Holomorphy of Osborn loops

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Abstract. Let $(L, \cdot)$ be any loop and let $A(L)$ be a group of automorphisms of $(L, \cdot)$ such that α and ϕ are elements of $A(L)$. It is shown that, for all $x, y, z \in L$, the $A(L)$ -holomorph $(H, \circ) = H(L)$ of $(L, \cdot)$ is an Osborn loop if and only if $x\alpha(yz \cdot x\phi^{-1}) = x\alpha(yx^\lambda \cdot x) \cdot zx\phi^{-1}$. Furthermore, it is shown that for all $x \in L$, $H(L)$ is an Osborn loop if and only if $(L, \cdot)$ is an Osborn loop, $(x\alpha \cdot x^\rho)x = x\alpha$, $x(x^\lambda \cdot x\phi^{-1}) = x\phi^{-1}$ and every pair of automorphisms in $A(L)$ is nuclear (i.e. $x\alpha \cdot x^\rho, x^\lambda \cdot x\phi \in N(L, \cdot)$). It is shown that if $H(L)$ is an Osborn loop, then $A(L, \cdot) = \mathcal{P}(L, \cdot) \cap \Lambda(L, \cdot) \cap \Phi(L, \cdot) \cap \Psi(L, \cdot)$ and for any $\alpha \in A(L)$, $\alpha = L_{e\pi} = R_{e\rho}^{-1}$ for some $\pi \in \Phi(L, \cdot)$ and some $\rho \in \Psi(L, \cdot)$. Some commutative diagrams are deduced by considering isomorphisms among the various groups of regular bijections (whose intersection is $A(L)$) and the nucleus of $(L, \cdot)$.

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1 Introduction

The holomorph of a loop is a loop according to Bruck [10]. Since then, the concept of holomorphy of loops has caught the attention of some researchers. Interestingly, Adéníran [1] and Robinson [53], Chein and Robinson [12], Adéníran et. al. [3], Chiboka and Solarin [14], [15], Bruck [10], Bruck and Paige [11], Robinson [52], Huthnance [23], Adéníran et. al. [4] and,

Jaiyéḡlá and Popoola [34] have respectively studied the holomorphic structures of Bol/Bruck loops, Moufang loops, central loops, conjugacy closed loops, inverse property loops, A-loops, extra loops, weak inverse property loops and generalized Bol loops.

After the discovery of Osborn loops by Osborn [50] and Huthnance [23], Osborn loops were formally introduced and studied by Basarab [5–9], in the 20th century. In this 21st century, the study of Osborn loops was recently revived by Kinyon [42], where he proposed some problems and heart burning questions. Some of this problems and questions have been solved, answered fully or partially or somewhat addressed in Jaiyéḡlá et. al. [39–41], Jaiyéḡlá and Adéníran [35–37] and Jaiyéḡlá [28, 30, 31, 33]. Some results on the application of Osborn loops to cryptography can be found in Jaiyéḡlá and Adéníran [38] and Jaiyéḡlá [29, 32].

Some popular varieties of Osborn loops are: extra loops, Moufang loops, CC-loops, universal WIPs and V.D. loops. Some studies on them can be found in Drápal [18–22], Csörgő and Drápal [17], Csörgő [16], Kinyon and Kunen [43, 44], Kinyon et. al. [45]. Some newly constructed Osborn loops can be found in Isere et. al. [24, 26, 27], Adéníran and Isere [2].

2 Preliminaries

Let G be a non-empty set. Define a binary operation $(\cdot)$ on G . If $x \cdot y \in G$ for all $x, y \in G$, then the pair $(G, \cdot)$ is called a groupoid or Magma.

If each of the equations:

$$a \cdot x = b \quad \text{and} \quad y \cdot a = b$$

has unique solution in G for x and y respectively, then $(G, \cdot)$ is called a quasigroup.

If there exists a unique element $e \in G$ called the identity element such that for all $x \in G$, $x \cdot e = e \cdot x = x$, $(G, \cdot)$ is called a loop. We write xy instead of $x \cdot y$, and stipulate that $\cdot$ has lower priority than juxtaposition among factors to be multiplied. For instance, $x \cdot yz$ stands for $x(yz)$.

For a groupoid $(G, \cdot)$, the right translation of x i.e. $R_x : G \rightarrow G$ is defined by $yR_x = y \cdot x$ while the left translation of x i.e. $L_x : G \rightarrow G$ is defined by $yL_x = x \cdot y$ for all $x, y \in G$.

It can now be seen that a groupoid $(G, \cdot)$ is a quasigroup if its left and right translation mappings are bijections or permutations. Since the left and right translation mappings of a loop are bijective, then the inverse mappings

L_x^{-1} and R_x^{-1} exist. Let

$$x \backslash y = yL_x^{-1} \quad \text{and} \quad x/y = xR_y^{-1}$$

and note that

$$x \backslash y = z \iff x \cdot z = y \quad \text{and} \quad x/y = z \iff z \cdot y = x.$$

Hence, $(G, \backslash)$ and $(G, /)$ are also quasigroups. Using the operations $(\backslash)$ and $(/)$, the definition of a loop can be stated as follows.

