f – Kenmotsu Metric as Conformal Ricci Soliton

H. G. Nagaraja and K. Venu

Abstract. In this paper, we study conformal Ricci solitons in f -Kenmotsu manifolds. We derive conditions for f -Kenmotsu metric to be a conformal Ricci soliton.

AMS Subject Classification (2000). 53C20, 53C44.

Keywords. conformal Ricci soliton, f -Kenmotsu manifold, Ricci flow, conformal pressure, η -Einstein

1 Introduction

Ricci solitons are self-similar solutions to the Ricci flow

$$\frac{\partial g}{\partial t} = -2S(g(t))$$

introduced by Hamilton [10] in 1982. These special solutions move under the Ricci flow simply by diffeomorphisms of the initial metric, that is

$$g(t) = \sigma(t)\phi_t^*(g(0)),$$

where $\sigma(t)$ is a smooth function of time and $g(t)$ is the pull back of the initial metric $g(0)$ by a one-parameter family of diffeomorphisms ϕ_t of the underlying manifold M , for all t such that the solution exists. Perelman used Ricci flow and its surgery to prove Poincare conjecture [15, 16].

The Ricci soliton is considered as natural generalization of Einstein metric and is defined on a Riemannian manifold (M, g) by

$$(\mathcal{L}_V g)(X, Y) + 2S(X, Y) + 2\lambda g(X, Y) = 0, \quad (1.1)$$

where $\mathcal{L}_V$ denote the Lie-derivative operator along a vector field V , λ is a scalar, g is a Riemannian metric, S is the Ricci tensor and X and Y are arbitrary vector fields on M . The Ricci soliton is said to be shrinking, steady and expanding according as $\lambda < 0$, $\lambda = 0$ and $\lambda > 0$, respectively.

The unit volume constraint plays a significant part in traditional Ricci flow equation. In 2004, Fischer [8] established the concept of conformal Ricci flow which is a variation of the classical Ricci flow equation that modifies the unit volume constraint of that equation to a scalar curvature constraint.

Let M be a smooth closed connected oriented n -manifold. The conformal Ricci flow equations on M are defined by

$$\frac{\partial g}{\partial t} = -2S - \left(p + \frac{2}{n}\right)g, \quad r(g) = -1,$$

where p is a non-dynamical scalar field (time dependent scalar field), $r(g)$ is the scalar curvature of the manifold and n is the dimension of the manifold M .

The conformal Ricci flow equations are analogous to the Navier-Stokes equations of fluid mechanics and because of this analogy the time dependent scalar field p is called a conformal pressure and, as for the real physical pressure in fluid mechanics that serves to maintain the incompressibility of the fluid, the conformal pressure serves as a Lagrange multiplier to conformally deform the metric flow so as to maintain the scalar curvature constraint.

The equilibrium points of the conformal Ricci flow equations are Einstein metrics with Einstein constant $\frac{-1}{n}$. Thus the conformal pressure p is zero at an equilibrium point and positive otherwise.

Basu and Bhattacharya [2] introduced the notion of conformal Ricci soliton equation as

$$(\mathcal{L}_V g)(X, Y) + 2S(X, Y) + \left[2\lambda - \left(p + \frac{2}{n}\right)\right]g(X, Y) = 0, \quad (1.2)$$

where λ is a constant.

In 1972, Kenmotsu [12] introduced and studied a new class of almost contact metric manifolds, later known as Kenmotsu manifolds. Olszaka and Rosca [14] studied f -Kenmotsu manifold, an almost contact metric manifold which is normal and locally conformal almost cosymplectic. Further they gave a geometric interpretation of f -Kenmotsu manifold and proved that a Ricci symmetric f -Kenmotsu manifold is an Einstein manifold.

Initially, Sharma [18] started the study of Ricci solitons in contact manifolds. Later Tripathi [20], Bejan and Crasmareanu [3], Ghosh [9], and Nagaraja and Premalatha [13] have extensively studied Ricci solitons in contact metric manifolds. Calin and Crasmareanu [5] devoted their work to the Eisenhart problem in a f -Kenmotsu manifold and its relationship with the Ricci soliton. Yildiz, et al. [21] generated some interesting results on Ricci solitons in 3-dimensional f -Kenmotsu manifolds. Recently, Basu, Bhattacharya and Dutta [1, 2] established some results on conformal Ricci solitons in Kenmotsu manifolds and trans-Sasakian manifolds.

