Uniqueness of a Polynomial and a Differential Monomial Sharing a Small Function

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Abstract. In this paper taking a question in [1] into background we investigate the uniqueness of a non-constant polynomial with the differential monomial generated by a non-constant meromorphic function f . Our result will also extend a result of *Banerjee-Majumder* [2] given earlier. An open question is also posed, in the paper, for future investigation.

AMS Subject Classification (2000). 30D35

Keywords. Meromorphic function, uniqueness, differential monomial, small function

1 A first section

Let f be a non-constant meromorphic function defined in the open complex plane $\mathbb{C}$. We adopt the standard notations of the Nevanlinna theory of meromorphic functions as explained in [4].

If for some $a \in \mathbb{C} \cup \{\infty\}$, f and g have the same set of a -points with the same multiplicities, we say that f and g share the value a CM (counting multiplicities) and if we do not consider the multiplicities then f, g are said to share the value a IM (ignoring multiplicities). When $a = \infty$ the zeros of $f - a$ means the poles of f .

It will be convenient to let J denote any set of positive real numbers of finite linear measure, not necessarily the same at each occurrence. For any

non-constant meromorphic function f , we denote by $S(r, f)$ any quantity satisfying

$$S(r, f) = o(T(r, f)) \quad (r \rightarrow \infty, r \notin J).$$

A meromorphic function $a = a(z) (\neq \infty)$ is called a small function with respect to f provided that $T(r, a) = S(r, f)$ as $r \rightarrow \infty, r \notin J$. If $a = a(z)$ is a small function we define that f and g share a IM or a CM according as $f - a$ and $g - a$ share 0 IM or 0 CM respectively. Also it is known to us that the hyper order of f , denoted by $\rho_2(f)$, is defined by

$$\rho_2(f) = \limsup_{r \rightarrow \infty} \frac{\log \log T(r, f)}{\log r}.$$

We start our discussion on a well known result of *Rubel* and *Yang* ([10]), where they proved that if a non-constant entire function f and f' share two distinct finite numbers a, b CM, then $f = f'$. This result is the starting point of the investigations about the relation between an entire or meromorphic function sharing some values with their derivatives.

In 1979, *Mues* and *Steinmetz* ([9]) obtained an analogous result for *IM* sharing. In this direction, in 1996, *Brück* ([3]) proposed his following famous conjecture.

Conjecture : *Let f be a non-constant entire function such that the hyper order $\rho_2(f)$ of f is not a positive integer or infinite. If f and f' share a finite value a CM, then $\frac{f' - a}{f - a} = c$, where c is a non-zero constant.*

Brück himself proved the conjecture for $a = 0$. For $a \neq 0$, *Brück* ([3]) obtained the following result in which additional supposition was required.

Theorem A. ([3]) *Let f be a non-constant entire function. If f and f' share the value 1 CM and if $N(r, 0; f') = S(r, f)$ then $\frac{f' - 1}{f - 1}$ is a nonzero constant.*

Next we recall the following definitions.

Definition 1.1. ([12]) *For $a \in \mathbb{C} \cup \{\infty\}$ and a positive integer p we denote by*

$$N_p(r, a; f) = \overline{N}(r, a; f) + \overline{N}(r, a; f | \geq 2) + \dots + \overline{N}(r, a; f | \geq p).$$

Clearly $N_1(r, a; f) = \overline{N}(r, a; f)$.

Definition 1.2. ([12]) *For $a \in \mathbb{C} \cup \{\infty\}$ and a positive integer p we put*

$$\delta_p(a, f) = 1 - \limsup_{r \rightarrow \infty} \frac{N_p(r, a; f)}{T(r, f)}.$$

Clearly

$$0 \leq \delta(a, f) \leq \delta_p(a, f) \leq \delta_{p-1}(a, f) \leq \dots \leq \delta_2(a, f) \leq \delta_1(a, f) = \Theta(a, f) \leq 1.$$

Definition 1.3. ([1]) For two positive integers n, p we define $\mu_p = \min\{n, p\}$ and $\mu_p^* = p + 1 - \mu_p$. Then it is clear that

$$N_p(r, 0; f^n) \leq \mu_p N_{\mu_p^*}(r, 0; f).$$

Definition 1.4. ([2]) Let z_0 be a zero of $f - a$ of multiplicity p and a zero of $g - a$ of multiplicity q . We denote by $\overline{N}_L(r, a; f)$ the counting function of those a -points of f and g where $p > q \geq 1$, by $N_E^{(1)}(r, a; f)$ the counting function of those a -points of f and g where $p = q = 1$ and by $\overline{N}_E^{(2)}(r, a; f)$ the counting function of those a -points of f and g where $p = q \geq 2$, each point in these counting functions is counted only once. In the same way we can define $\overline{N}_L(r, a; g)$, $N_E^{(1)}(r, a; g)$, $\overline{N}_E^{(2)}(r, a; g)$.

Definition 1.5. ([5]) Let k be a nonnegative integer or infinity. For $a \in \mathbb{C} \cup \{\infty\}$ we denote by $E_k(a; f)$ the set of all a -points of f , where an a -point of multiplicity m is counted m times if $m \leq k$ and $k + 1$ times if $m > k$. If $E_k(a; f) = E_k(a; g)$, we say that f, g share the value a with weight k .

The definition implies that if f, g share a value a with weight k then z_0 is an a -point of f with multiplicity $m (\leq k)$ if and only if it is an a -point of g with multiplicity $m (\leq k)$ and z_0 is an a -point of f with multiplicity $m (> k)$ if and only if it is an a -point of g with multiplicity $n (> k)$, where m is not necessarily equal to n .

We write f, g share (a, k) to mean that f, g share the value a with weight k . Clearly if f, g share (a, k) , then f, g share (a, p) for any integer p , $0 \leq p < k$. Also we note that f, g share a value a IM or CM if and only if f, g share $(a, 0)$ or (a, ∞) respectively.

