$ and

$$f^n(x) \rightarrow x^* \text{ as } n \rightarrow \infty, \text{ for all } x \in M.$$

A Picard operator f for which there exists a function $\psi : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ increasing, continuous in 0 and satisfying $\psi(0) = 0$, such that

$$d(x, x^*) \leq \psi(d(x, f(x))), \text{ for all } x \in X,$$

is called a ψ -Picard operator. In particular, if ψ has a linear form (i.e., $\psi(t) = ct, t \in \mathbb{R}_+$, for some $c > 0$), then f is called a c -Picard operator.

Definition 1.1.

Let (M, d) be a metric space and $f : M \rightarrow M$ be an operator. Then, f is called:

- (i) an α -contraction if $\alpha \in]0, 1[$ and

$$d(f(x), f(y)) \leq \alpha d(x, y), \text{ for every } x, y \in M.$$

- (ii) a Kannan operator if there exists $\beta \in]0, \frac{1}{2}[$ such that

$$d(f(x), f(y)) \leq \beta (d(x, f(x)) + d(y, f(y))), \text{ for every } x, y \in M.$$

(iii) a φ -contraction if $\varphi : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ is a comparison function (i.e., φ is increasing and $\varphi^n(t) \rightarrow 0$ as $n \rightarrow \infty$, for every $t \in \mathbb{R}_+$) and

$$d(f(x), f(y)) \leq \varphi(d(x, y)), \text{ for every } x, y \in M.$$

(iv) a Hardy-Rogers operator if there are $\alpha, \beta, \gamma \in \mathbb{R}_+$ with $\alpha + 2\beta + 2\gamma < 1$, such that, for every $x, y \in M$, we have

$$\begin{aligned} d(f(x), f(y)) \leq & \alpha d(x, y) + \beta (d(x, f(x)) + d(y, f(y))) \\ & + \gamma (d(x, f(y)) + d(y, f(x))). \end{aligned}$$

Example 1.1.

- 1) If (M, d) is a complete metric space and $f : M \rightarrow M$ is an α -contraction, then f is a $\frac{1}{1-\alpha}$ -Picard operator.
- 2) If (M, d) is a complete metric space and $f : M \rightarrow M$ is Kannan operator with constant β , then f is a $\frac{1-\beta}{1-2\beta}$ -Picard operator.
- 3) If (M, d) is a complete metric space and $f : M \rightarrow M$ is Hardy-Rogers operator with constants α, β, γ , then f is a $\frac{1-\beta-\gamma}{1-\alpha-2\beta-2\gamma}$ -Picard operator.
- 4) If (M, d) is a complete metric space and $f : M \rightarrow M$ is φ -contraction, such that φ is a strict comparison function (i.e., φ is a comparison function and $t - \varphi(t) \rightarrow \infty$ as $t \rightarrow \infty$), then the operator f is a ψ_φ -Picard operator, with $\psi_\varphi(t) := \sup\{s \in \mathbb{R}_+ : s - \varphi(s) \leq t\}$.

Other examples of Picard operator involving function spaces are given now.

Example 1.2.

Let $(B, |\cdot|)$ be a Banach space and $g \in C([a, b] \times B, B)$ be an operator. We suppose that there exists $l_g \in]0, 1[$ such that

$$|g(t, u) - g(t, v)| \leq l_g |u - v|, \forall t \in [a, b], \forall u, v \in B.$$

Consider $T : C([a, b], B) \rightarrow C([a, b], B)$ defined by

$$Tx(t) := g(t, x(t)), t \in [a, b].$$

If we consider the Banach space $\mathbb{B} := (C([a, b], B), \|\cdot\|_\infty)$, then the operator $T : \mathbb{B} \rightarrow \mathbb{B}$ is an l_g -contraction and, hence, a Picard operator.

Example 1.3.