2 Preliminaries

An odd-dimensional Riemannian manifold (M, g) is said to be an almost contact metric manifold if there exist on M a $(1, 1)$ -tensor field ϕ , a vector field ξ (called the structure vector field) and a 1-form η such that

$$\phi^2 = -I + \eta \otimes \xi, \quad \eta(\xi) = 1, \quad g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y). \quad (2.1)$$

In particular on an almost contact metric manifold, we also have

$$\phi\xi = 0, \quad \eta \circ \phi = 0, \quad (2.2)$$

$$g(X, \phi Y) = -g(\phi X, Y), \quad g(X, \xi) = \eta(X), \quad (2.3)$$

for all vector fields X, Y on M .

An almost contact metric manifold is said to be a f -Kenmotsu manifold [12] if

$$(\nabla_X \phi)Y = f[g(\phi X, Y)\xi - \eta(Y)\phi X], \quad (2.4)$$

where $f \in C^\infty(M)$ is such that $df \wedge \eta = 0$. If $f = \beta = \text{constant} \neq 0$, then the manifold is an β -Kenmotsu manifold. The 1-Kenmotsu manifold is a Kenmotsu manifold. If $f = 0$, then the manifold is cosymplectic.

Remark 2.1. [21] An f -Kenmotsu manifold is called regular if $f^2 + f' \neq 0$, where $f' = \xi f$.

For the f -Kenmotsu manifold, from (2.4), we derive

$$\nabla_X \xi = f[X - \eta(X)\xi], \quad (2.5)$$

where ∇ denotes the Riemannian connection of g . The condition $df \wedge \eta = 0$ holds if $\dim M \geq 5$. In general, that is not true if $\dim M = 3$ [14].

In a f -Kenmotsu manifold, we have

$$(\nabla_X \eta)Y = f(g(X, Y) - \eta(X)\eta(Y)), \quad (2.6)$$

$$R(X, Y)\xi = f^2(\eta(X)Y - \eta(Y)X) - XY(f) + Y(f)\eta(X)\xi \\ + YX(f) - X(f)\eta(Y)\xi, \quad (2.7)$$

$$S(X, \xi) = -[(n-1)f^2 + \xi f]\eta(X) - (n-2)X(f), \quad (2.8)$$

$$S(\xi, \xi) = -(n-1)[f^2 + \xi f], \quad (2.9)$$

$$Q\xi = -[(n-1)f^2 + \xi f]\xi - (n-2)\text{grad}f, \quad (2.10)$$

for any vector fields X, Y, Z , where R is the Riemannian curvature, S is Ricci curvature and Q is Ricci operator given by $g(QX, Y) = S(X, Y)$.

A f -Kenmotsu manifold M is said to be η -Einstein if its Ricci tensor is of the form

$$S(X, Y) = ag(X, Y) + b\eta(X)\eta(Y), \quad (2.11)$$

for any vector fields X, Y and a, b are functions on M . If $b = 0$, then the above equation reduces to

$$S(X, Y) = ag(X, Y).$$

We call it as an Einstein manifold.

3 Three-dimensional Kenmotsu manifold and Conformal Ricci soliton

In a 3-dimensional f -Kenmotsu manifold, we have

$$R(X, Y)Z = \left(\frac{r}{2} + 2f^2 + 2f'\right)[g(Y, Z)X - g(X, Z)Y] \\ - \left(\frac{r}{2} + 3f^2 + 3f'\right)[g(Y, Z)\eta(X)\xi - \eta(X)\eta(Z)Y] \\ - \left(\frac{r}{2} + 3f^2 + 3f'\right)[\eta(Y)\eta(Z)X - g(X, Z)\eta(Y)\xi], \quad (3.1)$$

$$S(X, Y) = \left(\frac{r}{2} + f^2 + f'\right)g(X, Y) - \left(\frac{r}{2} + 3f^2 + 3f'\right)\eta(X)\eta(Y), \quad (3.2)$$

where r is the scalar curvature. From (3.1) and (3.2) we have

$$R(X, Y)\xi = -(f^2 + f')[\eta(Y)X - \eta(X)Y], \quad (3.3)$$

$$S(X, \xi) = -2(f^2 + f')\eta(X). \quad (3.4)$$

Naturally, two situations appear regarding the vector field V : $V \in \text{span}\xi$ and $V \perp \xi$. We consider the case $V = \xi$.

