On Ψ Bounded Solutions for a Nonlinear Lyapunov Matrix Differential Equation on $\mathbb{R}$

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Dedicated to Professor Mihail Megan on the occasion of his 70th birthday

Abstract. Using Banach and Schauder - Tychonoff fixed point theorems, existence results for a nonlinear Lyapunov matrix differential equation on $\mathbb{R}$ are given. The obtained results generalize and extend the results from [5] and [18].

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1 Introduction

The purpose of present paper is to provide sufficient conditions for the existence and uniqueness and existence of at least one Ψ – bounded solution for the nonlinear Lyapunov matrix differential equation on $\mathbb{R}$

$$Z' = A(t)Z + ZB(t) + C(t) + F(t, Z) \quad (1.1)$$

with the help of Banach and Schauder-Tychonoff fixed point theorems. We first establish two results in connection with the existence and uniqueness and existence of at least one Ψ – bounded solution for the nonlinear matrix differential equation on $\mathbb{R}$ of the form

$$Z' = A(t)Z + C(t) + F(t, Z). \quad (1.2)$$

Second, using vectorization operator and Kronecker product of matrices, we treat the same problems for the nonlinear Lyapunov matrix differential equation on $\mathbb{R}$ of the form (1.1).

History of problem. A classical result in connection with boundedness of solutions of systems of ordinary differential equations

$$x' = A(t)x + c(t) + f(t, x) \quad (1.3)$$

was given by Coppel [5] (Chapter V, section 2, Theorem 4). The problem of Ψ -bounded solutions for systems of ordinary differential equations has been studied by many authors: [2], [3], [4], [8], [9], [12], [13], [14], [18] and for Lyapunov matrix differential equations, [6], [7], [10], [11], [16], [17].

The introduction of the matrix function Ψ in the study of solutions permits to obtain a mixed asymptotic behavior of the components of the solutions of the above equations.

2 Preliminaries

In this section we present some basic notations, definitions, hypotheses and results which are useful later on.

Let $\mathbb{R}^d$ be the Euclidean d -dimensional space. For $x = (x_1, x_2, \dots, x_d)^T \in \mathbb{R}^d$, let $\|x\| = \max\{|x_1|, |x_2|, \dots, |x_d|\}$ be the norm of x (here, T denotes transpose).

Let $\mathbb{M}_{d \times d}$ be the linear space of all real $d \times d$ matrices.

For $A = (a_{ij}) \in \mathbb{M}_{d \times d}$, we define the norm $|A|$ by $|A| = \sup_{\|x\| \leq 1} \|Ax\|$. It

is well-known that $|A| = \max_{1 \leq i \leq d} \left\{ \sum_{j=1}^d |a_{ij}| \right\}$.

By a solution of the equation (1.1) we mean a continuous differentiable $d \times d$ matrix function satisfying the equation (1.1) for all $t \in \mathbb{R}$.

In equation (1.1) we assume that the coefficients are continuous functions.

Let $\Psi_i : \mathbb{R}_+ \rightarrow (0, \infty)$, $i = 1, 2, \dots, d$, be continuous functions and

$$\Psi = \text{diag} [\Psi_1, \Psi_2, \dots, \Psi_d].$$

A matrix P is said to be a projection if $P^2 = P$.

Definition 2.1. ([12], [8]) A function $\varphi : R \rightarrow R^d$ is said to be Ψ -bounded on R if $\Psi(t)\varphi(t)$ is bounded on R (i.e. there exists $m > 0$ such that $\|\Psi(t)\varphi(t)\| \leq m$, for all $t \in R$).

Otherwise, is said that the function φ is Ψ -unbounded on R .

Definition 2.2. ([10]) A matrix function $M : \mathbb{R} \longrightarrow \mathbb{M}_{d \times d}$ is said to be Ψ -bounded on R if the matrix function $\Psi(t)M(t)$ is bounded on R (i.e. there exists $m > 0$ such that $|\Psi(t)M(t)| \leq m$, for all $t \in R$). Otherwise, is said that the matrix function M is Ψ -unbounded on R .

We now describe a few definitions and properties in connection with Kronecker product of matrices and vectorization operator.

Definition 2.3. ([1]) Let $A = (a_{ij}) \in M_{m \times n}$ and $B = (b_{ij}) \in M_{p \times q}$. The Kronecker product of A and B , written $A \otimes B$, is defined to be the partitioned matrix

$$A \otimes B = \begin{pmatrix} a_{11}B & a_{12}B & \cdots & a_{1n}B \\ a_{21}B & a_{22}B & \cdots & a_{2n}B \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1}B & a_{m2}B & \cdots & a_{mn}B \end{pmatrix}.$$

Obviously, $A \otimes B \in \mathbb{M}_{mp \times nq}$.

We next show the important rules of calculation of the Kronecker product.

Lemma 2.1. ([1]) The Kronecker product has the following properties and rules, provided that the dimension of the matrices are such that the various expressions exist:

- 1). $A \otimes (B \otimes C) = (A \otimes B) \otimes C$;
- 2). $(A \otimes B)^T = A^T \otimes B^T$;
- 3). $(A \otimes B) \cdot (C \otimes D) = (A \cdot C) \otimes (B \cdot D)$;
- 4). $(A \otimes B)^{-1} = A^{-1} \otimes B^{-1}$;
- 5). $A \otimes (B + C) = A \otimes B + A \otimes C$;
- 6). $(A + B) \otimes C = A \otimes C + B \otimes C$;

$$7). I_d \otimes A = \begin{pmatrix} A & O & \cdots & O \\ O & A & \cdots & O \\ \vdots & \vdots & \vdots & \vdots \\ O & O & \cdots & A \end{pmatrix};$$

$$8). (A(t) \otimes B(t))' = A'(t) \otimes B(t) + A(t) \otimes B'(t); (' \text{ denotes the derivative } \frac{d}{dt}).$$

Proof. See in [1]. □

Definition 2.4. ([15]) The application $\text{Vec} : \mathbb{M}_{m \times n} \longrightarrow \mathbb{R}^{mn}$, defined by

$$\text{Vec}(A) = (a_{11}, a_{21}, \cdots, a_{m1}, a_{12}, a_{22}, \cdots, a_{m2}, \cdots, a_{1n}, a_{2n}, \cdots, a_{mn})^T,$$

where $A = (a_{ij}) \in \mathbb{M}_{m \times n}$, is called the vectorization operator.

Lemma 2.2. ([10]) *The vectorization operator*

$$\mathcal{V}ec : \mathbb{M}_{m \times n} \longrightarrow \mathbb{R}^{mn}, \quad A \longrightarrow \mathcal{V}ec(A),$$

is a linear and one-to-one operator. In addition, $\mathcal{V}ec$ and $\mathcal{V}ec^{-1}$ are continuous operators.