With the notion of weighted sharing of values Lahiri-Sarkar ([6]) improved the result of Zhang ([11]). In ([12]) Zhang extended the result of Lahiri-Sarkar ([6]) and replaced the concept of value sharing by small function sharing.

In 2008, Zhang and Lü ([13]) obtained the following result.

Theorem B. ([13]) Let $k(\geq 1)$, $n(\geq 1)$ be integers and f be a non-constant meromorphic function. Also let $a(z) (\not\equiv 0, \infty)$ be a small function with respect to f . Suppose $f^n - a$ and $f^{(k)} - a$ share $(0, l)$. If $l = \infty$ and

$$(3 + k)\Theta(\infty, f) + 2\Theta(0, f) + \delta_{2+k}(0, f) > 6 + k - n \quad (1.1)$$

or $l = 0$ and

$$(6 + 2k)\Theta(\infty, f) + 4\Theta(0, f) + 2\delta_{2+k}(0, f) > 12 + 2k - n \quad (1.2)$$

then $f^n \equiv f^{(k)}$.

In ([13]), Zhang and Lü asked this question :

Question 1.1. *What will happen if f^n and $[f^{(k)}]^m$ share a small function ?*

In 2010 Banerjee and Majumder ([2]) answer the above open question affirmatively in the following manner.

Theorem C. ([2]) *Let $k(\geq 1)$, $n(\geq 1)$, $m(\geq 2)$ be integers and f be a non-constant meromorphic function. Also let $a(z)(\neq 0, \infty)$ be a small function with respect to f . Suppose $f^n - a$ and $[f^{(k)}]^m - a$ share $(0, l)$. If $l = 2$ and*

$$(3 + 2k)\Theta(\infty, f) + 2\Theta(0, f) + 2\delta_{1+k}(0, f) > 7 + 2k - n \quad (1.3)$$

or $l = 1$ and

$$\left(\frac{7}{2} + 2k\right)\Theta(\infty, f) + \frac{5}{2}\Theta(0, f) + 2\delta_{1+k}(0, f) > 8 + 2k - n \quad (1.4)$$

or $l = 0$ and

$$(6 + 3k)\Theta(\infty, f) + 4\Theta(0, f) + 3\delta_{1+k}(0, f) > 13 + 3k - n \quad (1.5)$$

then $f^n \equiv [f^{(k)}]^m$.

Here we observe that in the conditions (1.3)-(1.5) there was no influence of m .

Next we recall the following definition.

Definition 1.6. ([4]) *Let $n_{0j}, n_{1j}, \dots, n_{kj}$ be nonnegative integers.*

The expression $M_j[f] = (f)^{n_{0j}}(f^{(1)})^{n_{1j}} \dots (f^{(k)})^{n_{kj}}$ is called a differential monomial generated by f of degree $d_{M_j} = d(M_j) = \sum_{i=0}^k n_{ij}$ and weight $\Gamma_{M_j} = \sum_{i=0}^k (i+1)n_{ij}$.

The sum $P[f] = \sum_{j=1}^t b_j M_j[f]$ is called a differential polynomial generated by f of degree $\bar{d}(P) = \max\{d(M_j) : 1 \leq j \leq t\}$ and weight $\Gamma_P = \max\{\Gamma_{M_j} : 1 \leq j \leq t\}$, where $T(r, b_j) = S(r, f)$ for $j = 1, 2, \dots, t$.

The numbers $\underline{d}(P) = \min\{d(M_j) : 1 \leq j \leq t\}$ and k (the highest order of the derivative of f in $P[f]$) are called respectively the lower degree and order of $P[f]$.

$P[f]$ is said to be homogeneous if $\bar{d}(P) = \underline{d}(P)$.

$P[f]$ is called a linear differential polynomial generated by f if $\bar{d}(P) = 1$. Otherwise $P[f]$ is called a non-linear differential polynomial.

We denote by $Q = \max\{\Gamma_{M_j} - d(M_j) : 1 \leq j \leq t\} = \max\{n_{1j} + 2n_{2j} + \dots + kn_{kj} : 1 \leq j \leq t\}$.

Also for the sake of convenience for a differential monomial $M[f]$ we denote by $\lambda = \Gamma_M - d_M$.

Since the natural extension of $[f^{(k)}]^m$ is a differential monomial, it will be interesting to see whether *Theorem C* can remain true when $[f^{(k)}]^m$ is replaced by $M[f]$. In this direction, very recently *Banerjee - Chakraborty* ([1]) have improved *Theorem C* in the following way which in turn improve a recent result of Li-Huang [7] as well.

Theorem D. ([1]) Let $k(\geq 1)$, $n(\geq 1)$ be integers and f be a non-constant meromorphic function and $M[f]$ be a differential monomial of degree d_M and weight Γ_M and k is the highest derivative in $M[f]$. Also $a(z) (\neq 0, \infty)$ be a small function with respect to f . Suppose $f^n - a$ and $M[f] - a$ share $(0, l)$. If $l \geq 2$ and

$$(3 + \lambda)\Theta(\infty, f) + \mu_2\delta_{\mu_2^*}(0, f) + d_M\delta_{k+2}(0; f) > \Gamma_M + 3 + \mu_2 - n \quad (1.6)$$

or $l = 1$ and

$$\left(\frac{7}{2} + \lambda\right)\Theta(\infty, f) + \frac{1}{2}\Theta(0; f) + \mu_2\delta_{\mu_2^*}(0, f) + d_M\delta_{k+2}(0; f) > \Gamma_M + 4 + \mu_2 - n \quad (1.7)$$

or $l = 0$ and

$$(6 + 2\lambda)\Theta(\infty, f) + 2\Theta(0; f) + \mu_2\delta_{\mu_2^*}(0, f) + d_M\delta_{k+2}(0; f) + d_M\delta_{k+1}(0; f) > 2\Gamma_M + 8 + \mu_2 - n \quad (1.8)$$

then $f^n \equiv M[f]$.