From (2.5), it follows that

$$(\mathcal{L}_\xi g)(X, Y) = 2f[g(X, Y) - \eta(X)\eta(Y)]. \quad (3.5)$$

In view of (1.2) and (3.5), we get

$$S(X, Y) = Ag(X, Y) + f\eta(X)\eta(Y), \quad (3.6)$$

where $A = -\frac{1}{2} [2\lambda - (p + \frac{2}{3}) + 2f]$.

Thus we state the following:

Proposition 3.1. A 3-dimensional f -Kenmotsu manifold admitting conformal Ricci soliton (g, ξ, λ) is an η -Einstein manifold.

Suppose that 3-dimensional f -Kenmotsu manifold admits a conformal Ricci soliton. Since $2\lambda - (p + \frac{2}{3})$ is constant in the conformal Ricci soliton equation (1.2), we get $\nabla((2\lambda - (p + \frac{2}{3})))g = 0$. Thus, $\mathcal{L}_\xi g + 2S$ is parallel.

If we set

$$h(X, Y) = (\mathcal{L}_\xi g + 2S)(X, Y),$$

then for a 3-dimensional f -Kenmotsu manifold, we have

$$\begin{aligned} h(X, Y) = (r + 2f' + 2f + 2f^2)g(X, Y) - (r + 2f' + 2f + 6f^2)\eta(X)\eta(Y) \\ - 2Y(f)\eta(X) - 2X(f)\eta(Y), \end{aligned} \quad (3.7)$$

where $h(\xi, \xi)$ is given by the formula

$$h(\xi, \xi) = (\mathcal{L}_\xi g + 2S)(\xi, \xi) = -4(f^2 + f').$$

Thus, $\mathcal{L}_\xi g + 2S + [2\lambda - (p + \frac{2}{3})]g$ reduces to $(-4(f^2 + f') + [2\lambda - (p + \frac{2}{3})])g$. By using (1.2), we get

$$\lambda = 2(f^2 + f') + \frac{1}{2} \left(p + \frac{2}{3} \right).$$

Hence we state that:

Theorem 3.1. In a 3-dimensional regular f -Kenmotsu manifold admitting conformal Ricci soliton with f as a constant, the constant λ is always positive.

Corollary 3.1. In a 3-dimensional Kenmotsu manifold admitting conformal Ricci soliton, the constant λ is always positive.

Corollary 3.2. *In a 3-dimensional cosymplectic manifold admitting conformal Ricci soliton, the constant λ is always positive.*

Let V be pointwise collinear with ξ , i.e., $V = b\xi$, where b is a function on the 3-dimensional f -Kenmotsu manifold. Then

$$\left(\mathcal{L}_V g + 2S + \left[2\lambda - \left(p + \frac{2}{n} \right) \right] g \right) (X, Y) = 0,$$

implies that

$$g(\nabla_X b\xi, Y) + g(\nabla_Y b\xi, X) + 2S(X, Y) + \left[2\lambda - \left(p + \frac{2}{3} \right) \right] g(X, Y) = 0.$$

Applying the property of Lie derivative and Levi-Civita connection, we have

$$\begin{aligned} bg(\nabla_X \xi, Y) + (Xb)\eta(Y) + bg(\nabla_Y \xi, X) + (Yb)\eta(X) + 2S(X, Y) \\ + \left[2\lambda - \left(p + \frac{2}{3} \right) \right] g(X, Y) = 0. \end{aligned} \quad (3.8)$$

Substituting (2.5) in (3.8), we obtain

$$\begin{aligned} 2fbg(X, Y) - 2fb\eta(X)\eta(Y) + (Xb)\eta(Y) + (Yb)\eta(X) + 2S(X, Y) \\ + \left[2\lambda - \left(p + \frac{2}{3} \right) \right] g(X, Y) = 0. \end{aligned} \quad (3.9)$$