Proof. See Lemma 2, [10]. □

Remark 2.1. *Obviously, a function $F : \mathbb{R} \longrightarrow \mathbb{M}_{d \times d}$ is a continuous (differentiable) matrix function on R if and only if the function $f : R \longrightarrow \mathbb{R}^{d^2}$, defined by $f(t) = \mathcal{V}ec(F(t))$, is a continuous (differentiable) vector function on R .*

We recall that the vectorization operator $\mathcal{V}ec$ has the following properties as concerns the calculations.

Lemma 2.3. ([15]) *If $A, B, M \in \mathbb{M}_{n \times n}$, then*

- 1). $\mathcal{V}ec(AMB) = (B^T \otimes A) \cdot \mathcal{V}ec(M)$;
- 2). $\mathcal{V}ec(MB) = (B^T \otimes I_n) \cdot \mathcal{V}ec(M)$;
- 3). $\mathcal{V}ec(AM) = (I_n \otimes A) \cdot \mathcal{V}ec(M)$;
- 4). $\mathcal{V}ec(MA) = (M^T \otimes I_n) \cdot \mathcal{V}ec(I_n)$.

Proof. See [15], Chapter 2. □

The following lemmas play a vital role in the proofs of main results of present paper.

Lemma 2.4. ([10]) *The matrix function $Z(t)$ is a solution on R of (1.1) if and only if the vector function $z(t) = \mathcal{V}ec(Z(t))$ is a solution of the differential system*

$$z' = (I_d \otimes A(t) + B^T(t) \otimes I_d) z + c(t) + f(t, z), \quad (2.1)$$

where $c(t) = \mathcal{V}ec(C(t))$ and $f(t, z) = \mathcal{V}ec(F(t, Z))$, on the same interval R .

Proof. See Lemma 7, [10]. □

Definition 2.5. ([10]) *The above system (2.1) is called "corresponding Kronecker product system associated with (1.1)".*

Lemma 2.5. ([10]). *For every matrix function $M : R \longrightarrow \mathbb{M}_{d \times d}$,*

$$\frac{1}{d} \|\Psi(t)M(t)\| \leq \|(I_d \otimes \Psi(t)) \mathcal{V}ec(M(t))\|_{\mathbb{R}^{d^2}} \leq \|\Psi(t)M(t)\|, \quad \forall t \geq 0. \quad (2.2)$$

Proof. See Lemma 4, [10]. $\square$

Lemma 2.6. *The solutions of (1.1) are Ψ -bounded on R if and only if the solutions of the differential system (2.1) are $I_d \otimes \Psi$ -bounded on R .*

Proof. It results from above Lemma (2.5). $\square$

Lemma 2.7. ([10]). *Let $X(t)$ and $Y(t)$ be a fundamental matrices for the equations*

$$Z' = A(t)Z \quad (2.3)$$

$$Z' = ZB(t) \quad (2.4)$$

respectively.

Then, the matrix $Z(t) = Y^T(t) \otimes X(t)$ is a fundamental matrix for the linear differential system

$$z' = (I_d \otimes A(t) + B^T(t) \otimes I_d) z \quad (2.5)$$

(i.e. for homogeneous differential system associated with (2.1).

Proof. See Lemma 6, [10]. $\square$

3 Ψ -bounded solutions for the matrix differential equation (1.2)

The purpose of this section is to provide sufficient conditions for the existence and uniqueness and existence of at least one Ψ -bounded solution on $\mathbb{R}$ for the equation (1.2).

Theorem 3.1. *Suppose that:*

1). *There exist supplementary projections P_- , P_0 , $P_+ \in \mathbb{M}_{d \times d}$ and a positive constant K such that the fundamental matrix $X(t)$ for (2.3) satisfies the condition*

$$\begin{aligned} & \int_{-\infty}^t | \Psi(t)X(t)P_-X^{-1}(s)\Psi^{-1}(s) | ds + \\ & + | \int_0^t | \Psi(t)X(t)P_0X^{-1}(s)\Psi^{-1}(s) | ds | + \\ & + \int_t^\infty | \Psi(t)X(t)P_+X^{-1}(s)\Psi^{-1}(s) | ds \leq K, \end{aligned} \quad (3.1)$$

for all $t \geq 0$;

2). *The continuous function $F : R \times \mathbb{M}_{d \times d} \rightarrow \mathbb{M}_{d \times d}$ satisfies $F(t, O) = O$ and the Lypschitz condition*

$$| \Psi(t) (F(t, Z_1) - F(t, Z_2)) | \leq \gamma | \Psi(t) (Z_1 - Z_2) |,$$

for $t \in R$, $Z_1, Z_2 \in \mathbb{M}_{d \times d}$ with $|\Psi(t)Z_1| \leq \rho$, $|\Psi(t)Z_2| \leq \rho$ for $t \in R$, ($\rho > 0$ is given), where γ is a positive constant such that $\gamma K < 1$;

3). The continuous function $C : R \rightarrow \mathbb{M}_{d \times d}$ is Ψ -bounded on R such that

$$|C|_{\Psi} = \sup_{t \in R} |\Psi(t)C(t)| \leq \frac{\rho(1 - \gamma K)}{K}.$$

Then, the equation (1.2) has a unique Ψ -bounded solution $Z(t)$ on R for which $|\Psi(t)Z(t)| \leq \rho$, for all $t \in R$.

Proof. We prove this theorem by means of Banach fixed point theorem. Consider the space

$$C_{\Psi} = \{Z : R \rightarrow \mathbb{M}_{d \times d} \mid Z \text{ is continuous and } \Psi\text{-bounded on } R\}.$$

C_{Ψ} is a Banach space with respect to the norm $|Z|_{\Psi} = \sup_{t \in R} |\Psi(t)Z(t)|$.

Let the ball $S_{\rho} = \{Z \in C_{\Psi} \mid |Z|_{\Psi} \leq \rho\}$.

For $Z \in C_{\Psi}$, define the operator T by

$$\begin{aligned} (TZ)(t) &= \int_{-\infty}^t X(t)P_{-}X^{-1}(s)(C(s) + F(s, Z(s)))ds + \\ &+ \int_0^t X(t)P_0X^{-1}(s)(C(s) + F(s, Z(s)))ds - \\ &- \int_t^{\infty} X(t)P_{+}X^{-1}(s)(C(s) + F(s, Z(s)))ds. \end{aligned}$$

From hypotheses, TZ exists and is continuous differentiable on R .