In [1, Example 1.13], Banerjee-Chakraborty ([1]) have shown that f^n can't be replaced by an arbitrary polynomial $P[f] = a_0f^n + a_1f^{n-1} + \dots + a_n$ in *Theorem D* for IM sharing case.

Observing Example 1.13 in [1] we note that $f(z) = e^z$, $P(f) = f^2 + 2f$ and $M[f] = f^{(3)}$. So $P + 1 = (M + 1)^2$. Thus P and M share $(-1, 0)$. Also

$\Theta(\infty; f) = 1 = \Theta(0; f) = \delta_q(0; f)$. Here $p = n = 1$, $m = 1$, $w_p = 0$, $\mu_2 = 1$, $d_M = 1$, $\Gamma_M = 4$, $\lambda = 3$. Here

$$(6 + 2\lambda)\Theta(\infty, f) + 2\Theta(0; f) + \mu_2\delta_{\mu_2^*}(0, f) + d_M\delta_{k+2}(0; f) + d_M\delta_{k+1}(0; f) \\ = 17 > 16 = 2\Gamma_M + 8 + \mu_2 - n$$

so that (1.8) is satisfied but $M[f] \not\equiv P[f]$. On the basis of this observation in ([1]) following question was asked in [1]

Question 1.2. *Is it possible to replace f^n by arbitrary polynomial $P[f] = a_0f^n + a_1f^{n-1} + \dots + a_n$ in Theorem D for $l \geq 1$?*

In this paper, we will not only try to find the possible answer of the above Question 1.2, but also improve *Theorem D* to a large extent. We have observed that if we consider the general polynomial $P(f) = a_nf^n + \dots + a_0$ in the place of f^n in the line of the proof of *Theorem D*, we will get a different inequality in comparison to (1.8) such that Example 1.13 is not violating the new condition (1.14) given later on.

Through the paper we shall assume the following notations. Let

$$\mathcal{P}(w) = a_{n+m}w^{n+m} + \dots + a_nw^n + \dots + a_0 = a_{n+m} \prod_{i=1}^s (w - w_{p_i})^{p_i}$$

where $a_j (j = 0, 1, 2, \dots, n + m - 1)$, $a_{n+m} \neq 0$ and $w_{p_i} (i = 1, 2, \dots, s)$ are distinct finite complex numbers and $2 \leq s \leq n + m$ and $p_1, p_2, \dots, p_s$,

$s \geq 2$, n , m and k are all positive integers with $\sum_{i=1}^s p_i = n + m$. Also

let $p > \max_{p \neq p_i, i=1, \dots, r} \{p_i\}$, $r = s - 1$, where s and r are two positive integers.

Let $P(w_1) = a_{n+m} \prod_{i=1}^{s-1} (w_1 + w_p - w_{p_i})^{p_i} = b_q w_1^q + b_{q-1} w_1^{q-1} + \dots + b_0$, where

$a_{n+m} = b_q$, $w_1 = w - w_p$, $q = n + m - p$. Therefore, $\mathcal{P}(w) = w_1^p P(w_1)$.

Next we assume $P(w_1) = b_q \prod_{i=1}^r (w_1 - \alpha_i)^{p_i}$, where $\alpha_i = w_{p_i} - w_p$, ($i = 1, 2, \dots, r$), be distinct zeros of $P(w_1)$.

The following theorem is the main result of this paper which gives an affirmative answer of the questions of *Banerjee - Chakraborty* ([1]) and also the question posed by *Zhang-Lü* ([13]) in a more convenient way.

Theorem 1.1. *Let $k(\geq 1)$, $n(\geq 1)$, $p(\geq 1)$ and $m(\geq 0)$ be integers and f and $f_1 = f - w_p$ be two non-constant meromorphic functions and $M[f]$ be a differential monomial of degree d_M and weight Γ_M and k is the highest*

derivative in $M[f]$. Let $\mathcal{P}(z) = a_{m+n}z^{m+n} + \dots + a_nz^n + \dots + a_0, a_{m+n} \neq 0$, be a polynomial in z of degree $m+n$ such that $\mathcal{P}(f) = f_1^p P(f_1)$. Also let $a(z) (\neq 0, \infty)$ be a small function with respect to f . Suppose $\mathcal{P}(f) - a$ and $M[f] - a$ share $(0, l)$. If $l \geq 2$ and

$$(3 + \lambda)\Theta(\infty, f) + \mu_2\delta_{\mu_2^*}(w_p, f) + d_M\delta_{2+k}(0, f) > \Gamma_M + \mu_2 + 3 - p \quad (1.9)$$

or $l = 1$ and

$$\begin{aligned} & \left(\frac{7}{2} + \lambda\right)\Theta(\infty, f) + \frac{1}{2}\Theta(w_p, f) + \mu_2\delta_{\mu_2^*}(w_p, f) + d_M\delta_{2+k}(0, f) \\ & > \Gamma_M + \mu_2 + 4 + \frac{(m+n) - 3p}{2} \end{aligned} \quad (1.10)$$

or $l = 0$ and

$$\begin{aligned} & (6 + 2\lambda)\Theta(\infty, f) + 2\Theta(w_p, f) + \mu_2\delta_{\mu_2^*}(w_p, f) + d_M \left(\sum_{i=1}^2 \delta_{k+i}(0, f) \right) \\ & > 2\Gamma_M + \mu_2 + 8 + 2(m+n) - 3p \end{aligned} \quad (1.11)$$

then $\mathcal{P}(f) \equiv M[f]$.

The following Corollary can easily be deduced from the above theorem which is an extension and improvement of the Theorem D. It is clear that for $P(z) = 1$ i.e., $m = 0$, we get exactly Theorem E from Corollary 1.1.

