Ricci Solitons in β -Kenmotsu Manifolds

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Abstract. The object of the present paper is to study Ricci soliton in β -Kenmotsu manifolds. Here it is proved that a symmetric parallel second order covariant tensor in a β -Kenmotsu manifold is a constant multiple of the metric tensor. Using this result, it is shown that if $(\mathcal{L}_V g + 2S)$ is ∇ -parallel where V is a given vector field, then the structure (g, V, λ) yields a Ricci soliton. Further, by virtue of this result, we found the conditions of Ricci soliton in β -Kenmotsu manifold to be shrinking, steady and expanding respectively. Next, Ricci soliton for 3-dimensional β -Kenmotsu manifold are discussed with an example.

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1 Introduction

The Ricci flow is an intrinsic geometric flow which was introduced by Hamilton in 1982 ([25], [26]). Ricci flow on a smooth, compact and without boundary Riemannian manifold M equipped with a Riemannian metric g satisfies the following geometric evolution equation

$$\frac{\partial g}{\partial t} = -2S, \quad (1.1)$$

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where S is Ricci curvature tensor depending on g .

Hamilton himself and many other researchers like Cao [16], Yau [31], Chow and others [5], Perelman ([14],[15]), Morgan and Tian [19] developed the theory of Ricci flow. On the other hand Hamilton [27] introduced a more general notion of Ricci soliton in the context of metric paracontact geometry. More precisely, Ricci soliton is the natural generalization of Einstein metric and is defined on a Riemannian manifold.

In a Riemannian manifold (M, g) , g is called a Ricci soliton if

$$(\mathcal{L}_V g)(X, Y) + 2S(X, Y) + 2\lambda g(X, Y) = 0 \quad (1.2)$$

for any vector fields X, Y and V on M , where $\mathcal{L}_V$ denote the Lie derivative operator along the vector field V , S is the Ricci tensor and λ is a constant. The metric satisfying (1.2) are very interesting in the field of Physics and are often referred as quasi-Einstein ([32],[33],[12]).

The Ricci soliton is said to be shrinking, steady and expanding according as $\lambda < 0$, $\lambda = 0$ and $\lambda > 0$ respectively.

In the paracontact geometry, Ricci soliton firstly was studied by Calvaruso and Perrone [12]. Recently, Bejan and Crasmareanu [8] studied Ricci solitons on 3-dimensional normal paracontact manifold.

It is known that [22] if a positive definite Riemannian manifold (M, g) admits a second order parallel symmetric covariant tensor other than a constant multiple of the metric tensor, then it is reducible. The necessary and sufficient condition for the existence of such tensor was given by Levy [18].

The generalization of Levy's results is given by Sharma ([28],[29]). He shown that a second order parallel (not necessarily symmetric and non-singular) tensor on an n -dimensional ($n > 2$) space of constant curvature is a constant multiple of the metric tensor. He also proved that there is no non-zero parallel 2-form in a Sasakian manifold. Das [21] studied second order parallel tensor on an almost contact metric manifold and found that on an α -K-contact manifold (α being non-zero real constant) a second order symmetric parallel tensor is a constant multiple of the associative positive definite Riemannian metric tensor. It is also proved that in an α -Sasakian manifold there is no non-zero parallel 2-form. The study of Ricci solitons in K-contact manifolds was started by Sharma [30] and in the continuation of this Ghosh, Sharma and Cho [2] studied gradient Ricci soliton of a non-Sasakian (κ, μ) -contact manifold. Generally in a P-Sasakian manifold the structure vector field ξ is not killing, that is $(\mathcal{L}_V g) \neq 0$ but in K-contact manifold ξ is a killing vector field, that is $(\mathcal{L}_V g) = 0$. Recently in [34], De have studied Ricci soliton in P-Sasakian manifolds. Barua and De [4] have studied Ricci solitons in Riemannian manifolds. Since then, several other studied

Ricci soliton in various contact manifolds: Eisenhart problem to Ricci soliton in f -Kenmotsu manifold [6], Eta-Ricci solitons on para-Kenmotsu manifolds [3], on contact and Lorentzian manifolds ([6],[9],[28]), on Sasakian manifold ([1],[7]), on α -Sasakian [13], on Kenmotsu manifold [10], etc.

Motivated by the above studies, in this paper we treat Ricci soliton in β -Kenmotsu manifolds. The paper is structured as follows. After introduction, section 2 is a brief review of β -Kenmotsu manifold. Section 3 is devoted to the study of parallel symmetric second order tensor in β -Kenmotsu manifolds and Ricci soliton in β -Kenmotsu manifolds. So we obtain a relation between symmetric parallel second order covariant tensor and metric tensor in β -Kenmotsu manifold. In the second problem of this section we studied the necessary and sufficient condition of a Ricci semi-symmetric β -Kenmotsu manifold to be an Einstein manifold. We also analyzed the behavior of Ricci soliton in an n -dimension β -Kenmotsu manifold and η -Einstein manifolds. Section 4 is devoted to study Ricci soliton in 3-dimensional β -Kenmotsu manifold with an example.

2 Preliminaries

An n -dimensional differential smooth manifold M is said to be almost contact metric manifold [11] if it admits a $(1,1)$ tensor field φ , a contravariant vector field ξ , a 1-form η and a Riemannian metric g which satisfy

$$\varphi^2(X) = -X + \eta(X)\xi, \quad \eta(\xi) = 1, \quad \varphi\xi = 0, \quad \eta(\varphi X) = 0, \quad (2.1)$$

$$g(\varphi X, \varphi Y) = g(X, Y) - \eta(X)\eta(Y), \quad g(X, \xi) = \eta(X), \quad (2.2)$$

for all vector fields X, Y on M .

An almost contact metric manifold $M(\varphi, \xi, \eta, g)$ is said to be β -Kenmotsu manifold [20] if

$$(\nabla_X \varphi)(Y) = \beta[g(\varphi X, Y)\xi - \eta(Y)\varphi X]. \quad (2.3)$$

From (2.3), we have

$$\nabla_X \xi = \beta[X - \eta(X)\xi], \quad (2.4)$$

where $\beta \in C^\infty(M)$ and ∇ denote the Riemannian connection of g . If $\beta = 1$ then β -Kenmotsu manifold is called Kenmotsu manifold and if β is constant then it is called homothetic Kenmotsu manifold.

In an n -dimensional β -Kenmotsu manifold, the following relations hold [20]:

$$R(X, Y)\xi = -\beta^2[\eta(Y)X - \eta(X)Y] + (X\beta)[Y - \eta(Y)\xi] - (Y\beta)[X - \eta(X)\xi], \quad (2.5)$$

$$R(\xi, X)Y = (\beta^2 + \xi\beta)[\eta(Y)X - g(X, Y)\xi], \quad (2.6)$$

$$\begin{aligned} \eta(R(X, Y)Z) &= \beta^2[g(X, Z)\eta(Y) - g(Y, Z)\eta(X)] - (X\beta)[g(Y, Z) \\ &\quad - \eta(Y)\eta(Z)] + (Y\beta)[g(X, Z) - \eta(X)\eta(Z)], \end{aligned} \quad (2.7)$$

$$S(X, \xi) = -(2n\beta^2 + \xi\beta)\eta(X) - (2n - 1)(X\beta), \quad (2.8)$$

$$S(\xi, \xi) = -2n(\beta^2 + \xi\beta), \quad (2.9)$$

$$Q\xi = -(2n\beta^2 + \xi\beta)\xi - (2n - 1)\text{grad}\beta, \quad (2.10)$$

for any vector field X, Y, Z on M , where R is the Riemannian curvature tensor, S is the Ricci tensor of type (0,2) and Q is the Ricci operator.

3 Parallel symmetric second order tensors and Ricci solitons in β -Kenmotsu manifolds

Let h denote a (0,2) type symmetric tensor field which is parallel with respect to ∇ that is $\nabla h = 0$. Then it follows that ([28],[24]):

$$\nabla^2 h(X, Y; Z, W) - \nabla^2 h(X, Y; W, Z) = 0, \quad (3.1)$$

which gives

$$h(R(X, Y)Z, W) + h(Z, R(X, Y)W) = 0. \quad (3.2)$$

Taking $Z = W = \xi$ in (3.2) and using (2.5), we have

$$\begin{aligned} &\beta^2[\eta(Y)h(X, \xi) - \eta(X)h(Y, \xi)] - (X\beta)[h(Y, \xi) - \eta(Y)h(\xi, \xi)] \\ &\quad + (Y\beta)[h(X, \xi) - \eta(X)h(\xi, \xi)] = 0. \end{aligned} \quad (3.3)$$

With $X = \xi$ in (3.3) and by the symmetry of h , we have

$$(\beta^2 - \xi\beta)[\eta(Y)h(\xi, \xi) - h(Y, \xi)] = 0. \quad (3.4)$$

Since $\beta^2 - \xi\beta \neq 0$, so by (3.4), we have

$$h(Y, \xi) = \eta(Y)h(\xi, \xi). \quad (3.5)$$

Differentiating (3.5) covariantly with respect to X , we have

$$\begin{aligned} (\nabla_X h)(Y, \xi) + h(\nabla_X Y, \xi) + h(Y, \nabla_X \xi) \\ = [(\nabla_X \eta)(Y) + \eta(\nabla_X Y)]h(\xi, \xi) \\ + \eta(Y)[(\nabla_X h)(\xi, \xi) + 2h(\nabla_X \xi, \xi)]. \end{aligned} \quad (3.6)$$

By using (2.4), (3.5) and the parallel condition $\nabla h = 0$ in (3.6), we have

$$h(X, Y) = g(X, Y)h(\xi, \xi). \quad (3.7)$$

which implies that the parallelism of h gives $h(\xi, \xi)$ is a constant, via (3.5). So we have the following theorem.

Theorem 3.1. *A symmetric parallel second order covariant tensor in β -Kenmotsu manifold is a constant multiple of the metric tensor.*

Corollary 3.2. *A locally Ricci symmetric ($\nabla S = 0$) β -Kenmotsu manifold is an Einstein manifold.*

Remark 3.1. The following statements for β -Kenmotsu manifold are equivalent:

- (i) Einstein,
- (ii) locally Ricci symmetric,
- (iii) Ricci semi-symmetric, that is $R \cdot S = 0$.

The implication $(i) \rightarrow (ii) \rightarrow (iii)$ is trivial. Now we prove that the implication $(iii) \rightarrow (i)$ in the more general frame work of β -Kenmotsu manifold. Since $R \cdot S = 0$, means exactly (3.2) with h replaced by S , that is

$$(R(X, Y) \cdot S)(U, V) = -S(R(X, Y)U, V) - S(U, R(X, Y)V). \quad (3.8)$$

Taking $R \cdot S = 0$ and putting $X = \xi$ in (3.8), we have

$$S(R(\xi, Y)U, V) + S(U, R(\xi, Y)V) = 0. \quad (3.9)$$

In view of (2.6) and $\beta^2 + \xi\beta \neq 0$, the above equation becomes

$$\begin{aligned} \{\eta(U)S(Y, V) - g(Y, U)S(\xi, V)\} \\ + \{\eta(V)S(U, Y) - g(Y, V)S(U, \xi)\} = 0. \end{aligned} \quad (3.10)$$

Putting $U = \xi$ in (3.10) and by using (2.1), (2.8) and (2.9), we obtain

$$\begin{aligned} S(Y, V) = -2n(\beta^2 + \xi\beta)g(Y, V) + (2n - 1)(Y\beta)\eta(V) \\ - (2n - 1)(V\beta)\eta(Y). \end{aligned} \quad (3.11)$$

If $\omega(X) = g(X, \rho) = X\beta = g(\text{grad}\beta, X)$ for all X , then (3.11) yields

$$S(Y, V) = -2n(\beta^2 + \xi\beta)g(Y, V) + (2n - 1)\{\eta(V)\omega(Y) - \eta(Y)\omega(V)\}. \quad (3.12)$$

From (3.12), it follows that a Ricci semi-symmetric β -Kenmotsu manifold is an Einstein manifold if and only if

$$\eta(V)\omega(Y) = \eta(Y)\omega(V), \quad (3.13)$$

that is, the vector field ξ and $\rho = \text{grad}\beta$ are parallel.

This leads to the following theorem.

Theorem 3.3. *A Ricci semi-symmetric β -Kenmotsu manifold (M, g) is an Einstein manifold if and only if the structure vector field ξ and the scalar potential of the structure function β are parallel.*

Corollary 3.4. *If on a β -Kenmotsu manifold the tensor field $(\mathcal{L}_V g + 2S)$ is ∇ -parallel, then (g, V, λ) gives a Ricci soliton.*

Proof. A Ricci soliton in β -Kenmotsu manifold defined by (1.1) which gives $(\mathcal{L}_V g + 2S)$ is parallel. By theorem (3.1) it is clear that a symmetric parallel (0,2) tensor in β -Kenmotsu manifold is a constant multiple of metric tensor. Hence $(\mathcal{L}_V g + 2S)$ is a constant multiple of metric tensor g that is $(\mathcal{L}_V g + 2S)(X, Y) = g(X, Y)h(\xi, \xi)$, where $h(\xi, \xi)$ is a non-zero constant. It is the application of the theorem (3.1) to Ricci soliton. $\square$

Theorem 3.5. *If a metric g in β -Kenmotsu manifold is a Ricci soliton with $V = \xi$, then it is η -Einstein.*

Proof. Taking $V = \xi$ in (1.2), we obtain

$$(\mathcal{L}_\xi g)(X, Y) + 2S(X, Y) + 2\lambda g(X, Y) = 0. \quad (3.14)$$

Substituting

$$(\mathcal{L}_\xi g)(X, Y) = g(\nabla_X \xi, Y) + g(X, \nabla_Y \xi), \quad (3.15)$$

in (3.14) and by use of (2.4), we obtain

$$S(X, Y) = -(\beta + \lambda)g(X, Y) + \beta\eta(X)\eta(Y),$$

hence the result. $\square$

Theorem 3.6. *A Ricci soliton (g, ξ, λ) in an n -dimensional β -Kenmotsu manifold can not be steady but is expanding.*

Proof. In linear algebra either the vector field $V \in \text{Span } \xi$ or $V \perp \xi$. But the second case $V \perp \xi$ seems to be complex to analyze in practice. For this reason we investigate for the case $V = \xi$.

By a simple computation of $(\mathcal{L}_V g + 2S)$, we obtain

$$(\mathcal{L}_\xi g)(X, Y) = 0. \quad (3.16)$$

From (1.2), we have

$$h(X, Y) = -2\lambda g(X, Y).$$

On putting $X = Y = \xi$, we obtain from above relation as

$$h(\xi, \xi) = -2\lambda, \quad (3.17)$$

where

$$h(\xi, \xi) = (\mathcal{L}_\xi g)(\xi, \xi) + 2S(\xi, \xi). \quad (3.18)$$

Using (2.9) and (3.16) we get from above

$$h(\xi, \xi) = -4n(\beta^2 + \xi\beta). \quad (3.19)$$

By virtue of (3.17) and (3.19), it follows that

$$\lambda = 2n(\beta^2 + \xi\beta).$$

Since β is some non-zero function, we have $\lambda \neq 0$ and so Ricci soliton in an n -dimension β -Kenmotsu manifold can not be steady but is expanding because $\lambda > 0$. $\square$

Theorem 3.7. *If an n -dimensional β -Kenmotsu manifold is η -Einstein then the Ricci soliton in β -Kenmotsu manifold that is (g, ξ, λ) , where $\lambda = 2n\beta^2 + \xi\beta$ with varying scalar curvature can not be steady but it is expanding.*

Proof. The proof consists of three parts.

- (i) We prove that β -Kenmotsu manifold is η -Einstein,
- (ii) We prove that the Ricci soliton in β -Kenmotsu manifold is consisting of varying scalar curvature,
- (iii) We prove that the Ricci soliton in β -Kenmotsu manifold is expanding.

First we prove that the β -Kenmotsu manifold is η -Einstein: The metric g is called η -Einstein if there exists two real functions a and b such that the Ricci tensor of g is given by the general equation

$$S(X, Y) = ag(X, Y) + b\eta(X)\eta(Y). \quad (3.20)$$

Let $e_i, i = 1, 2, \dots, n$ be an orthonormal basis of the tangent space at any point of the manifold. Then putting $X = Y = e_i$ in (3.20) and taking summation over i , we get

$$r = na + b. \quad (3.21)$$

Again putting $X = Y = \xi$ in (3.20) then by use of (2.9), we have

$$a + b = -(2n\beta^2 + \xi\beta). \quad (3.22)$$

Then from (3.21) and (3.22), we have

$$a = \frac{r + (2n\beta^2 + \xi\beta)}{n - 1}, \quad b = -\left[\frac{r + n(2n\beta^2 + \xi\beta)}{n - 1}\right]. \quad (3.23)$$

Substituting the values of a and b in (3.20), we have

$$\begin{aligned} S(X, Y) &= \left[\frac{r + (2n\beta^2 + \xi\beta)}{n - 1}\right] g(X, Y) \\ &\quad - \left[\frac{r + n(2n\beta^2 + \xi\beta)}{n - 1}\right] \eta(X)\eta(Y), \end{aligned} \quad (3.24)$$

the above equation shows that β -Kenmotsu manifold is an η -Einstein manifold.

Now, we have to show that the scalar curvature r is not a constant and it is varying. For an n -dimensional β -Kenmotsu manifold the symmetric parallel covariant tensor $h(X, Y)$ of type $(0, 2)$ is given by

$$h(X, Y) = (\mathcal{L}_\xi g)(X, Y) + 2S(X, Y). \quad (3.25)$$

Using (3.16) and (3.24) in (3.25), we have

$$\begin{aligned} h(X, Y) &= 2\left[\beta + \frac{r + (2n\beta^2 + \xi\beta)}{n - 1}\right] g(X, Y) \\ &\quad - 2\left[\beta + \frac{r + n(2n\beta^2 + \xi\beta)}{n - 1}\right] \eta(X)\eta(Y). \end{aligned} \quad (3.26)$$

Differentiating (3.26) covariantly with respect to Z , we have

$$\begin{aligned} (\nabla_Z h)(X, Y) &= 2 \left[Z\beta + \frac{(\nabla_Z r) + 4nZ\beta + Z(\xi\beta)}{n-1} \right] g(X, Y) \\ &- 2 \left[Z\beta + \frac{(\nabla_Z r) + n(4nZ\beta + Z(\xi\beta))}{n-1} \right] \eta(X)\eta(Y) \\ &- 2\beta \left[\beta + \frac{r + n(2n\beta^2 + \xi\beta)}{n-1} \right] \{g(Z, X)\eta(Y) + g(Z, Y)\eta(X)\}. \end{aligned} \quad (3.27)$$

By substituting $Z = \xi$ and $X = Y \in (\text{Span } \xi)^\perp$ in (3.27) and using $\nabla h = 0$, we have

$$\nabla_\xi r = -(n-1)\nabla_\xi \beta - [4n\nabla_\xi \beta + \nabla_\xi(\xi\beta)]. \quad (3.28)$$

On integrating (3.28), we have

$$r = -(5n-1)\beta - \xi\beta + c, \quad (3.29)$$

where c is some integral constant. Thus from (3.29) we have r is a varying scalar curvature.

Finally, we have to check the nature of the soliton that is Ricci soliton in β -Kenmotsu manifold.

From (1.2), we have $h(X, Y) = -2\lambda g(X, Y)$, then putting $X = Y = \xi$, we have

$$h(\xi, \xi) = -2\lambda. \quad (3.30)$$

On putting $X = Y = \xi$ in (3.26), we have

$$h(\xi, \xi) = -2(2n\beta^2 + \xi\beta). \quad (3.31)$$

Equating (3.30) and (3.31), we have

$$\lambda = 2n\beta^2 + \xi\beta. \quad (3.32)$$

Since, $\lambda \neq 0$ because β is smooth function and $\lambda > 0$, that is the Ricci soliton in β -Kenmotsu manifold is expending. $\square$

4 Ricci solitons in 3-Dimensional β -Kenmotsu manifold

Theorem 4.1. *In a Ricci soliton (g, ξ, λ) where $\lambda = 6\beta^2 + \frac{1}{2}\xi\beta$ of 3-dimensional β -Kenmotsu manifold with varying scalar curvature can not be steady but it is expending.*

Proof. The proof consists of three parts.

- (i) We prove that the Riemannian curvature tensor of 3-dimensional β -Kenmotsu manifold is η -Einstein manifold,
- (ii) We prove that the Ricci soliton in 3-dimensional β -Kenmotsu manifold is consisting of varying scalar curvature,
- (iii) We prove that the Ricci soliton in a 3-dimensional β -Kenmotsu manifold is expanding.

The Riemannian curvature tensor of 3-dimensional β -Kenmotsu manifold is given by

$$\begin{aligned} R(X, Y, Z) &= g(Y, Z)QX - g(X, Z)QY + S(Y, Z)X \\ &\quad - S(X, Z)Y - \frac{r}{2}\{g(Y, Z)X - g(X, Z)Y\}. \end{aligned} \quad (4.1)$$

Putting $Z = \xi$ in (4.1) and by using (2.5) and (2.8), we have

$$\begin{aligned} &-\beta^2\{\eta(Y)X - \eta(X)Y\} + (X\beta)\{Y - \eta(Y)\xi\} - (Y\beta)\{X - \eta(X)\xi\} \\ &= \eta(Y)QX - \eta(X)QY - (6\beta^2 + \xi\beta)\eta(Y)X - 5(Y\beta)X \\ &\quad + (6\beta^2 + \xi\beta)\eta(X)Y + 5(X\beta)Y - \frac{r}{2}\{\eta(Y)X - \eta(X)Y\}. \end{aligned} \quad (4.2)$$

Again putting $Y = \xi$ in (4.2) and by using (2.1) and (2.10), we have

$$\begin{aligned} QX &= \left(\frac{r}{2} + 5\beta^2 + 5\xi\beta\right)X - \left(\frac{r}{2} + 11\beta^2 + \xi\beta\right)\xi \\ &\quad + 5(\text{grad } \beta)\eta(X) - 5(X\beta)\xi. \end{aligned} \quad (4.3)$$

By taking inner product of (4.3) with Y , we get

$$\begin{aligned} S(X, Y) &= \left(\frac{r}{2} + 5\beta^2 + 5\xi\beta\right)g(X, Y) - \left(\frac{r}{2} + 11\beta^2 + \xi\beta\right)\eta(X)\eta(Y) \\ &\quad + 5(Y\beta)\eta(X) - 5(X\beta)\eta(Y). \end{aligned} \quad (4.4)$$

Interchanging X and Y in (4.4), we have

$$\begin{aligned} S(Y, X) &= \left(\frac{r}{2} + 5\beta^2 + 5\xi\beta\right)g(Y, X) - \left(\frac{r}{2} + 11\beta^2 + \xi\beta\right)\eta(Y)\eta(X) \\ &\quad + 5(X\beta)\eta(Y) - 5(Y\beta)\eta(X). \end{aligned} \quad (4.5)$$

Adding (4.4) and (4.5), we have

$$\begin{aligned} S(X, Y) &= \left(\frac{r}{2} + 5\beta^2 + 5\xi\beta\right)g(X, Y) \\ &\quad - \left(\frac{r}{2} + 11\beta^2 + \xi\beta\right)\eta(X)\eta(Y). \end{aligned} \quad (4.6)$$

This shows that a 3-dimensional β -Kenmotsu manifold is η -Einstein manifold.

Now, we would like to show that the scalar curvature r is not a constant that is r is varying.

For a 3-dimensional β -Kenmotsu manifold the symmetric parallel covariant tensor $h(X, Y)$ of type $(0, 2)$ is given by

$$h(X, Y) = (\mathcal{L}_\xi g)(X, Y) + 2S(X, Y). \quad (4.7)$$

By using (3.16) and (4.6) in (4.7), we have

$$h(X, Y) = (r + 10\beta^2 + 10\xi\beta)g(X, Y) - (r + 22\beta^2 + 2\xi\beta)\eta(X)\eta(Y). \quad (4.8)$$

Differentiating the above equation covariantly with respect to Z , we have

$$\begin{aligned} (\nabla_Z h)(X, Y) &= \{\nabla_Z r + 20\beta(Z\beta) + 10Z(\xi\beta)\}g(X, Y) \\ &\quad - \{\nabla_Z r + 44\beta(Z\beta) + 2Z(\xi\beta)\}\eta(X)\eta(Y) \\ &\quad - (r + 22\beta^2 + 2\xi\beta)\{(\nabla_Z \eta)(X)\eta(Y) - \eta(X)(\nabla_Z \eta)(Y)\}. \end{aligned} \quad (4.9)$$

Substituting $Z = \xi$ and $X = Y \in (Span \xi)^\perp$ in (4.9) and by virtue of $\nabla h = 0$, we have

$$\{\nabla_\xi r + 10\nabla_\xi(\beta^2) + 10\nabla_\xi(\xi\beta)\} = 0. \quad (4.10)$$

On integrating (4.10), we have

$$r = -10(\beta^2 + \xi\beta) + c. \quad (4.11)$$

where c is integral constant. Thus from (4.11), we have r a variable scalar curvature.

Finally, we have to check the nature of the Ricci soliton (g, ξ, η) in 3-dimensional β -Kenmotsu manifold.

From (1.2), we have

$$h(X, Y) = -2\lambda g(X, Y). \quad (4.12)$$

On putting $X = Y = \xi$ in (4.12), we have

$$h(\xi, \xi) = -2\lambda. \quad (4.13)$$

On taking $X = Y = \xi$ in (4.8), we have

$$h(\xi, \xi) = -12\beta^2 - \xi\beta. \quad (4.14)$$

Equating (4.13) and (4.16), we have

$$\lambda = 6\beta^2 + \frac{1}{2}\xi\beta. \quad (4.15)$$

Since from (4.15), $\lambda \neq 0$ and $\lambda > 0$, therefore Ricci soliton (g, ξ, η) in 3-dimensional β -Kenmotsu manifold is expanding. $\square$

Example 4.1. Let $M = \{(x, y, z) \in R^3 : z \neq 0\}$, where (x, y, z) are standard co-ordinate in R^3 . Let $\{E_1, E_2, E_3\}$ be linearly independent vector fields given by

$$E_1 = z^2 \frac{\partial}{\partial x}, \quad E_2 = z^2 \frac{\partial}{\partial y}, \quad E_3 = \frac{\partial}{\partial z}.$$

Let g be the Riemannian metric defined by

$$\begin{aligned} g(E_1, E_2) &= g(E_1, E_3) = g(E_2, E_3) = 0, \\ g(E_1, E_1) &= g(E_2, E_2) = g(E_3, E_3) = 1. \end{aligned}$$

Let η be a 1-form defined by $\eta(U) = g(U, E_3)$ for any $U \in \chi(M)$ and φ be the $(1, 1)$ -tensor field defined by

$$\varphi E_1 = -E_2, \quad \varphi E_2 = E_1 \quad \text{and} \quad \varphi E_3 = 0.$$

Then using the linearity of φ on g , we have

$$\eta(E_3) = 1, \quad \varphi^2 U = -U + \eta(U)E_3,$$

and $g(\varphi U, \varphi W) = g(U, W) - \eta(U)\eta(W)$, for any $U, W \in \chi(M)$. Thus for $E_3 = \xi$, (ϕ, ξ, η, g) defines an almost contact metric structure on M .

Let ∇ be the Riemannian connection of g , then we have

$$[E_1, E_2] = 0, \quad [E_1, E_3] = -\frac{2}{z}E_1 \quad \text{and} \quad [E_2, E_3] = -\frac{2}{z}E_2.$$

Koszul formula is given by

$$\begin{aligned} 2g(\nabla_X Y, Z) &= X(g(Y, Z)) + Y(g(Z, X)) - Z(g(X, Y)) \\ &\quad - g(X, [Y, Z]) - g(Y, [X, Z]) + g(Z, [X, Y]). \end{aligned}$$

By using the Koszul formula for the Riemannian metric g , we can easily calculate

$$\begin{aligned} \nabla_{E_1} E_1 &= \frac{2}{z}E_3, & \nabla_{E_1} E_2 &= 0, & \nabla_{E_1} E_3 &= -\frac{2}{z}E_1, \\ \nabla_{E_2} E_1 &= 0, & \nabla_{E_2} E_2 &= \frac{2}{z}E_3, & \nabla_{E_2} E_3 &= -\frac{2}{z}E_2, \\ \nabla_{E_3} E_1 &= 0, & \nabla_{E_3} E_2 &= 0, & \nabla_{E_3} E_3 &= 0. \end{aligned}$$

From the above it can be easily seen that (ϕ, ξ, η, g) is β -Kenmotsu structure on M and satisfy

$$(\nabla_X \varphi)Y = \beta[g(\varphi X, Y)\xi - \eta(Y)\varphi X], \quad \nabla_X \xi = \beta[X - \eta(X)\xi], \quad (4.16)$$

where $\beta = -\frac{2}{z}$. Hence structure (ϕ, ξ, η, g) defines a β -Kenmotsu structure. Thus M equipped with β -Kenmotsu structure is a β -Kenmotsu manifold. The tangent vector X and Y on M are expressed as linear combination of E_1, E_2, E_3 , that is

$$\begin{aligned} X &= a_1 E_1 + a_2 E_2 + a_3 E_3, \\ Y &= b_1 E_1 + b_2 E_2 + b_3 E_3, \end{aligned}$$

where a_i and b_i , ($i = 1, 2, 3$) are scalars. Using $\beta = -\frac{2}{z}$ in (4.11), we have

$$r = -\frac{60}{z^2} + c,$$

which shows that, the scalar curvature r is not constant.

Using $\beta = -\frac{2}{z}$ in (4.15), we have

$$\lambda = \frac{25}{z^2},$$

this implies that $\lambda > 0$, that is the Ricci soliton in 3-dimensional β -Kenmotsu manifold is expanding.

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