For $Z \in S_{\rho}$ and $t \in R$, we have

$$\begin{aligned} |\Psi(t)(TZ)(t)| &= \left| \int_{-\infty}^t \Psi(t)X(t)P_{-}X^{-1}(s)\Psi^{-1}(s)\Psi(s)(C(s) + F(s, Z(s)))ds + \right. \\ &+ \int_0^t \Psi(t)X(t)P_0X^{-1}(s)\Psi^{-1}(s)\Psi(s)(C(s) + F(s, Z(s)))ds - \\ &- \left. \int_t^{\infty} \Psi(t)X(t)P_{+}X^{-1}(s)\Psi^{-1}(s)\Psi(s)(C(s) + F(s, Z(s)))ds \right| \leq \\ &\leq \int_{-\infty}^t |\Psi(t)X(t)P_{-}X^{-1}(s)\Psi^{-1}(s)| |\Psi(s)(C(s) + F(s, Z(s)))| ds + \\ &+ \left| \int_0^t \Psi(t)X(t)P_0X^{-1}(s)\Psi^{-1}(s) \right| |\Psi(s)(C(s) + F(s, Z(s)))| ds + \\ &+ \int_t^{\infty} |\Psi(t)X(t)P_{+}X^{-1}(s)\Psi^{-1}(s)| |\Psi(s)(C(s) + F(s, Z(s)))| ds \leq \\ &\leq K \cdot \sup_{t \in R} |\Psi(s)(C(s) + F(s, Z(s)))| \leq \\ &\leq K \cdot \left(\frac{\rho(1 - \gamma K)}{K} + \gamma |Z|_{\Psi} \right) = \rho(1 - \gamma K) + \gamma K \rho = \rho. \end{aligned}$$

It follows that $TZ \in S_\rho$ and hence,

$$TS_\rho \subset S_\rho.$$

On the other hand, for $Z_1, Z_2 \in S_\rho$ and $t \in R$, we have

$$\begin{aligned} & | \Psi(t) ((TZ_1)(t) - (TZ_2)(t)) | = \\ & = | [\int_{-\infty}^t \Psi(t)X(t)P_-X^{-1}(s)\Psi^{-1}(s)\Psi(s)(C(s) + F(s, Z_1(s))) ds + \\ & + \int_0^t \Psi(t)X(t)P_0X^{-1}(s)\Psi^{-1}(s)\Psi(s)(C(s) + F(s, Z_1(s))) ds - \\ & - \int_t^\infty \Psi(t)X(t)P_+X^{-1}(s)\Psi^{-1}(s)\Psi(s)(C(s) + F(s, Z_1(s))) ds] - \\ & - [\int_{-\infty}^t \Psi(t)X(t)P_-X^{-1}(s)\Psi^{-1}(s)\Psi(s)(C(s) + F(s, Z_2(s))) ds + \\ & + \int_0^t \Psi(t)X(t)P_0X^{-1}(s)\Psi^{-1}(s)\Psi(s)(C(s) + F(s, Z_2(s))) ds - \\ & - \int_t^\infty \Psi(t)X(t)P_+X^{-1}(s)\Psi^{-1}(s)\Psi(s)(C(s) + F(s, Z_2(s))) ds] | = \\ & = | \int_{-\infty}^t \Psi(t)X(t)P_-X^{-1}(s)\Psi^{-1}(s)\Psi(s)(F(s, Z_1(s)) - F(s, Z_2(s))) ds + \\ & + \int_0^t \Psi(t)X(t)P_0X^{-1}(s)\Psi^{-1}(s)\Psi(s)(F(s, Z_1(s)) - F(s, Z_2(s))) ds - \\ & - \int_t^\infty \Psi(t)X(t)P_+X^{-1}(s)\Psi^{-1}(s)\Psi(s)(F(s, Z_1(s)) - F(s, Z_2(s))) ds | \leq \\ & \leq \int_{-\infty}^t | \Psi(t)X(t)P_-X^{-1}(s)\Psi^{-1}(s) || \Psi(s)(F(s, Z_1(s)) - F(s, Z_2(s))) | ds + \\ & + | \int_0^t | \Psi(t)X(t)P_0X^{-1}(s)\Psi^{-1}(s) || \Psi(s)(F(s, Z_1(s)) - F(s, Z_2(s))) | ds | + \\ & + \int_t^\infty | \Psi(t)X(t)P_+X^{-1}(s)\Psi^{-1}(s) || \Psi(s)(F(s, Z_1(s)) - F(s, Z_2(s))) | ds \leq \\ & \leq \gamma \int_{-\infty}^t | \Psi(t)X(t)P_-X^{-1}(s)\Psi^{-1}(s) || \Psi(s)(Z_1(s) - Z_2(s)) | ds + \\ & + \gamma | \int_0^t | \Psi(t)X(t)P_0X^{-1}(s)\Psi^{-1}(s) || \Psi(s)(Z_1(s) - Z_2(s)) | ds | + \\ & + \gamma \int_t^\infty | \Psi(t)X(t)P_+X^{-1}(s)\Psi^{-1}(s) || \Psi(s)(Z_1(s) - Z_2(s)) | ds \leq \\ & \leq \gamma K \sup_{t \in R} | \Psi(s)(Z_1(s) - Z_2(s)) | = \gamma K | Z_1 - Z_2 |_\Psi. \end{aligned}$$

It follows that

$$\|TZ_1 - TZ_2\|_{\Psi} \leq \gamma K \|Z_1 - Z_2\|_{\Psi}.$$

Therefore, T is a contraction operator on S_{ρ} . Hence, by Banach fixed point theorem, T has a unique fixed point $Z \in S_{\rho}$. From $Z = TZ$, it follows that Z is continuous differentiable on $\mathbb{R}$ and then, for $t \in \mathbb{R}$,

$$\begin{aligned} Z'(t) &= (TZ)'(t) = \\ &= \int_{-\infty}^t X'(t)P_-X^{-1}(s)(C(s) + F(s, Z(s)))ds + \\ &+ X(t)P_-X^{-1}(t)(C(t) + F(t, Z(t))) + \\ &+ \int_0^t X'(t)P_0X^{-1}(s)(C(s) + F(s, Z(s)))ds + \\ &+ X(t)P_0X^{-1}(t)(C(t) + F(t, Z(t))) - \\ &- \int_t^{\infty} X'(t)P_+X^{-1}(s)(C(s) + F(s, Z(s)))ds + \\ &+ X(t)P_+X^{-1}(t)(C(t) + F(t, Z(t))) = \\ &= A(t)(TZ)(t) + X(t)(P_- + P_0 + P_+)X^{-1}(t)(C(t) + F(t, Z(t))) = \\ &= A(t)Z(t) + C(t) + F(t, Z(t)). \end{aligned}$$

Thus, $Z(t)$ is a solution of equation (1.2).

In conclusion, the equation (1.2) has a unique Ψ -bounded solution $Z(t)$ on $\mathbb{R}$ for which $\|\Psi(t)Z(t)\| \leq \rho$, for all $t \in \mathbb{R}$. $\square$

Remark 3.1. Theorem generalizes the Theorem 4 ([5], Ch. 5, s. 2) and Theorem 2.1, [18] from systems of differential equations to matrix differential equations and extends them for case $P_0 \neq 0$.

The next simple example is an illustration of Theorem.

Example 3.1. Consider the equation (1.2) with

$$A(t) = \text{diag}[-2t, 1, -1], \quad C(t) = \text{diag}\left[\frac{\alpha}{1+t^2}, \alpha e^{-t} \sin t, \alpha e^t \cos t\right]$$

and

$$F(t, Z) = \text{diag}[\ln(1 + a \|z_{11}\|), \sin az_{22}, \arctg az_{33}],$$

where $Z = (z_{ij}) \in \mathbb{M}_{3 \times 3}$ and a, α are real constants such that

$$0 < a < \left[1 + \int_0^1 e^{s^2} ds\right]^{-1}.$$

Then, $X(t) = \text{diag} [e^{t^2}, e^t, e^{-t}]$, $t \in R$, is a fundamental matrix for (2.3).