Let $(B, |\cdot|)$ be a Banach space and $K \in C([a, b] \times [a, b] \times B, B)$ be an operator. We suppose that there exists $l_K \in]0, 1[$ such that

$$|K(t, s, u) - K(t, s, v)| \leq l_K |u - v|, \forall t, s \in [a, b], \forall u, v \in B.$$

Consider $S : C([a, b], B) \rightarrow C([a, b], B)$ defined by

$$Sx(t) := \int_a^t K(t, s, x(s))ds, t \in [a, b].$$

If we consider the Banach space $\mathbb{B} := (C([a, b], B), \|\cdot\|_\tau)$ (where $\|\cdot\|_\tau$ is a Bi-elecki type norm with $\tau > l_K$), then the operator $S : \mathbb{B} \rightarrow \mathbb{B}$ is $\frac{l_K}{\tau}$ -contraction and, hence, a Picard operator. Moreover, S is a strong contraction in the sense of Goebel. Recall that if $(B, \|\cdot\|)$ is a Banach space, then an operator $f : B \rightarrow B$ is called a strong contraction if for every $\epsilon > 0$ there exists a norm $\|\cdot\|_\epsilon$ on B , equivalent with $\|\cdot\|$, such that

$$\|f(x) - f(y)\|_\epsilon \leq \epsilon \|x - y\|_\epsilon, \text{ for every } x, y \in B.$$

For details see [13] and [34].

2 Results concerning Problem I.

For a better understanding of the problem, we present several examples.

Example 2.1.

If (M, d) is a complete metric space and $f : M \rightarrow M$ is an α -contraction, then, for Problem I, we have the following result.

Theorem 2.1. *Let (M, d) is a complete metric space, $f : M \rightarrow M$ be an α -contraction and $g : M \rightarrow M$ be an operator for which there exists $\eta > 0$ such that $d(f(x), g(x)) \leq \eta$, for every $x \in M$. Denote by x_f^* the unique fixed point of f . Then*

$$d(g^n(x), x_f^*) \leq \frac{\eta}{1 - \alpha} + \frac{\alpha^n}{1 - \alpha} d(x, f(x)), \forall x \in M, \forall n \in \mathbb{N}^*.$$

Proof. For $x \in M$, we have that

$$\begin{aligned} d(g^n(x), x_f^*) &\leq d(f^n(x), x_f^*) + d(f^n(x), g^n(x)) \leq \\ &\frac{\alpha^n}{1 - \alpha} d(x, f(x)) + d(f^n(x), g^n(x)). \end{aligned}$$

On the other hand

$$\begin{aligned} d(f^n(x), g^n(x)) &\leq d(f(g^{n-1}(x)), g(g^{n-1}(x))) + d(f(g^{n-1}(x)), f(f^{n-1}(x))) \leq \\ &\eta + \alpha d(f^{n-1}(x), g^{n-1}(x)) \leq \eta + \alpha\eta + \alpha^2 d(f^{n-2}(x), g^{n-2}(x)) \leq \dots \leq \frac{\eta}{1 - \alpha}. \end{aligned}$$

Thus, the conclusion follows by the above two relations. $\square$

Example 2.2.

Let (M, d) be a complete metric space and $f : M \rightarrow M$ be a φ -contraction. Suppose that, in addition, φ satisfies the following assumptions:

- (a) $t - \varphi(t) \rightarrow \infty$, as $t \rightarrow \infty$ (i.e., φ is a strict comparison function);
- (b) $\varphi(t_1 + t_2) \leq \varphi(t_1) + \varphi(t_2)$, $\forall t_1, t_2 \in \mathbb{R}_+$.

Then, for Problem I, we have the following result.

Theorem 2.2. *Let (M, d) be a complete metric space and $f : M \rightarrow M$ be a strong φ -contraction satisfying (a)-(b) from above. Denote by x_f^* the unique fixed point of f . If $g : M \rightarrow M$ is an operator for which there exists $\eta > 0$ such that $d(f(x), g(x)) \leq \eta$ for every $x \in M$, then we have that*

$$d(g^n(x), x_f^*) \leq \theta(\eta) + \varphi^n(\psi_\varphi(d(x, f(x)))), \forall x \in M, \forall n \in \mathbb{N}^*,$$

where $\theta(t) := \sum_{n=0}^{\infty} \varphi^n(t)$ and $\psi_\varphi(t) = \sup\{s \in \mathbb{R}_+ : s - \varphi(s) \leq t\}$, for $t \in \mathbb{R}_+$.