Replacing Y by ξ and using (3.4) in the above equation, we get

$$(Xb) + (\xi b)\eta(X) - 4(f^2 + f')\eta(X) + \left[2\lambda - \left(p + \frac{2}{3} \right) \right] \eta(X) = 0. \quad (3.10)$$

Setting $X = \xi$ in (3.10), we get

$$\xi b = 2[f^2 + f'] - \lambda + \frac{1}{2} \left(p + \frac{2}{3} \right).$$

Substituting this quantity in (3.10), we have

$$\begin{aligned} (Xb) + \left(2[f^2 + f'] - \lambda + \frac{1}{2} \left(p + \frac{2}{3} \right) \right) \eta(X) - 4(f^2 + f')\eta(X) \\ - \left[2\lambda - \left(p + \frac{2}{3} \right) \right] \eta(X) = 0, \end{aligned}$$

which implies

$$Xb = \left(2[f^2 + f'] - \lambda + \frac{1}{2} \left(p + \frac{2}{3} \right) \right) \eta(X). \quad (3.11)$$

Applying exterior differentiation in (3.11), we have

$$\left(2[f^2 + f'] - \lambda + \frac{1}{2}\left(p + \frac{2}{3}\right)\right) d\eta = 0. \quad (3.12)$$

Since $d\eta \neq 0$, we have

$$2[f^2 + f'] - \lambda + \frac{1}{2}\left(p + \frac{2}{3}\right) = 0. \quad (3.13)$$

Using (3.13) in (3.12), we get

$$Xb = 0.$$

Theorem 3.2. *If in a 3-dimensional f -Kenmotsu manifold admitting conformal Ricci soliton, V is pointwise collinear with ξ , then b is a constant.*

It follows from (3.9) that

$$S(X, Y) = -\frac{1}{2}\left(2\lambda - \left(p + \frac{2}{3}\right) + 2fb\right)g(X, Y) + fb\eta(X)\eta(Y). \quad (3.14)$$

Putting $X = Y = e_i$ in (3.14), where $\{e_i\}$ is orthonormal basis of the tangent space TM and summing over i we get,

$$r = -\frac{3}{2}\left(2\lambda - \left(p + \frac{2}{3}\right) + 2fb\right) + fb.$$

Now for conformal Ricci soliton $r = -1$, so putting this value in the above equation we get

$$\lambda = \frac{1}{2}p + \frac{2}{3}(1 - fb).$$

So we assert the following:

Corollary 3.3. *Let (M, g) be a 3-dimensional f -Kenmotsu manifold where g is a conformal Ricci soliton. If the vector field $V = b\xi$ with $fb = 1$ then g is a steady Ricci soliton if and only if conformal pressure p vanishes.*

We see that the 3-dimensional f -Kenmotsu metric is a conformal Ricci soliton. By using (3.2) and (3.5), we get

$$\begin{aligned} &\left(\mathcal{L}_\xi g + 2S + \left[2\lambda - \left(p + \frac{2}{3}\right)\right]g\right)(X, Y) = \\ &2\left(f + \lambda - \frac{1}{2}\left(p + \frac{2}{3} + \frac{r}{2} + f^2\right)\right)g(X, Y) - 2\left(f + \frac{r}{2} + 3f^2\right)\eta(X)\eta(Y). \end{aligned} \quad (3.15)$$

For $\lambda = 2f^2 + \frac{1}{2}(p + \frac{2}{3})$, (3.15) becomes

$$\begin{aligned} & \left(\mathcal{L}_\xi g + 2S + \left[2\lambda - \left(p + \frac{2}{3} \right) \right] g \right) (X, Y) = \\ & 2 \left(f + \frac{r}{2} + 3f^2 \right) (g(X, Y) - \eta(X)\eta(Y)). \end{aligned} \quad (3.16)$$

It follows from (3.16) that,

M admits a conformal Ricci soliton $(g, \xi, 2f^2 + \frac{1}{2}(p + \frac{2}{3}))$ if and only if $r = -2f(3f + 1)$. Thus, we arrive at the following assertion:

Theorem 3.3. *For a constant f , 3-dimensional f -Kenmotsu metric g becomes a conformal Ricci soliton $(g, \xi, (2f^2 + \frac{1}{2}(p + \frac{2}{3})))$ if and only if $r = -2f(3f + 1)$.*