Consider $\Psi(t) = \text{diag} [1, e^t, e^{-t}]$, $t \in R$.

There exist supplementary projections

$$P_- = \text{diag} [0, 0, 1], \quad P_0 = \text{diag} [1, 0, 0], \quad \text{and} \quad P_+ = \text{diag} [0, 1, 0]$$

such that:

- $\Psi(t)X(t)P_-X^{-1}(s)\Psi^{-1}(s) = \text{diag} [0, 0, e^{-2(t-s)}]$ and then

$$\int_{-\infty}^t |\Psi(t)X(t)P_-X^{-1}(s)\Psi^{-1}(s)| ds \leq \int_{-\infty}^t e^{-2(t-s)} ds = \frac{1}{2};$$

- $\Psi(t)X(t)P_+X^{-1}(s)\Psi^{-1}(s) = \text{diag} [0, e^{2(t-s)}, 0]$ and then

$$\int_t^{\infty} |\Psi(t)X(t)P_+X^{-1}(s)\Psi^{-1}(s)| ds \leq \int_t^{\infty} e^{2(t-s)} ds = \frac{1}{2};$$

- $\Psi(t)X(t)P_0X^{-1}(s)\Psi^{-1}(s) = \text{diag} [e^{-t^2+s^2}, 0, 0]$ and then

$$\begin{aligned} & \left| \int_0^t |\Psi(t)X(t)P_0X^{-1}(s)\Psi^{-1}(s)| ds \right| = \left| \int_0^t e^{-t^2+s^2} ds \right| = \\ & = \int_0^{|t|} e^{-t^2+s^2} ds \leq 1 + \int_0^1 e^{s^2} ds. \end{aligned}$$

and then, the condition (3.1) is satisfied with $K = 1 + \int_0^1 e^{s^2} ds$.

After that, for $t \in R$ and for $Z', Z'' \in \mathbb{M}_{3 \times 3}$, we have

$$\begin{aligned} & |\Psi(t)(F(t, Z') - F(t, Z''))| \leq \\ & \leq \max\{a |z'_{11} - z''_{11}|, ae^t |z'_{22} - z''_{22}|, ae^{-t} |z'_{33} - z''_{33}|\} = \\ & = a \cdot \max\{|z'_{11} - z''_{11}|, e^t |z'_{22} - z''_{22}|, e^{-t} |z'_{33} - z''_{33}|\} = \\ & = a \cdot |\Psi(t)(Z' - Z'')|. \end{aligned}$$

and then, the condition 2) of Theorem is satisfied.

At least, for the matrix $C(t)$ we have that

$$|C| = \sup_{t \in R} |\Psi(t)C(t)| = |\alpha|.$$

From Theorem, it follows that for $\rho \geq \frac{|\alpha|K}{1-aK}$, the equation (1.2) has a unique Ψ -bounded solution $Z(t)$ for which $|\Psi(t)Z(t)| \leq \rho$, for all $t \in R$.

Theorem 3.2. *Suppose that:*

1). *There exist supplementary projections P_- , $P_+ \in \mathbb{M}_{d \times d}$ and a positive constants K_1 , K_2 , α and β such that the fundamental matrix $X(t)$ for (2.3) satisfies the conditions*

$$| \Psi(t)X(t)P_-X^{-1}(s)\Psi^{-1}(s) | \leq K_1 e^{-\alpha(t-s)}, \text{ for } s \leq t$$

$$| \Psi(t)X(t)P_+X^{-1}(s)\Psi^{-1}(s) | \leq K_2 e^{-\beta(s-t)}, \text{ for } t \leq s$$

2). *The continuous function $F : R \times \mathbb{M}_{d \times d} \rightarrow \mathbb{M}_{d \times d}$ satisfies the condition*

$$| \Psi(t)F(t, Z) | \leq \gamma | \Psi(t)Z |,$$

for $t \in R$, $Z \in \mathbb{M}_{d \times d}$ with $| \Psi(t)Z | \leq \rho$ for $t \in R$ ($\rho > 0$ is given), where γ is a positive constant such that $\gamma \left(\frac{K_1}{\alpha} + \frac{K_2}{\beta} \right) < 1$;

3). *The continuous function $C : R \rightarrow \mathbb{M}_{d \times d}$ is Ψ -bounded on R such that*

$$| C |_{\Psi} = \sup_{t \in R} | \Psi(t)C(t) | \leq \frac{\rho \left[1 - \gamma \left(\frac{K_1}{\alpha} + \frac{K_2}{\beta} \right) \right]}{\frac{K_1}{\alpha} + \frac{K_2}{\beta}}.$$

Then, the equation (1.2) has at least one Ψ -bounded solution $Z(t)$ on R for which $| \Psi(t)Z(t) | \leq \rho$, for all $t \in R$.

Proof. We prove this treorem by means of Schauder-Tychonoff fixed point theorem.

For this, let C_{Ψ} denote the set of all matrix functions $Z(t)$ which are continuous and Ψ -bounded on R , and S_{ρ} be the subset formed by those functions $Z(t)$ such that $| Z |_{\Psi} = \sup_{t \in R} | \Psi(t)Z(t) | \leq \rho$.

For $Z \in C_{\Psi}$, define the operator T by

$$(TZ)(t) = \int_{-\infty}^t X(t)P_-X^{-1}(s)(C(s) + F(s, Z(s)))ds - \\ - \int_t^{\infty} X(t)P_+X^{-1}(s)(C(s) + F(s, Z(s)))ds,$$

This operator have the following two properties:

i). T is continuous, in the sense that if $Z_n \in S_{\rho}$ ($n = 1, 2, \dots$) and $Z_n \rightarrow Z$ uniformly on every compact subinterval J of R , then $TZ_n \rightarrow TZ$ uniformly on every compact subinterval J of R .

Indeed, let $Z_n \in S_{\rho}$ ($n = 1, 2, \dots$) and $Z_n \rightarrow Z$ uniformly on every compact subinterval $J = [p, q]$ of R . For an arbitrary small $\varepsilon > 0$, choose $\tau > 0$ so large that

$$\tau > \max \left\{ -\frac{1}{\alpha} \ln \frac{\alpha \varepsilon}{8\rho\gamma K_1}, -\frac{1}{\beta} \ln \frac{\beta \varepsilon}{8\rho\gamma K_2} \right\}.$$

Since $F(t, Z)$ is uniformly continuous for $t \in [p - \tau, q + \tau]$ and $|\Psi(t)Z| \leq \rho$, it follows that the sequence $U_n(t) = \Psi(t)(F(t, Z_n(t)) - F(t, Z(t)))$ tends to zero uniformly on $[p - \tau, q + \tau]$. Thus, there exists $n_0 \in \mathbb{N}$ such that $|U_n(t)| < \frac{\varepsilon}{4\tau \max\{K_1, K_2\}}$, for $n \geq n_0$ and $t \in [p - \tau, q + \tau]$.

For $t \in J$ and $n \geq n_0$, consider

$$\begin{aligned}
 & |\Psi(t)((TZ_n)(t) - (TZ)(t))| = \\
 & = |[\int_{-\infty}^t \Psi(t)X(t)P_-X^{-1}(s)\Psi^{-1}(s)\Psi(s)(C(s) + F(s, Z_n(s)))ds - \\
 & - \int_t^\infty \Psi(t)X(t)P_+X^{-1}(s)\Psi^{-1}(s)\Psi(s)(C(s) + F(s, Z_n(s)))ds] - \\
 & - [\int_{-\infty}^t \Psi(t)X(t)P_-X^{-1}(s)\Psi^{-1}(s)\Psi(s)(C(s) + F(s, Z(s)))ds - \\
 & - \int_t^\infty \Psi(t)X(t)P_+X^{-1}(s)\Psi^{-1}(s)\Psi(s)(C(s) + F(s, Z(s)))ds]| = \\
 & = |[\int_{-\infty}^t \Psi(t)X(t)P_-X^{-1}(s)\Psi^{-1}(s)\Psi(s)(F(s, Z_n(s)) - F(s, Z(s)))ds - \\
 & - \int_t^\infty \Psi(t)X(t)P_+X^{-1}(s)\Psi^{-1}(s)\Psi(s)(F(s, Z_n(s)) - F(s, Z(s)))ds]| \leq \\
 & \leq \int_{-\infty}^t |\Psi(t)X(t)P_-X^{-1}(s)\Psi^{-1}(s)| |\Psi(s)(F(s, Z_n(s)) - F(s, Z(s)))| ds + \\
 & + \int_t^\infty |\Psi(t)X(t)P_+X^{-1}(s)\Psi^{-1}(s)| |\Psi(s)(F(s, Z_n(s)) - F(s, Z(s)))| ds \leq \\
 & \leq K_1 \int_{-\infty}^t e^{-\alpha(t-s)} |U_n(s)| ds + K_2 \int_t^\infty e^{-\beta(s-t)} |U_n(s)| ds = \\
 & = K_1 \int_{-\infty}^{t-\tau} e^{-\alpha(t-s)} |U_n(s)| ds + K_1 \int_{t-\tau}^t e^{-\alpha(t-s)} |U_n(s)| ds + \\
 & + K_2 \int_{t+\tau}^\infty e^{-\beta(s-t)} |U_n(s)| ds + K_2 \int_t^{t+\tau} e^{-\beta(s-t)} |U_n(s)| ds \leq \\
 & \leq 2\rho\gamma(K_1 \int_{-\infty}^{t-\tau} e^{-\alpha(t-s)} ds + K_2 \int_{t+\tau}^\infty e^{-\beta(s-t)} ds) + \max\{K_1, K_2\} \int_{t-\tau}^{t+\tau} |U_n(s)| ds < \\
 & < 2\rho\gamma\left(K_1 \cdot \frac{e^{-\alpha\tau}}{\alpha} + K_2 \cdot \frac{e^{-\beta\tau}}{\beta}\right) + \max\{K_1, K_2\} \cdot \frac{\varepsilon}{4\tau \max\{K_1, K_2\}} \cdot 2\tau < \varepsilon.
 \end{aligned}$$

This shows that $TZ_n \rightarrow TZ$ uniformly on every compact subinterval of $\mathbb{R}$.

Thus, T is continuous.

ii). the functions in the image set TS_ρ are equicontinuous and bounded at every point of J .

Indeed, from $Z \in S_\rho$, we have

$$\begin{aligned}
& | \Psi(t) (TZ) (t) | = \\
& = | \int_{-\infty}^t \Psi(t) X(t) P_- X^{-1}(s) \Psi^{-1}(s) \Psi(s) (C(s) + F(s, Z(s))) ds - \\
& - \int_t^{\infty} \Psi(t) X(t) P_+ X^{-1}(s) \Psi^{-1}(s) \Psi(s) (C(s) + F(s, Z(s))) ds | \leq \\
& \leq \int_{-\infty}^t | \Psi(t) X(t) P_- X^{-1}(s) \Psi^{-1}(s) | | \Psi(s) (C(s) + F(s, Z(s))) | ds + \\
& + \int_t^{\infty} | \Psi(t) X(t) P_+ X^{-1}(s) \Psi^{-1}(s) | | \Psi(s) (C(s) + F(s, Z(s))) | ds \leq \\
& \leq K_1 \int_{-\infty}^t e^{-\alpha(t-s)} | \Psi(s) (C(s) + F(s, Z(s))) | ds + \\
& + K_2 \int_t^{\infty} e^{-\beta(s-t)} | \Psi(s) (C(s) + F(s, Z(s))) | ds \leq \\
& \leq K_1 \int_{-\infty}^t e^{-\alpha(t-s)} (| \Psi(s) C(s) | + \gamma | \Psi(s) Z(s) |) ds + \\
& + K_2 \int_t^{\infty} e^{-\beta(s-t)} (| \Psi(s) C(s) | + \gamma | \Psi(s) Z(s) |) ds \leq \\
& \leq \left(\frac{\rho [1 - \gamma (\frac{K_1}{\alpha} + \frac{K_2}{\beta})]}{\frac{K_1}{\alpha} + \frac{K_2}{\beta}} + \gamma \rho \right) \left(\frac{K_1}{\alpha} + \frac{K_2}{\beta} \right) = \rho.
\end{aligned}$$

Hence, the functions in the image set TS_ρ are uniformly bounded at every point of J .

On the other hand, we have

$$\begin{aligned}
& (TZ)'(t) = \\
& = \int_{-\infty}^t X'(t) P_- X^{-1}(s) (C(s) + F(s, Z(s))) ds + \\
& + X(t) P_- X^{-1}(t) (C(t) + F(t, Z(t))) + \\
& - \int_t^{\infty} X'(t) P_+ X^{-1}(s) (C(s) + F(s, Z(s))) ds + \\
& + X(t) P_+ X^{-1}(t) (C(t) + F(t, Z(t))) = \\
& = A(t) (TZ) (t) + X(t) (P_- + P_+) X^{-1}(t) (C(t) + F(t, Z(t))) = \\
& = A(t) (TZ) (t) + C(t) + F(t, Z(t)),
\end{aligned}$$

which shows that $(TZ) (t)$ is a solution of equation $W' = A(t)W + C(t) + F(t, Z(t))$.

It follows that the derivatives $(TZ)'(t)$ are uniformly bounded on any compact subinterval J of $\mathbb{R}$. Thus, the functions in TS_ρ are echicontinuous on any compact subinterval J of $\mathbb{R}$.