Proof. If φ is a strict comparison function, then the function $\psi_\varphi : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ defined by $\psi_\varphi(t) = \sup\{s \in \mathbb{R}_+ : s - \varphi(s) \leq t\}$ is increasing and has the property that $\psi_\varphi(t) \rightarrow 0_+$ as $t \rightarrow 0_+$. Then, for each $x \in M$, we have

$$d(g^n(x), x_f^*) \leq d(f^n(x), x_f^*) + d(f^n(x), g^n(x)).$$

On one hand

$$d(f^n(x), x_f^*) \leq \varphi^n(d(x, x_f^*))$$

with

$$d(x, x_f^*) \leq \psi_\varphi(d(x, f(x))).$$

On the other hand

$$\begin{aligned} d(f^n(x), g^n(x)) &\leq d(f(g^{n-1}(x)), g(g^{n-1}(x))) + d(f(g^{n-1}(x)), f(f^{n-1}(x))) \leq \\ &\eta + \varphi(d(f^{n-1}(x), g^{n-1}(x))) \leq \eta + \varphi(\eta + \varphi(d(f^{n-2}(x), g^{n-2}(x)))) \leq \\ &\eta + \varphi(\eta) + \varphi^2(d(f^{n-2}(x), g^{n-2}(x))) \leq \dots \leq \sum_{n=0}^{\infty} \varphi^n(\eta) = \theta(\eta). \end{aligned}$$

Thus, the conclusion follows by the above two relations. $\square$

We formulate now two open problems.

Problem A. *For which generalized contractions (Kannan operators, Hardy-Rogers operators, ...) in complete metric spaces we have similar results to the above ones ?*

Problem B. *A second open question is to study the above problem in the case of generalized metric spaces.*

Concerning the last problem, we have the following result in complete $\mathbb{R}_+^m$ -metric spaces.

Example 2.3.

If (M, d) is a complete $\mathbb{R}_+^m$ -metric space and $f : M \rightarrow M$ is an A -contraction (i.e., $A \in \mathcal{M}_{m,m}$ is a matrix convergent to zero and $d(f(x), f(y)) \leq Ad(x, y)$, $\forall x, y \in M$), then we have the following result.

Theorem 2.3. *Let (M, d) is a complete $\mathbb{R}_+^m$ -metric space, $f : M \rightarrow M$ be an A -contraction and $g : M \rightarrow M$ be an operator for which there exists $\eta = (\eta_1, \dots, \eta_m) \in \mathbb{R}_+^m$ with $\eta_i > 0$ for every $i \in \{1, \dots, m\}$, such that $d(f(x), g(x)) \leq \eta$, for every $x \in M$. Denote by x_f^* the unique fixed point of f . Then, we have that*

$$d(g^n(x), x_f^*) \leq (I_m - A)^{-1}\eta + A^n(I_m - A)^{-1}d(x, f(x)), \forall x \in M, \forall n \in \mathbb{N}^*.$$

Remark 2.1.

Consider Problem I with $f : M \rightarrow M$ be a ψ -Picard operator. In this case, we have

$$d(g^n(x), x_f^*) \leq \psi(d(g^n(x), f(g^n(x)))), \forall x \in M, \forall n \in \mathbb{N}^*.$$

By this relation, we obtain that, if g is continuous and the sequence $(g^n(x))_{n \in \mathbb{N}}$ is f -asymptotically regular (i.e., $d(g^n(x), f(g^n(x))) \rightarrow 0$ as $n \rightarrow \infty$ for all $x \in M$), then g is a Picard operator with $x_g^* = x_f^*$.

3 Results concerning Problem II.

We start this section by the following result of Browder-Petryshyn (see [7]), given in terms of Picard operators.

Theorem 3.1. *Let B be a Banach space and $f : B \rightarrow B$ be a bounded linear operator. Suppose that f is Picard. Let $g : B \rightarrow B$ be a constant operator, i.e., there exists $y \in B$ such that $g(x) = y$, for every $x \in B$. Then, for each $y \in (1_B - f)(B)$ the operator $f + g$ is Picard.*

The main result of this section is the following.

Theorem 3.2. *Let B be a Banach space and $f, g : B \rightarrow B$ be two operators.*

We suppose:

(1) *f is a strong contraction with respect to equivalent norms $\|\cdot\|_\epsilon$, where $0 < \epsilon < 1$;*

(2) *g is an l_g -contraction with respect to each $\|\cdot\|_\epsilon$, for $0 < \epsilon < 1$.*

Then, the operator $f + g$ is Picard.

Proof. Let $\epsilon > 0$ such that $l_g + \epsilon < 1$. Then, the operator $f + g$ is a $(l_g + \epsilon)$ -contraction with respect to $\|\cdot\|_\epsilon$. $\square$

Example 3.1.

Let $(B, |\cdot|)$ be a Banach space and T be given by Example 1.2. Since there exists $l_g \in]0, 1[$ such that

$$|g(t, u) - g(t, v)| \leq l_g |u - v|, \forall t \in [a, b], \forall u, v \in B,$$

we get that the operator T is an l_g -contraction with respect to a Bielecki norm $\|\cdot\|_\tau$ on $C([a, b], B)$, for every $\tau > 0$. Let S be the operator given in Example 1.3. Then, $T + S$ is a Picard operator on $(C([a, b], B), \|\cdot\|_\infty)$. Indeed, $T + S$ is a $(l_g + \frac{l_K}{\tau})$ -contraction with respect to $\|\cdot\|_\tau$, where $\tau > 0$ is such that $l_g + \frac{l_K}{\tau} < 1$. Since the norms $\|\cdot\|_\infty$ and $\|\cdot\|_\tau$ are Lipschitz equivalent, the conclusion follows.

For other examples of this type see [32]. These examples suggest the following result.

Theorem 3.3. *Let $(B, |\cdot|)$ be a Banach space and $\mathbb{B} := (C([a, b], B), \|\cdot\|_\tau)$ be the Banach space of continuous abstract functions on $[a, b]$ with values in B endowed with a Bielecki norm $\|\cdot\|_\tau$ corresponding to $\tau > 0$. Let $T, S : \mathbb{B} \rightarrow \mathbb{B}$ be two operators. We suppose that:*

(1) *there exists $l_T \in]0, 1[$ such that*

$$|Tx(t) - Ty(t)| \leq l_T |x(t) - y(t)|, \forall t \in [a, b], \forall x, y \in \mathbb{B};$$

(2) *there exists $c > 0$ such that*

$$|Sx(t) - Sy(t)| \leq \frac{c}{\tau} \|x - y\|_\tau e^{\tau(t-a)}, \forall t \in [a, b], \forall x, y \in \mathbb{B} \text{ and } \tau > 0.$$

Then, the operators T, S and $T + S$ are Picard in $((C([a, b], B), \|\cdot\|_\infty)$

Proof. We will prove that $T + S$ is a Picard operator. Indeed, for all $t \in [a, b]$, $x, y \in \mathbb{B}$ and $\tau > 0$, we have

$$|(T + S)x(t) - (T + S)y(t)| \leq l_T |x(t) - y(t)| + \frac{c}{\tau} \|x - y\|_\tau e^{\tau(t-a)}.$$

Then

$$\|(T + S)x - (T + S)y\|_\tau \leq (l_T + \frac{c}{\tau})\|x - y\|_\tau.$$

Since $l_T < 1$, there exists $\tau > 0$ such that $l_T + \frac{c}{\tau} < 1$. This shows that $T + S$ is a contraction and hence a Picard operator. $\square$

Example 3.2.

Let $f, g : \mathbb{R} \rightarrow \mathbb{R}$ be two mappings defined by $f(x) = \frac{1}{3}x$ and $g(x) = -x$. In this case, f is a Picard operator, g is not a Picard operator, but $f + g : \mathbb{R} \rightarrow \mathbb{R}$ is a Picard operator.

By the above considerations, the following open question also arises.

Problem C. Let B be a Banach space and $f : B \rightarrow B$ be an operator. In which conditions the operator $1_B - f$ is Picard ? Some basic references for this problem are [8], [9], [25], ...

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