Corollary 1.1. Let $k(\geq 1)$, $n(\geq 1)$ and $m(\geq 0)$ be integers and f be a non-constant meromorphic function and $M[f]$ be a differential monomial of degree d_M and weight Γ_M and k is the highest derivative in $M[f]$. Let $P(z) = a_mz^m + \dots + a_0, a_m \neq 0$, be a polynomial in z of degree m . Also let $a(z) (\neq 0, \infty)$ be a small function with respect to f . Suppose $f^n P(f) - a$ and $M[f] - a$ share $(0, l)$. If $l \geq 2$ and

$$(3 + \lambda)\Theta(\infty, f) + \mu_2\delta_{\mu_2^*}(w_p, f) + d_M\delta_{2+k}(0, f) > \Gamma_M + \mu_2 + 3 - n \quad (1.12)$$

or $l = 1$ and

$$\begin{aligned} & \left(\frac{7}{2} + \lambda\right)\Theta(\infty, f) + \frac{1}{2}\Theta(w_p, f) + \mu_2\delta_{\mu_2^*}(w_p, f) + d_M\delta_{2+k}(0, f) \\ & > \Gamma_M + \mu_2 + 4 + \frac{m}{2} - n \end{aligned} \quad (1.13)$$

or $l = 0$ and

$$\begin{aligned} & (6 + 2\lambda)\Theta(\infty, f) + 2\Theta(w_p, f) + \mu_2\delta_{\mu_2^*}(w_p, f) + d_M \left(\sum_{i=1}^2 \delta_{k+i}(0, f) \right) \\ & > 2\Gamma_M + \mu_2 + 8 + 2m - n \end{aligned} \quad (1.14)$$

then $f^n P(f) \equiv M[f]$.

We see that in case of Example 1.13 in [1] we have $\Theta(\infty; f) = 1 = \Theta(0; f) = \delta_q(0; f)$, $p = n = 1$, $m = 1$, $w_p = 0$, $\mu_2 = 1$, $d_M = 1$, $\Gamma_M = 4$, $\lambda = 3$. So when $l = 0$, we get

$$\begin{aligned} & (6 + 2\lambda)\Theta(\infty, f) + 2\Theta(w_p, f) + \mu_2\delta_{\mu_2^*}(w_p, f) + d_M \left(\sum_{i=1}^2 \delta_{k+i}(0, f) \right) \\ &= 17 < 18 = 2\Gamma_M + \mu_2 + 8 + 2m - n, \end{aligned}$$

thus (1.14) ceases to hold and hence Example 1.13 [1] is not violating Corollary 1.1.

However the following question is still open.

Question 1.3. *Is it possible to extend Theorem 1.1 up to differential polynomial $P[f]$ instead of differential monomial $M[f]$?*

Following example shows that in Theorem 1.1 $a(z) \not\equiv 0$ is essential.

Example 1.1. Let us take $f(z) = e^{Nz}$ where $N \neq 0, \pm 1$ and $\mathcal{P}(f) = f^3$, $M[f] = f^{(2)}$. Then $\mathcal{P}(f)$ and $M[f]$ share $a = 0$ (or, ∞). Here $m = 0, p = n = 1, w_p = 0, d_M = 1, \mu_2 = 1, \Gamma_M = 3$ and $\lambda = 2$. Also $\Theta(\infty; f) = 1 = \Theta(0; f)$ and $\delta_q(0; f) = 1, \forall q \in \mathbb{N}$. Thus we see that the deficiency conditions stated in Theorem 1.1 are satisfied but $\mathcal{P}(f) \not\equiv M[f]$.

The next example shows that the deficiency conditions stated in Theorem 1.1 are not necessary.

Example 1.2. Let $f(z) = A \cos z + B \sin z, AB \neq 0$. Then $\overline{N}(r, f) = S(r, f)$ and

$$\overline{N}(r, 0; f) = \overline{N} \left(r, \frac{A + iB}{A - iB}; e^{2iz} \right) \sim T(r, f).$$

Here $m = 0, p = n = 1, w_p = 0, d_M = 1, \mu_2 = 1, \Gamma_M = 4k + 1$ and $\lambda = 4k$. Again $\Theta(\infty, f) = 1$ and $\Theta(0, f) = \delta_p(0, f) = 0$. Let $m = 0$, hence $\mathcal{P}(f) = f$.

Therefore it is clear that $M[f] = f^{(4k)}$, for $k \in \mathbb{N}$ and $\mathcal{P}(f)$ share $a(z)$ and the deficiency conditions in Theorem 1.1 are not satisfied, but $\mathcal{P}(f) \equiv M$.

The following three examples show that the conditions (1.9) - (1.11) in Theorem 1.1 can not be removed.

Example 1.3. Let $f(z) = e^{Nz}$, where N is a non-zero integer. For $n \geq 2$ let

$$\mathcal{P}(f) = -N^{2n} \sum_{r=0}^{2n-1} (-1)^r \binom{2n}{r} f^{2n-r} \quad \text{and} \quad M[f] = f^{(2n)}.$$

Then it is clear that

$$\mathcal{P}(f) - N^{2n} = -N^{2n}(e^{Nz} - 1)^{2n} \quad \text{and} \quad M[f] - N^{2n} = N^{2n}(e^{Nz} - 1).$$

Thus we see that $\mathcal{P}(f)$ and $M[f]$ share $(N^{2n}, 0)$. Here $n + m = 2n$, $p = 1$, $w_p = 0$, $d_M = 1$, $\Gamma_M = 2n + 1$, $\mu_2 = 1$ and $\lambda = 2n$. Also $\Theta(\infty; f) = 1 = \Theta(0; f)$ and $\delta_q(0; f) = 1, \forall q \in \mathbb{N}$.
So for $l = 0$

$$\begin{aligned} & (6 + 2\lambda)\Theta(\infty, f) + 2\Theta(w_p, f) + \mu_2\delta_{\mu_2^*}(w_p, f) + d_M \left(\sum_{i=1}^2 \delta_{k+i}(0, f) \right) \\ &= 4n + 11 \leq 8n + 8 = 2\Gamma_M + \mu_2 + 8 + 2(m + n) - 3p \end{aligned}$$

and we see that $\mathcal{P}(f) \not\equiv M[f]$.

Example 1.4. Let $f(z) = -\sin(\alpha z) + a - \frac{a}{\alpha^{4k}}$, $k \in \mathbb{N}$; where $\alpha \neq 0, \alpha^{4k} \neq 1$ and $a \in \mathbb{C} - \{0\}$. Let $p = n = 1$, $w_p = 0$ and $m = 0$. Then let $\mathcal{P}(f) = f$. Again let $M[f] = f^{(4k)}$. Then $M[f] = -\alpha^{4k} \sin(\alpha z)$. Here $m = 0, \mu_2 = 1, \Gamma_M = 4k + 1, d_M = 1$ and $\lambda = 4k$. Again $\Theta(\infty; f) = 1$ and

$$\overline{N}(r, 0; f) = \overline{N}\left(r, a - \frac{a}{\alpha^{4k}}; \sin(\alpha z)\right) \sim T(r, f).$$

Therefore,

$$\Theta(0; f) = 0 = \delta_q(0; f), \forall q \in \mathbb{N}.$$

Also it is clear that $\mathcal{P}(f)$ and $M[f]$ share (a, l) ($l \geq 0$) but none of the inequalities (1.9), (1.10) and (1.11) of Theorem 1.1 is satisfied and $\mathcal{P}(f) \not\equiv M[f]$.

Example 1.5. Let $f(z) = e^{\beta z} + a - \frac{a}{\beta^2}$; where $a \neq 0, \infty$ and $\beta \neq 0, \pm 1$. Let $p = n = 1$, $w_p = 0$ and $m = 0$. Then let $\mathcal{P}(f) = f$. Again let $M[f] = f^{(2)}$. Then $M[f] = \beta^2 e^{\beta z}$. Here $m = 0, \mu_2 = 1, \Gamma_M = 3, d_M = 1$ and $\lambda = 2$. Again $\Theta(\infty; f) = 1$ and

$$\overline{N}(r, 0; f) = \overline{N}\left(r, \frac{a}{\beta^2} - a; e^{\beta z}\right) \sim T(r, f).$$

Therefore,

$$\Theta(0; f) = 0 = \delta_q(0; f), \forall q \in \mathbb{N}.$$

Also it is clear that $\mathcal{P}(f)$ and $M[f]$ share (a, l) ($l \geq 0$) but none of the inequalities (1.9), (1.10) and (1.11) of Theorem 1.1 is satisfied and $\mathcal{P}(f) \not\equiv M[f]$.

2 Lemmas

In this section we present some Lemmas which will be needed in the sequel. Let F, G be two non-constant meromorphic functions. Henceforth we shall denote by H the following function.

$$H = \left(\frac{F''}{F'} - \frac{2F'}{F-1} \right) - \left(\frac{G''}{G'} - \frac{2G'}{G-1} \right). \quad (2.1)$$

Lemma 2.1. [8] *Let f be a non-constant meromorphic function and let*

$$R(f) = \frac{\sum_{i=0}^n a_i f^i}{\sum_{j=0}^m b_j f^j}$$

be an irreducible rational function in f with constant coefficients $\{a_i\}$ and $\{b_j\}$ where $a_n \neq 0$ and $b_m \neq 0$. Then

$$T(r, R(f)) = dT(r, f) + S(r, f),$$

where $d = \max\{n, m\}$.

Lemma 2.2. ([1]) *For any non-constant meromorphic function f ,*

$$N \left(r, \infty; \frac{M}{f^{d_M}} \right) \leq d_M N(r, 0; f) + \lambda \overline{N}(r, \infty; f) + S(r, f).$$

Lemma 2.3. ([1]) *For the differential monomial $M[f]$,*

$$N_p(r, 0; M[f]) \leq d_M N_{p+k}(r, 0; f) + \lambda \overline{N}(r, \infty; f) + S(r, f).$$

Lemma 2.4. *Let f be a non-constant meromorphic function and $a(z)$ be a small function in f . Let us define $F = \frac{\mathcal{P}(f)}{a} = \frac{f_1^p P(f_1)}{a}$ and $G = \frac{M[f]}{a}$. Then $FG \not\equiv 1$.*

Proof. On contrary suppose $FG \equiv 1$ i.e

$$f_1^p P(f_1) M[f] = a^2.$$

From above it is clear that the function f can't have any zeros and poles. Therefore

$$\overline{N}(r, 0; f) = S(r, f) = \overline{N}(r, \infty; f).$$

So by the First Fundamental Theorem and Lemma 2.1, we have

$$\begin{aligned}
 (m+n+d_M)T(r, f) &= T\left(r, \frac{a^2}{f_1^p P(f_1) f^{d_M}}\right) + S(r, f) \\
 &\leq T\left(r, \frac{M[f]}{f^{d_M}}\right) + S(r, f) \\
 &\leq m\left(r, \frac{M[f]}{f^{d_M}}\right) + N\left(r, \frac{M[f]}{f^{d_M}}\right) + S(r, f) \\
 &\leq N\left(r, \frac{M[f]}{f^{d_M}}\right) + S(r, f).
 \end{aligned}$$