From i) and ii), all the conditions of the Schauder-Tychonoff theorem are satisfied. Hence, the operator T has at least one fixed point $Z(t)$ in S_ρ . But the fixed point of T is just the solution of the integral equation

$$Z = TZ$$

in S_ρ , i.e. , of the matrix differential equation (1.2), with the required properties. $\square$

Remark 3.2. In a particular case, our result reduces to Theorem 2.2 obtained in [18].

Indeed, if

$$F(t, Z) = \begin{pmatrix} f_1(t, z) & f_1(t, z) & \cdots & f_1(t, z) \\ f_2(t, z) & f_2(t, z) & \cdots & f_2(t, z) \\ \vdots & \vdots & \vdots & \vdots \\ f_d(t, z) & f_d(t, z) & \cdots & f_d(t, z) \end{pmatrix}, C(t) = \begin{pmatrix} c_1(t) & c_1(t) & \cdots & c_1(t) \\ c_2(t) & c_2(t) & \cdots & c_2(t) \\ \vdots & \vdots & \vdots & \vdots \\ c_d(t) & c_d(t) & \cdots & c_d(t) \end{pmatrix}$$

it is easy to see that the solutions of the equation (1.2) is

$$Z(t) = \begin{pmatrix} z_1(t) & z_1(t) & \cdots & z_1(t) \\ z_2(t) & z_2(t) & \cdots & z_2(t) \\ \vdots & \vdots & \vdots & \vdots \\ z_d(t) & z_d(t) & \cdots & z_d(t) \end{pmatrix},$$

where $z(t) = (z_1(t), z_2(t), \dots, z_d(t))^T$ is the solution of the equation (1.3) with

$$c(t) = (c_1(t), c_2(t), \dots, c_d(t))^T \text{ and } f(t, z) = (f_1(t, z), f_1(t, z), \dots, f_1(t, z))^T.$$

In this case, the solution $z(t)$ is Ψ -bounded on $\mathbb{R}$ iff the corresponding solution $Z(t)$ is Ψ -bounded on $\mathbb{R}$.

Thus, the Theorem generalizes the result from [18], from systems of differential equations to matrix differential equations.

4 Ψ – bounded solutions for the Lyapunov matrix differential equation (1)

The purpose of this section is to provide sufficient conditions for the existence and uniqueness and existence of at least one Ψ – bounded solution on $\mathbb{R}$ for the Lyapunov matrix differential equation (1.1).

Theorem 4.1. *Suppose that:*

1). *There exist supplementary projections $P_-, P_0, P_+ \in \mathbb{M}_{d \times d}$ and a positive constant K such that the fundamental matrices $X(t)$ and $Y(t)$ for (2.3) and (2.4) respectively satisfy the condition*

$$\begin{aligned} & \int_{-\infty}^t | \left(Y^T(t) (Y^T)^{-1}(s) \right) \otimes (\Psi(t)X(t)P_-X^{-1}(s)\Psi^{-1}(s)) | ds + \\ & + | \int_0^t | \left(Y^T(t) (Y^T)^{-1}(s) \right) \otimes (\Psi(t)X(t)P_0X^{-1}(s)\Psi^{-1}(s)) | ds | + \\ & + \int_t^\infty | \left(Y^T(t) (Y^T)^{-1}(s) \right) \otimes (\Psi(t)X(t)P_+X^{-1}(s)\Psi^{-1}(s)) | ds \leq K, \end{aligned}$$

for all $t \geq 0$;

2). *The continuous function $F : R \times \mathbb{M}_{d \times d} \rightarrow \mathbb{M}_{d \times d}$ satisfies the Lipschitz condition*

$$| \Psi(t) (F(t, Z_1) - F(t, Z_2)) | \leq \frac{\gamma}{d} | \Psi(t) (Z_1 - Z_2) |,$$

for $t \in R$, $Z_1, Z_2 \in \mathbb{M}_{d \times d}$ with $| \Psi(t)Z_1 | \leq \rho$, $| \Psi(t)Z_2 | \leq \rho$ for $t \in R$, ($\rho > 0$ is given), where γ is a positive constant such that $\gamma K < 1$;

3). *The continuous matrix function $C : R \rightarrow \mathbb{M}_{d \times d}$ is Ψ – bounded on R such that*

$$| C |_\Psi = \sup_{t \in R} | \Psi(t)C(t) | \leq \frac{\rho(1 - \gamma K)}{K}.$$

Then, the Lyapunov matrix differential equation (1.1) has a unique Ψ – bounded solution $Z(t)$ on R for which $| \Psi(t)Z(t) | \leq \rho d$, for all $t \in R$.

Proof. From Lemma 2.4, one know that $Z(t)$ is a solution of (1.1) iff the vector function $z(t) = \text{Vec}(Z(t))$ is a solution of the corresponding Kronecker product system associated with (1.1), i.e. of the differential system (2.1).

From Lemma (2.7), one know that $U(t) = Y^T(t) \otimes X(t)$ is a fundamental matrix for the differential system (2.5).

Now, the hypotheses of the Theorem ensure the hypotheses of Theorem 3.1 (variant for systems) for the system (2.1). Indeed:

i). Since

$$\begin{aligned} & \left(Y^T(t) (Y^T)^{-1}(s) \right) \otimes (\Psi(t)X(t)P_-X^{-1}(s)\Psi^{-1}(s)) = \\ & = (I \otimes \Psi(t)) \cdot (Y^T(t) \otimes X(t)) \cdot (I \otimes P_-) \cdot \left((Y^T)^{-1}(s) \otimes X^{-1}(s) \right) \cdot (I \otimes \Psi^{-1}(s)) \end{aligned}$$

(see Lemma 2.1) and similarly for P_0 and P_+ , the hypothesis 1) of Theorem 3.1 is satisfied;

ii). Since

$$\begin{aligned} & \| (I \otimes \Psi(t)) \cdot (f(t, z_1) - f(t, z_2)) \|_{R^{d^2}} = \\ & = \| (I \otimes \Psi(t)) \cdot \mathcal{V}ec(F(t, Z_1) - F(t, Z_2)) \|_{R^{d^2}} \leq \\ & \leq | \Psi(t) (F(t, Z_1) - F(t, Z_2)) | \leq \frac{\gamma}{d} | \Psi(t) (Z_1 - Z_2) | \leq \\ & \leq \gamma \| (I \otimes \Psi(t)) \cdot \mathcal{V}ec(Z_1 - Z_2) \|_{R^{d^2}} = \\ & = \gamma \| (I \otimes \Psi(t)) \cdot (z_1 - z_2) \|_{R^{d^2}}, \end{aligned}$$

for all z_1, z_2 with $\| (I \otimes \Psi(t)) \cdot z_i \|_{R^{d^2}} = \| (I \otimes \Psi(t)) \cdot \mathcal{V}ec(Z_i) \|_{R^{d^2}} \leq | \Psi(t)Z_i | \leq \rho$,

(see Lemmas 2.1 and 2.5) and $\gamma K < 1$, the hypothesis 2) of Theorem 3.1 is satisfied;

iii). Since

$$\begin{aligned} \| c \|_{R^{d^2}} &= | C | = \sup_{t \in R} | (I \otimes \Psi(t)) \mathcal{V}ec(C(t)) | \leq \\ &\leq \sup_{t \in R} | \Psi(t)C(t) | \leq \frac{\rho(1-\gamma K)}{K}, \end{aligned}$$

the hypothesis 3) of Theorem 3.1 is satisfied.

At this stage we appeal to Theorem 3.1 to deduce that the system (2.1) has a unique $I \otimes \Psi(t)$ - bounded solution $z(t)$ on $\mathbb{R}$ for which $\| (I \otimes \Psi(t)) \cdot z(t) \|_{R^{d^2}} \leq \rho$.

From Lemma 2.4 again, the matrix function $Z(t) = \mathcal{V}ec^{-1}(z(t))$ is unique solution of (1.1) on $\mathbb{R}$ such that (see Lemma 2.5) $| \Psi(t)Z(t) | \leq \rho d$, for all $t \in \mathbb{R}$. $\square$

Remark 4.1. The Theorem extends the Theorem 2.1, [18] and Theorem 3.1 above to Lyapunov matrix differential equation (1.1).

Theorem 4.2. *Suppose that:*

1). *There exist supplementary projections $P_-, P_+ \in \mathbb{M}_{d \times d}$ and a positive*

constants K_1, K_2, α and β such that the fundamental matrices $X(t)$ and $Y(t)$ for (2.3) and (2.4) respectively satisfy the condition

$$| \left(Y^T(t) (Y^T)^{-1}(s) \right) \otimes (\Psi(t)X(t)P_-X^{-1}(s)\Psi^{-1}(s)) | \leq K_1 e^{-\alpha(t-s)}, \text{ for } s \leq t$$

$$| \left(Y^T(t) (Y^T)^{-1}(s) \right) \otimes (\Psi(t)X(t)P_+X^{-1}(s)\Psi^{-1}(s)) | \leq K_2 e^{-\beta(s-t)}, \text{ for } t \leq s$$

2). The continuous function $F : R \times \mathbb{M}_{d \times d} \rightarrow \mathbb{M}_{d \times d}$ satisfies the condition

$$| \Psi(t)F(t, Z) | \leq \frac{\gamma}{d} | \Psi(t)Z |,$$

for $t \in R, Z \in \mathbb{M}_{d \times d}$ with $| \Psi(t)Z | \leq \rho$ for $t \in R$ ($\rho > 0$ is given), where γ is a positive constant such that $\gamma \left(\frac{K_1}{\alpha} + \frac{K_2}{\beta} \right) < 1$;

3). The continuous matrix function $C : R \rightarrow \mathbb{M}_{d \times d}$ is Ψ -bounded on R such that

$$| C |_{\Psi} = \sup_{t \in R} | \Psi(t)C(t) | \leq \frac{\rho \left[1 - \gamma \left(\frac{K_1}{\alpha} + \frac{K_2}{\beta} \right) \right]}{\frac{K_1}{\alpha} + \frac{K_2}{\beta}}.$$

Then, the Lyapunov matrix differential equation (1.1) has at least one Ψ -bounded solution $Z(t)$ on R for which $| \Psi(t)Z(t) | \leq \rho d$.

Proof. From Lemma 2.4, one know that $Z(t)$ is a solution of (1.1) iff the vector function $z(t) = \mathcal{V}ec(Z(t))$ is a solution of the corresponding Kronecker product system associated with (1.1), i.e. of the differential system (2.1).

From Lemma (2.7), one know that $U(t) = Y^T(t) \otimes X(t)$ is a fundamental matrix for the differential system (2.5).

Now, the hypotheses of the Theorem ensure the hypotheses of Theorem 3.2 (variant for systems) for the system (2.1). Indeed:

i). Since

$$\begin{aligned} & \left(Y^T(t) (Y^T)^{-1}(s) \right) \otimes (\Psi(t)X(t)P_-X^{-1}(s)\Psi^{-1}(s)) = \\ & = (I \otimes \Psi(t)) \cdot (Y^T(t) \otimes X(t)) \cdot (I \otimes P_-) \cdot \left((Y^T)^{-1}(s) \otimes X^{-1}(s) \right) \cdot (I \otimes \Psi^{-1}(s)) \end{aligned}$$

(see Lemma 2.1) and similarly for P_+ , the hypothesis 1) of Theorem 3.2 is satisfied;

ii). Since

$$\begin{aligned} & \| (I \otimes \Psi(t)) \cdot f(t, z) \|_{R^{d^2}} = \| (I \otimes \Psi(t)) \cdot \mathcal{V}ec F(t, Z) \|_{R^{d^2}} \leq \\ & \leq | \Psi(t)F(t, Z) | \leq \frac{\gamma}{d} | \Psi(t)Z | \leq \gamma \| (I \otimes \Psi(t)) \cdot \mathcal{V}ec(Z) \|_{R^{d^2}} = \\ & = \gamma \| (I \otimes \Psi(t)) \cdot z \|_{R^{d^2}}, \end{aligned}$$

for $t \in R$ and $z \in R^{d^2}$, (see Lemmas 2.1 and 2.5) and $\gamma\left(\frac{K_1}{\alpha} + \frac{K_2}{\beta}\right) < 1$, the hypothesis 2) of Theorem 3.2 is satisfied;

iii). Since

$$\begin{aligned} \|c\|_{R^{d^2}} &= \|\mathcal{V}ec(C(t))\|_{R^{d^2}} = \sup_{t \in R} |(I \otimes \Psi(t)) \mathcal{V}ec(C(t))| \leq \\ &\leq \sup_{t \in R} |\Psi(t)C(t)| \leq \frac{\rho(1 - \gamma(\frac{K_1}{\alpha} + \frac{K_2}{\beta}))}{\frac{K_1}{\alpha} + \frac{K_2}{\beta}}, \end{aligned}$$

the hypothesis 3) of Theorem 3.2 is satisfied.

At this stage we appeal to Theorem 3.2 to deduce that the system (2.1) has at least one $I \otimes \Psi(t)$ -bounded solution $z(t)$ on R for which $\|(I \otimes \Psi(t)) \cdot z(t)\|_{R^{d^2}} \leq \rho$.

From Lemma 2.4 again, the matrix function $Z(t) = \mathcal{V}ec^{-1}(z(t))$ is a solution of (1.1) on R such that (see Lemma 2.5), $|\Psi(t)Z(t)| \leq \rho d$, for all $t \in R$. $\square$

Remark 4.2. The Theorem extends the Theorem 2.2 [18] and Theorem 3.2 above to Lyapunov matrix differential equation (1.1).

The next simple example is an illustration of Theorem.