Definition 2.1. A loop $(G, \cdot, /, \backslash, e)$ is a set G together with three binary operations $(\cdot)$, $(/)$, $(\backslash)$ and one nullary operation e such that

- (i) $x \cdot (x \backslash y) = y$, $(y/x) \cdot x = y$ for all $x, y \in G$,
- (ii) $x \backslash (x \cdot y) = y$, $(y \cdot x)/x = y$ for all $x, y \in G$ and
- (iii) $x \backslash x = y/y$ or $e \cdot x = x$ for all $x, y \in G$.

We also stipulate that $(/)$ and $(\backslash)$ have higher priority than $(\cdot)$ among factors to be multiplied. For instance, $x \cdot y/z$ and $x \cdot y \backslash z$ stand for $x(y/z)$ and $x \cdot (y \backslash z)$ respectively.

In a loop $(G, \cdot)$ with identity element e , the left inverse element of $x \in G$ is the element $xJ_\lambda = x^\lambda \in G$ such that

$$x^\lambda \cdot x = e$$

while the right inverse element of $x \in G$ is the element $xJ_\rho = x^\rho \in G$ such that

$$x \cdot x^\rho = e.$$

A loop is called an Osborn loop if it obeys any of the three identities

$$x(yz \cdot x) = (x^\lambda \backslash y) \cdot zx \tag{2.1}$$

$$x(yz \cdot x) = x(yx^\lambda \cdot x) \cdot zx \tag{2.2}$$

$$x(yz \cdot x) = x(yx \cdot x^\rho) \cdot zx \tag{2.3}$$

Given any two sets X and Y . The statement ' $f : X \rightarrow Y$ ' is defined as ' $f(x) = y$, $x \in X$, $y \in Y$ ' will be expressed as ' $f : X \rightarrow Y \uparrow f(x) = y$ '.

Let $(G, \cdot)$ be a loop and let A, B and C be three bijective mappings, that map G onto G . The identity mapping on G will be denoted by I . The triple $\alpha = (A, B, C)$ is called an autotopism of $(G, \cdot)$ if and only if

$$xA \cdot yB = (x \cdot y)C \quad \forall x, y \in G.$$

Such triples form a group $AUT(G, \cdot)$ called the autotopism group of $(G, \cdot)$ under the binary operation of componentwise composition. That is, for $(A_1, B_1, C_1), (A_2, B_2, C_2) \in AUT(G, \cdot)$, $(A_1, B_1, C_1)(A_2, B_2, C_2) = (A_1A_2, B_1B_2, C_1C_2)$.

If $A = B = C$, then A is called an automorphism of the loop $(G, \cdot)$. Such bijections form a group $AUM(G, \cdot)$ called the automorphism group of $(G, \cdot)$. Let G and H be groups such that $\varphi : G \rightarrow H$ is an isomorphism. If $\varphi(g) = h$, then this would be expressed as $g \stackrel{\varphi}{\cong} h$.

Definition 2.2. Let $(Q, \cdot)$ be a loop and $A(Q) \leq AUM(Q, \cdot)$ be a group of automorphisms of the loop $(Q, \cdot)$. Let $H = A(Q) \times Q$. Define $\circ$ on H as

$$(\alpha, x) \circ (\beta, y) = (\alpha\beta, x\beta \cdot y) \text{ for all } (\alpha, x), (\beta, y) \in H.$$

$(H, \circ)$ is a loop and is called the A -holomorph of $(Q, \cdot)$.

The right nucleus of $(L, \cdot)$ is defined by $N_\rho(L, \cdot) = \{x \in L \mid zy \cdot x = z \cdot yx \ \forall y, z \in L\}$. The left nucleus of $(L, \cdot)$ is defined by $N_\lambda(L, \cdot) = \{x \in L \mid x \cdot yz = xy \cdot z \ \forall y, z \in L\}$. The middle nucleus of $(L, \cdot)$ is defined by $N_\mu(L, \cdot) = \{x \in L \mid zx \cdot y = z \cdot xy \ \forall y, z \in L\}$. The nucleus of $(L, \cdot)$ is defined by $N(L, \cdot) = N_\rho(L, \cdot) \cap N_\lambda(L, \cdot) \cap N_\mu(L, \cdot)$. The centrum of $(L, \cdot)$ is defined by $C(L, \cdot) = \{a \in L : ax = xa \ \forall x \in L\}$ while its center is defined by $Z(L, \cdot) = N(L, \cdot) \cap C(L, \cdot)$.

Let $(G, \cdot)$ be a quasigroup. Then

1. a bijection U is called autotopic if there exists $(U, V, W) \in AUT(G, \cdot)$; the set of all such mappings forms a group $\Sigma(G, \cdot)$.
2. a bijection U is called ρ -regular if there exists $(I, U, U) \in AUT(G, \cdot)$; the set of all such mappings forms a group $\mathcal{P}(G, \cdot)$.
3. a bijection U is called λ -regular if there exists $(U, I, U) \in AUT(G, \cdot)$; the set of all such mappings forms a group $\Lambda(G, \cdot) \leq \Sigma(G, \cdot)$.
4. a bijection U is called μ -regular if there exists a bijection U' such that $(U, U'^{-1}, I) \in AUT(G, \cdot)$. U' is called the adjoint of U . The set of all μ -regular mappings forms a group $\Phi(G, \cdot) \leq \Sigma(G, \cdot)$. The set of all adjoint mapping forms a group $\Psi(G, \cdot)$.

Theorem 2.1. (*Jaiyéglá [34]*) