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A Simple Proof of the Chuang's Inequality

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Abstract. The purpose of this paper is to present a short proof of the Chuang's inequality.

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1 Introduction Definitions and Result

Originally, the Chuang's inequality is a standard estimate in Nevanlinna theory but later on this inequality is used as a valuable tool in the study of value distribution of differential polynomials ([5]).

Recently, using this inequality, some sufficient conditions are obtained for which two differential polynomials sharing a small function satisfies the conclusions of Brück conjecture ([1], [2], [3], [4]).

At this point, we recall some notations and definitions to proceed further.

It will be convenient to let E denote any set of positive real numbers of finite linear measure, not necessarily the same at each occurrence. Also we use I to denote any set of infinite linear measure of $0 < r < \infty$.

Let f be a non-constant meromorphic function in the open complex plane $\mathbb{C}$. For any non-constant meromorphic function f, we denote by $S(r, f)$ any quantity satisfying

$$S(r, f) = o(T(r, f)) \qquad (r \longrightarrow \infty, r \notin E).$$

Now we recall the following definitions.

Definition 1.1. A meromorphic function $a = a(z)(\not\equiv \infty)$ is called a small function with respect to f provided that $T(r,a) = S(r,f)$ as $r \longrightarrow \infty, r \notin E$.

Definition 1.2. ([8])Let $n_{0j}, n_{1j}, \ldots, n_{kj}$ be non negative integers.

The expression $M_j[f] = (f)^{n_{0j}}(f^{(1)})^{n_{1j}} \ldots (f^{(k)})^{n_{kj}}$ is called a differential monomial generated by f of degree $d(M_j) = \sum\limits_{i=0}^{k} n_{ij}$ and weight $\Gamma_{M_j} = \sum\limits_{i=0}^{k}(i+1)n_{ij}$.

The sum $P[f] = \sum\limits_{j=1}^{t} b_j M_j[f]$ is called a differential polynomial generated by f of degree $\overline{d}(P) = max\{d(M_j) : 1 \leq j \leq t\}$ and weight $\Gamma_P = max\{\Gamma_{M_j} : 1 \leq j \leq t\}$, where $T(r,b_j) = S(r,f)$ for $j = 1, 2, \ldots, t$.

The numbers $\underline{d}(P) = min\{d(M_j) : 1 \leq j \leq t\}$ and k (the highest order of the derivative of f in $P[f]$) are called respectively the lower degree and order of $P[f]$.

$P[f]$ is said to be homogeneous if $\overline{d}(P)$=$\underline{d}(P)$.

$P[f]$ is called a Linear Differential Polynomial generated by f if $\overline{d}(P) = 1$. Otherwise, $P[f]$ is called Non-linear Differential Polynomial.

We also denote by $\mu = max\ \{\Gamma_{M_j} - d(M_j) : 1 \leq j \leq t\} = max\ \{n_{1j} + 2n_{2j} + \ldots + kn_{kj} : 1 \leq j \leq t\}$.

Now we are position to state the Chuang's inequality.

Theorem A. ([7]) Let f be a non-constant meromorphic function and $P[f]$ be a differential polynomial. Then

$$m\left(r, \frac{P[f]}{f^{\overline{d}(P)}}\right) \leq (\overline{d}(P) - \underline{d}(P))m\left(r, \frac{1}{f}\right) + S(r,f).$$

In this paper, we give a short proof of the above inequality with some restrictions.

Theorem 1.1. Let f be a non-constant meromorphic function and $P[f]$ be a differential polynomial. Then

$$m\left(r, \frac{P[f]}{f^{\overline{d}(P)}}\right) \leq (\overline{d}(P) - \underline{d}(P))m\left(r, \frac{1}{f}\right) + S(r,f), \tag{1.1}$$

as $r \to \infty$ and $r \notin E_0$ where E_0 is a set whose linear measure is not greater than 2.

2 Lemmas

We prove the result, using the lemma of logarithmic derivative.

Lemma 2.1 (*Lemma of Logarithmic Derivative*)**.** ([9]) Suppose that $f(z)$ is a non-constant meromorphic function in whole complex plane. Then

$$m\left(r, \frac{f'}{f}\right) = S(r, f) \text{ as } r \to \infty \text{ and } r \notin E_0,$$

where E_0 is a set whose linear measure is not greater than 2.

Lemma 2.2. ([9]) Suppose that $f(z)$ is a non-constant meromorphic function in whole complex plane and l is natural number. Then

$$m\left(r, \frac{f^{(l)}}{f}\right) = S(r, f) \text{ as } r \to \infty \text{ and } r \notin E_0,$$

where E_0 is a set whose linear measure is not greater than 2.

3 Proof of Chuang's inequality

Proof of theorem 1.1. Suppose that $P[f] = \sum_{j=1}^{t} b_j M_j[f]$ be a differential polynomial generated by a non-constant meromorphic function f. Further suppose that $m_j = d(M_j)$ for $j = 1, 2, ..., t$.

Without loss of any generality, we can assume that $m_1 \leq m_2 \leq ... \leq m_t$.

We have to prove the inequality (1.1) by induction on t.

If $t = 1$, then in view of Lemma 2.2, the inequality (1.1) follows. Next we assume that the inequality holds for $t = l(\geq 2)$. Now we have to show that the inequality (1.1) holds for $t = l + 1$.

For this, assume

$$P[f] = \sum_{j=1}^{l+1} b_j M_j[f] = Q[f] + bM[f],$$

where $Q[f] = \sum\limits_{j=1}^{l} b_j M_j[f]$, $M[f] = M_{l+1}[f]$ and $b = b_{l+1}$.

Then $m_1 \leq m_2 \leq ... \leq m_l \leq m_{l+1}$ and by hypothesis

$$m\left(r, \frac{Q[f]}{f^{\overline{d}(Q)}}\right) \leq (\overline{d}(Q) - \underline{d}(Q))m\left(r, \frac{1}{f}\right) + S(r, f) \text{ as } r \to \infty \text{ and } r \notin E_0,$$

where E_0 is a set whose linear measure is not greater than 2.

Thus

$$\begin{aligned}
m\left(r,\frac{P[f]}{f^{\overline{d}(P)}}\right) &= m\left(r,\frac{Q[f]+bM[f]}{f^{\overline{d}(P)}}\right)\\
&\leq m\left(r,\frac{Q[f]}{f^{\overline{d}(P)}}\right)+m\left(r,\frac{M[f]}{f^{\overline{d}(P)}}\right)+S(r,f)\\
&\leq \left(\overline{d}(Q)-\underline{d}(Q)\right)m\left(r,\frac{1}{f}\right)+\left(\overline{d}(P)-\overline{d}(Q)\right)m\left(r,\frac{1}{f}\right)\\
&+ \left(\overline{d}(P)-d(M)\right)m\left(r,\frac{1}{f}\right)+S(r,f)\\
&\leq \left(\overline{d}(P)-\underline{d}(P)\right)m\left(r,\frac{1}{f}\right)\\
&+ \left(\overline{d}(P)+\underline{d}(P)-\overline{d}(Q)-d(M)\right)m\left(r,\frac{1}{f}\right)+S(r,f)\\
&\leq \left(\overline{d}(P)-\underline{d}(P)\right)m\left(r,\frac{1}{f}\right)+S(r,f)
\end{aligned}$$

as $r\to\infty$ and $r\notin E_0$, where E_0 is a set whose linear measure is not greater than 2, and $\left(\overline{d}(P)+\underline{d}(P)-\overline{d}(Q)-d(M)\right)=m_{l+1}+m_1-m_l-m_{l+1}\leq 0$.

Thus by the principle of Mathematical Induction, the inequality follows.

□

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