Example 4.1. Consider the nonlinear *Lyapunov matrix differential equation* (1.1) with

$$\begin{aligned} A(t) &= \begin{pmatrix} 2 & 0 \\ 0 & -2 \end{pmatrix}, \quad B(t) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \\ C(t) &= \begin{pmatrix} 0 & ae^{4t} \cos |t| \\ ae^{-3t} \sin t^2 & ae^{-3|t|} \end{pmatrix} \end{aligned}$$

and

$$F(t, Z) = m \begin{pmatrix} \sin z_1 \sin t & z_2 \cos t \\ z_3 \sin z_4 & \frac{2}{\pi} z_4 \arctg t \end{pmatrix},$$

where $t \in R$, $Z = \begin{pmatrix} z_1 & z_2 \\ z_3 & z_4 \end{pmatrix} \in \mathbb{M}_{2 \times 2}$ and a, m are real constants, $0 < |m| < \frac{1}{4}$.

Then,

$$X(t) = \begin{pmatrix} e^{2t} & 0 \\ 0 & e^{-2t} \end{pmatrix} \text{ and } Y(t) = \begin{pmatrix} e^t & 0 \\ 0 & 1 \end{pmatrix}$$

are fundamental matrices for (2.3) and (2.4) respectively.

Consider

$$\Psi(t) = \begin{pmatrix} e^{-4t} & 0 \\ 0 & e^{3t} \end{pmatrix}, \quad t \in R.$$

There exist supplementary projections

$$P_- = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \text{ and } P_+ = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

such that

$$| (Y^T(t) (Y^T)^{-1}(s)) \otimes (\Psi(t)X(t)P_-X^{-1}(s)\Psi^{-1}(s)) | = e^{-\alpha(t-s)}, \text{ for } s \leq t$$

and

$$| (Y^T(t) (Y^T)^{-1}(s)) \otimes (\Psi(t)X(t)P_+X^{-1}(s)\Psi^{-1}(s)) | = e^{-(s-t)}, \text{ for } t \leq s.$$

Thus, the condition 1) of Theorem is satisfied with $K_1 = K_2 = 1$ and $\alpha = \beta = 1$.

After that, for $t \in R$, $Z = \begin{pmatrix} z_1 & z_2 \\ z_3 & z_4 \end{pmatrix} \in \mathbb{M}_{2 \times 2}$, we have

$$| \Psi(t)F(t, Z) | = \left| m \begin{pmatrix} e^{-4t} \sin z_1 \sin t & e^{-4t} z_2 \cos t \\ e^{3t} z_3 \sin z_4 & \frac{2}{\pi} e^{3t} z_4 \arctg t \end{pmatrix} \right| \leq | m | | \Psi(t)Z |.$$

Thus, the condition 2) of Theorem is satisfied with $\gamma = 2 | m |$ and $d = 2$.
At least, for the matrix $C(t)$ we have that

$$| C | = \sup_{t \in R} | \Psi(t)C(t) | = \sup_{t \in R} \left| \begin{pmatrix} 0 & a \cos |t| \\ a \sin t^2 & ae^{-3|t|+3t} \end{pmatrix} \right| = 2 | a |.$$

Now, from Theorem 4.2, it follows that for all $\rho \geq \frac{8|a|}{1-4|m|}$, the equation (1.1) has at least one solution $Z(t)$ for which $| \Psi(t)Z(t) | \leq \rho$, for all $t \in R$.

References

- [1] **R. Bellman**, Introduction to Matrix Analysis, McGraw-Hill Book Company, Inc. New York, 1960 (translated in Romanian).
- [2] **P. N. Boi**, Existence of Ψ - bounded solutions on R for nonhomogeneous linear differential equations, *Electron. J. Diff. Eqns.* **2007** (2007), 52:1-10.
- [3] **P. N. Boi**, On the Ψ -dichotomy for homogeneous linear differential equations, *Electronic Journal of Differential Equations* **2006** (2006), 40: 1-12.

- [4] **A. Constantin**, Asymptotic properties of solutions of differential equations, *Analele Universității din Timișoara, Seria Științe Matematice* **30** (2-3) (1992), 183-225.
- [5] **W. A. Coppel**, Stability and Asymptotic Behavior of Differential Equations, D. C. Heath and Company, Boston, 1965.
- [6] **A. Diamandescu**, On the Ψ - boundedness of the solutions of a nonlinear Lyapunov matrix differential equation, *Applied Sciences* **19** (2017), 31-40.
- [7] **A. Diamandescu**, On the Ψ - boundedness of the solutions of linear nonhomogeneous Lyapunov matrix differential equations, *Differential Geometry - Dynamical Systems* **19** (2017), 35-44.
- [8] **A. Diamandescu**, A note on the existence of Ψ - bounded solutions for a system of differential equations on $\mathbb{R}$, *Electronic Journal of Differential Equations* **2008** (2008), 128: 1-11.
- [9] **A. Diamandescu**, Ψ - bounded solutions for linear differential systems with Lebesgue Ψ - integrable functions on $\mathbb{R}$ as right-hand sides, *Electronic Journal of Differential Equations* **2009** (2009), 5: 1-12.
- [10] **A. Diamandescu**, Ψ - bounded solutions for a Lyapunov matrix differential equation, *Electronic Journal of Qualitative Theory Differential Equations* **17** (2009), 1-11.
- [11] **A. Diamandescu**, Existence of Ψ - bounded solutions for nonhomogeneous Lyapunov matrix differential equations on $\mathbb{R}$, *Electronic Journal of Qualitative Theory Differential Equations* **42** (2010), 1-9.
- [12] **A. Diamandescu**, On the Ψ - boundedness of the solutions of a linear system of ordinary differential equations, *Analele Științifice ale Universității "Al. I. Cuza" Iași, XLVIII*, s. I, Matematică, (2) (2002), 269-286.
- [13] **T. G. Hallam**, On asymptotic equivalence of the bounded solutions of two systems of differential equations, *Mich. Math. Journal* **16** (1969), 353-363.
- [14] **T. Hara, T. Yoneyama, T. Itoh**, Asymptotic Stability Criteria for Nonlinear Volterra Integro-Differential Equations, *Funkcialaj Ekvacioj* **33** (1990), 39-57.

- [15] **J. R. Magnus, H. Neudecker**, Matrix Differential Calculus with Applications in Statistics and Econometrics, John Wiley & Sons Ltd, Chichester, 1999.
- [16] **M. S. N. Murty, G. Suresh Kumar**, On Ψ - boundedness and Ψ - stability of matrix Lyapunov systems, *J. Appl. Math. Comput.* **26** (2008), 67-84.
- [17] **M. S. N. Murty, G. Suresh Kumar**, On Ψ - bounded solutions for non-homogeneous matrix Lyapunov systems on $\mathbb{R}$, *Electronic Journal of Qualitative Theory Differential Equations* **62** (2009), 1-12.
- [18] **Ch. Vasavi, T. Srinivasa Rao, G. Suresh Kumar**, On Ψ - bounded solutions for semi-linear differential equations on $\mathbb{R}$, *International Journal of Advances in Engineering, Science and Technology* **3**, No. 1 (2013), 22-25.

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