Then using Lemma 2.2 and from above inequality, we get

$$(m+n+d_M)T(r, f) \leq d_M N(r, 0; f) + \lambda \overline{N}(r, f) + S(r, f) \leq S(r, f),$$

which is not possible. $\square$

Lemma 2.5. ([2]) *Let F and G share $(1, l)$ and $\overline{N}(r, F) = \overline{N}(r, G)$ and $H \not\equiv 0$, where F, G and H are defined as earlier. Then*

$$\begin{aligned}
 &N(r, \infty; H) \\
 &\leq \overline{N}(r, \infty; F) + \overline{N}(r, 0; F| \geq 2) + \overline{N}(r, 0; G| \geq 2) + \overline{N}_0(r, 0; F') \\
 &\quad + \overline{N}_0(r, 0; G') + \overline{N}_L(r, 1; F) + \overline{N}_L(r, 1; G) + S(r, f).
 \end{aligned}$$

Lemma 2.6. ([1]) *Let F and G share $(1, l)$. Then*

$$\overline{N}_L(r, 1; F) \leq \frac{1}{l+1} \overline{N}(r, \infty; F) + \frac{1}{l+1} \overline{N}(r, 0; F) + S(r, F) \quad \text{if } l \geq 1,$$

and

$$\overline{N}_L(r, 1; F) \leq \overline{N}(r, \infty; F) + \overline{N}(r, 0; F) + S(r, F) \quad \text{if } l = 0.$$

Lemma 2.7. ([1]) *Let F and G share $(1, l)$ and $H \not\equiv 0$. Then*

$$\begin{aligned}
 &\overline{N}(r, 1; F) + \overline{N}(r, 1; G) \\
 &\leq N(r, \infty; H) + \overline{N}_E^{(2)}(r, 1; F) + \overline{N}_L(r, 1; F) + \overline{N}_L(r, 1; G) \\
 &\quad + \overline{N}(r, 1; G) + S(r, f).
 \end{aligned}$$

The following lemma can be proved in the line of proof of Lemma 2.14 [1].

Lemma 2.8. *Let f be a non-constant meromorphic function and $a(z)$ be a small function of f . Let $F = \frac{\mathcal{P}(f)}{a} = \frac{f_1^p P(f_1)}{a}$ and $G = \frac{M[f]}{a}$ such that F and G shares $(1, \infty)$. Then one of the following cases holds:*

$$1. \quad T(r) \leq N_2(r, 0; F) + N_2(r, 0; G) + \overline{N}(r, \infty; F) + \overline{N}(r, \infty; G) + \overline{N}_L(r, \infty; F) \\ + \overline{N}_L(r, \infty; G) + S(r),$$

$$2. \quad F \equiv G,$$

$$3. \quad FG \equiv 1.$$

where $T(r) = \max\{T(r, F), T(r, G)\}$ and $S(r) = o(T(r))$, $r \in I$, I is a set of infinite linear measure of $r \in (0, \infty)$.

3 Proof of the theorem

Proof of Theorem 1.1. Let $F = \frac{\mathcal{P}(f)}{a} = \frac{f_1^p P(f_1)}{a}$ and $G = \frac{M[f]}{a}$. Then $F - 1 = \frac{f_1^p P(f_1) - a}{a}$ and $G - 1 = \frac{M[f] - a}{a}$. Since $\mathcal{P}(f)$ and $M[f]$ share (a, l) , it follows that F and G share $(1, l)$ except the zeros and poles of $a(z)$. Now we consider the following cases.

Case 1 Let $H \not\equiv 0$.

Subcase 1.1. Let $l \geq 1$.

Using the Second Fundamental Theorem and Lemmas 2.7, 2.5 we get

$$\begin{aligned} & T(r, F) + T(r, G) \\ & \leq \overline{N}(r, \infty; F) + \overline{N}(r, \infty; G) + \overline{N}(r, 0; F) + \overline{N}(r, 0; G) + \overline{N}(r, 1; F) \\ & \quad + \overline{N}(r, 1; G) - \overline{N}_0(r, 0; F') - \overline{N}_0(r, 0; G') + S(r, F) + S(r, G) \\ & \leq \overline{N}(r, \infty; F) + \overline{N}(r, \infty; G) + \overline{N}(r, 0; F) + \overline{N}(r, 0; G) + N(r, H) \\ & \quad + \overline{N}_E^{(2)}(r, 1; F) + \overline{N}_L(r, 1; F) + \overline{N}_L(r, 1; G) + \overline{N}(r, 1; G) \\ & \quad - \overline{N}_0(r, 0; F') - \overline{N}_0(r, 0; G') + S(r, f) \\ & \leq 2\overline{N}(r, \infty; F) + \overline{N}(r, \infty; G) + N_2(r, 0; F) + N_2(r, 0; G) + \overline{N}_E^{(2)}(r, 1; F) \\ & \quad + 2\overline{N}_L(r, 1; F) + 2\overline{N}_L(r, 1; G) + \overline{N}(r, 1; G) + S(r, f). \end{aligned}$$

Subsubcase 1.1.1. Suppose $l = 1$.

Then from the above inequality and using Lemmas 2.6, 2.3 we get

$$\begin{aligned}
& T(r, F) + T(r, G) \\
& \leq 2\overline{N}(r, \infty; F) + \overline{N}(r, \infty; G) + N_2(r, 0; F) + N_2(r, 0; G) + \overline{N}_E^{(2)}(r, 1; F) \\
& \quad + 2\overline{N}_L(r, 1; F) + 2\overline{N}_L(r, 1; G) + \overline{N}(r, 1; G) + S(r, f) \\
& \leq 3\overline{N}(r, \infty; F) + N_2(r, 0; f_1^p P(f_1)) + N_2(r, 0; M[f]) + \frac{1}{2}\overline{N}(r, \infty; F) \\
& \quad + \frac{1}{2}\overline{N}(r, 0; F) + N(r, 1; G) + S(r, f) \\
& \leq 3\overline{N}(r, \infty; f) + \mu_2 N_{\mu_2^*}(r, w_p; f) + (m + n - p)T(r, f) + d_M N_{k+2}(r, 0; f) \\
& \quad + \lambda \overline{N}(r, \infty; f) + \frac{1}{2}\overline{N}(r, \infty; F) + \frac{1}{2}\overline{N}(r, 0; F) + N(r, 1; G) + S(r, f) \\
& \leq \left(\frac{7}{2} + \lambda\right) \overline{N}(r, \infty; f) + \mu_2 N_{\mu_2^*}(r, w_p; f) + (m + n - p)T(r, f) \\
& \quad + d_M N_{k+2}(r, 0; f) + \frac{1}{2}\overline{N}(r, w_p; f) + \frac{1}{2}(m + n - p)T(r, f) + N(r, 1; G) \\
& \quad + S(r, f). \\
& \leq \left(\frac{7}{2} + \lambda\right) \overline{N}(r, \infty; f) + \mu_2 N_{\mu_2^*}(r, w_p; f) + \frac{3}{2}(m + n - p)T(r, f) \\
& \quad + d_M N_{k+2}(r, 0; f) + \frac{1}{2}\overline{N}(r, w_p; f) + T(r, G) + S(r, f).
\end{aligned}$$

i.e., in view of Lemma 2.1, for any given $\varepsilon > 0$

$$\begin{aligned}
& (m + n)T(r, f) \\
& \leq \left(\frac{7}{2} + \lambda\right) \overline{N}(r, \infty; f) + \mu_2 N_{\mu_2^*}(r, w_p; f) + \frac{3}{2}(m + n - p)T(r, f) \\
& \quad + d_M N_{k+2}(r, 0; f) + \frac{1}{2}\overline{N}(r, w_p; f) + S(r, f). \\
& \leq \left\{ \left(\frac{7}{2} + \lambda\right) - \left(\frac{7}{2} + \lambda\right) \Theta(\infty, f) + \frac{1}{2} - \frac{1}{2}\Theta(w_p, f) + \mu_2 - \mu_2 \delta_{\mu_2^*}(w_p, f) \right. \\
& \quad \left. + d_M - d_M \delta_{2+k}(0, f) + \frac{3}{2}(m + n - p) + \varepsilon \right\} T(r, f) + S(r, f).
\end{aligned}$$

i.e.,

$$\begin{aligned}
& \left\{ \left(\frac{7}{2} + \lambda\right) \Theta(\infty, f) + \frac{1}{2}\Theta(w_p, f) + \mu_2 \delta_{\mu_2^*}(w_p, f) + d_M \delta_{2+k}(0, f) - \varepsilon \right\} T(r, f) \\
& \leq \left(\Gamma_M + \mu_2 + 4 + \frac{(m + n) - 3p}{2} \right) T(r, f) + S(r, f),
\end{aligned}$$

which is a contradiction.

Subsubcase 1.1.2. Suppose $l \geq 2$.

Here by using Lemma 2.3, we obtained

$$\begin{aligned}
& T(r, F) + T(r, G) \\
& \leq 2\overline{N}(r, \infty; F) + \overline{N}(r, \infty; G) + N_2(r, 0; F) + N_2(r, 0; G) + \overline{N}_E^{(2)}(r, 1; F) \\
& \quad + 2\overline{N}_L(r, 1; F) + 2\overline{N}_L(r, 1; G) + \overline{N}(r, 1; G) + S(r, f) \\
& \leq 3\overline{N}(r, \infty; f) + \mu_2 N_{\mu_2^*}(r, w_p; f) + (m + n - p)T(r, f) + d_M N_{k+2}(r, 0; f) \\
& \quad + \lambda \overline{N}(r, \infty; f) + N(r, 1; G) + S(r, f) \\
& \leq (3 + \lambda)\overline{N}(r, \infty; f) + \mu_2 N_{\mu_2^*}(r, w_p; f) + (m + n - p)T(r, f) \\
& \quad + d_M N_{k+2}(r, 0; f) + T(r, G) + S(r, f).
\end{aligned}$$

So, in view of Lemma 2.1, for any given $\varepsilon > 0$ we have

$$\begin{aligned}
& (m + n)T(r, f) \\
& \leq (3 + \lambda)\overline{N}(r, \infty; f) + \mu_2 N_{\mu_2^*}(r, w_p; f) + (m + n - p)T(r, f) \\
& \quad + d_M N_{2+k}(r, 0; f) + S(r, f) \\
& \leq \{(3 + \lambda) - (3 + \lambda)\Theta(\infty, f) + \mu_2 - \mu_2 \delta_{\mu_2^*}(w_p, f) + d_M - d_M \delta_{2+k}(0, f) \\
& \quad + (m + n - p) + \varepsilon\}T(r, f) + S(r, f)
\end{aligned}$$

i.e.,

$$\begin{aligned}
& \left\{ (3 + \lambda)\Theta(\infty, f) + \mu_2 \delta_{\mu_2^*}(w_p, f) + d_M \delta_{2+k}(0, f) - \varepsilon \right\} T(r, f) \\
& \leq (\Gamma_M + \mu_2 + 3 - p)T(r, f) + S(r, f),
\end{aligned}$$

which is a contradiction.

Subcase 1.2. Suppose $l = 0$.

Then in view of the Second Fundamental Theorem and Lemmas 2.7, 2.5 we get

$$\begin{aligned}
& T(r, F) + T(r, G) \\
& \leq \overline{N}(r, \infty; F) + \overline{N}(r, 0; F) + \overline{N}(r, 1; F) + \overline{N}(r, \infty; G) + \overline{N}(r, 0; G) \\
& \quad + \overline{N}(r, 1; G) - \overline{N}_0(r, 0; F') - \overline{N}_0(r, 0; G') + S(r, F) + S(r, G) \\
& \leq \overline{N}(r, \infty; F) + \overline{N}(r, 0; F) + \overline{N}(r, \infty; G) + \overline{N}(r, 0; G) + N(r, \infty; H) \\
& \quad + \overline{N}_E^{(2)}(r, 1; F) + \overline{N}_L(r, 1; F) + \overline{N}_L(r, 1; G) + \overline{N}(r, 1; G) \\
& \quad - \overline{N}_0(r, 0; F') - \overline{N}_0(r, 0; G') + S(r, F) + S(r, G) \\
& \leq 2\overline{N}(r, \infty; F) + \overline{N}(r, \infty; G) + N_2(r, 0; F) + N_2(r, 0; G) + \overline{N}_E^{(2)}(r, 1; F) \\
& \quad + 2\overline{N}_L(r, 1; F) + 2\overline{N}_L(r, 1; G) + \overline{N}(r, 1; G) + S(r, f)
\end{aligned}$$

Again by Lemmas 2.6, 2.3 we get from above

$$\begin{aligned}
& T(r, F) + T(r, G) \\
& \leq 2\overline{N}(r, \infty; F) + \overline{N}(r, \infty; G) + \mu_2 N_{\mu_2^*}(r, w_p, f) + (m + n - p)T(r, f) \\
& \quad + N_2(r, 0; G) + 2(\overline{N}(r, \infty; F) + \overline{N}(r, 0; F)) + \overline{N}(r, \infty; G) + \overline{N}(r, 0; G) \\
& \quad + \overline{N}_E^2(r, 1; F) + \overline{N}_L(r, 1; G) + \overline{N}(r, 1; G) + S(r, f) \\
& \leq 4\overline{N}(r, \infty; F) + \mu_2 N_{\mu_2^*}(r, w_p, f) + (m + n - p)T(r, f) + N_2(r, 0; G) \\
& \quad + 2\overline{N}(r, \infty; G) + \overline{N}(r, 0; G) + 2\overline{N}(r, 0; F) + T(r, G) + S(r, f)
\end{aligned}$$

i.e., for any given $\varepsilon > 0$

$$\begin{aligned}
& (m + n)T(r, f) \\
& \leq (6 + 2\lambda)\overline{N}(r, \infty; f) + 2\overline{N}(r, w_p; f) + 2(m + n - p)T(r, f) \\
& \quad + \mu_2 N_{\mu_2^*}(r, w_p, f) + (m + n - p)T(r, f) + d_M N_{1+k}(r, 0; f) \\
& \quad + d_M N_{2+k}(r, 0; f) + S(r, f) \\
& \leq \left\{ (6 + 2\lambda) - (6 + 2\lambda)\Theta(\infty, f) + 2 - 2\Theta(w_p, f) + \mu_2 - \mu_2 \delta_{\mu_2^*}(w_p, f) \right. \\
& \quad \left. + 2d_M - d_M \left(\sum_{i=1}^2 \delta_{k+i}(0, f) \right) + 3(m + n - p) + \varepsilon \right\} T(r, f) + S(r, f).
\end{aligned}$$

i.e.,

$$\begin{aligned}
& \left\{ (6 + 2\lambda)\Theta(\infty, f) + 2\Theta(w_p, f) + \mu_2 \delta_{\mu_2^*}(w_p, f) \right. \\
& \quad \left. + d_M \left(\sum_{i=1}^2 \delta_{k+i}(0, f) \right) - \varepsilon \right\} T(r, f) \\
& \leq (2\Gamma_M + \mu_2 + 8 + 2(m + n) - 3p)T(r, f) + S(r, f),
\end{aligned}$$

which is a contradiction.

Case 2. Let $H \equiv 0$.

On Integration we get,

$$\frac{1}{G-1} \equiv \frac{A}{F-1} + B, \tag{3.1}$$

where $A (\neq 0)$, B are complex constants.

It is clear that F and G share $(1, \infty)$. Also by construction of F and G we see that F and G share $(\infty, 0)$ also.

So using Lemma 2.3 and condition (1.9), we obtain

$$\begin{aligned}
& N_2(r, 0; F) + N_2(r, 0; G) + \overline{N}(r, \infty; F) + \overline{N}(r, \infty; G) + \overline{N}_L(r, \infty; F) \\
& + \overline{N}_L(r, \infty; G) + S(r) \\
& \leq \mu_2 N_{\mu_2^*}(r, w_p; f) + N_2(r, 0; P(f)) + d_M N_{2+k}(r, 0; f) + \lambda \overline{N}(r, \infty; f) \\
& + 3\overline{N}(r, \infty; f) + S(r) \\
& \leq \mu_2 N_{\mu_2^*}(r, w_p; f) + (m + n - p)T(r, f) + d_M N_{2+k}(r, 0; f) \\
& + (3 + \lambda)\overline{N}(r, \infty; f) + S(r) \\
& \leq \{(3 + \lambda + d_M + \mu_2 + m + n - p) - ((\lambda + 3)\Theta(\infty, f) + \mu_2 \delta_{\mu_2^*}(w_p, f)) \\
& + d_M \delta_{2+k}(0, f)\} \\
& \quad T(r, f) + S(r) \\
& \leq \{(3 + \Gamma_M + \mu_2 + m + n - p) - (3 + \Gamma_M + \mu_2 - p)\}T(r, f) + S(r) \\
& \leq (m + n)T(r, f) + S(r) \\
& < T(r, F) + S(r).
\end{aligned}$$

Hence inequality (1) of Lemma 2.8 does not hold. Again in view of Lemma 2.4, we get $FG \not\equiv 1$. Therefore $F \equiv G$ i.e, $\mathcal{P}(f) \equiv M[f]$. $\square$

Acknowledgement

This research work is supported by the Council Of Scientific and Industrial Research, Extramural Research Division, CSIR Complex, Pusa, New Delhi-110012, India, under the sanction project no. 25(0229)/14/EMR-II.

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Received: 30.05.2016

Accepted: 25.07.2016