References

- [1] **A. Barros, R. Batista, and E. Ribeiro Jr**, Conformal almost Ricci solitons with constant scalar curvature are gradient, *Monstsh Mat*, **174**, (2014), 29-39.
- [2] **N. Basu and A. Bhattacharyya**, Conformal Ricci soliton in Kenmotsu manifold, *Global Journal of Advanced Research on Classical and Modern Geometries*, **4**, (2015), 15-21.
- [3] **C. L. Bejan and M. Crasmareanu**, Ricci solitons in manifolds with quasi-constant curvature, *Publ. Math. Debrecen*, **78**, (2011), 235-243.
- [4] **A. Bhattacharyya and N. Basu**, Some curvature identities on gradient shrinking conformal Ricci soliton, *Analele Stiintifice Ale Universitatii Al. I. Cuza Din Iasi(S.N) Mathematica, Tomul LXI*, **61**, (2015), 245-252.
- [5] **C. Calin and M. Crasmareanu**, From the Eisenhart problem to the Ricci solitons in f -Kenmotsu manifolds, *Bull. Malay. Math. Soc.*, **33**, (2010), 361-368.
- [6] **X. Cao**, Compact gradient shrinking Ricci solitons with positive curvature operator, *J. Geom. Anal.*, **17**, (2007), 425-433.
- [7] **B. Chow, P. Lu, and L. Ni**, Hamilton's Ricci flow, *Graduate Studies in Mathematics, American Mathematical Society, Providence, RI; Science Press, New York*, **77**, (2006)
- [8] **A. E. Fisher**, An introduction to conformal Ricci flow, *Class. quantum Grav.*, **21**, (2004), S171-S218.
- [9] **A. Ghosh**, Kenmotsu 3-metric as a Ricci soliton, *Chaos, Solitons and Fractals*, **44**, (2011), 647-650.
- [10] **R. S. Hamilton**, Three manifold with positive Ricci curvature, *J. Differential Geom.*, **17**, (1982), 255-306.
- [11] **R. S. Hamilton**, The Ricci flow on surfaces, *Contemporary Mathematics*, **71**, (1988), 237-261.
- [12] **K. Kenmotsu**, A class of almost contact Riemannian manifolds, *Tohoku Math. J.*, **24**, (1972), 93-103.

- [13] **H. G. Nagaraja and C. R. Premalatha**, Ricci solitons in Kenmotsu manifolds, *Journal of Mathematical Analysis*, **3**, (2012), 18-24.
- [14] **Z. Olszak and R. Rosco**, Normal locally conformal almost cosymplectic manifolds, *Publ. Math. Debrecen*, **39**, (1991), 315-323.
- [15] **G. Perelman**, The entropy formula for geometry and its applications, *arXiv:0211159v1*, [mathDG], (2002.)
- [16] **G. Perelman**, Ricci flow on surgery on three manifolds, *arXiv:0303109v1*, [mathDG], (2003)
- [17] **S. Pigola, M. Rigoli, M. Rimoldi, and A. G. Setti**, Ricci almost solitons, *arXiv:1003.2945v1*, [mathDG], (2010)
- [18] **B. B. Sinha and R. Sharma**, On para-A-Einstein manifolds, *Publications De L'Institut Mathematique, Nouvelle serie*, **34-48**, (1983), 211-215.
- [19] **P. Topping**, *Lectures on the Ricci flow*, Cambridge university press, Cambridge, UK, 2006.
- [20] **M. M. Tripathi**, Ricci solitons in contact metric manifolds, *arXiv:0801.4222v1*, [mathDG], (2008)
- [21] **A. Yildiz, U.C. De, and M. Turan**, On 3-dimensional f-Kenmotsu manifolds and Ricci solitons, *Ukrain. Math. J.*, **65**, (2013), 620-628.

H. G. Nagaraja and K. Venu

(H. G. Nagaraja and K. Venu) Department of Mathematics, Bangalore University,
Central College Campus, Bengaluru-560001, INDIA

E-mail: hgnraj@yahoo.com

E-mail: venuk.math@gmail.com

Received: 18.05.2016

Accepted: 21.12.2016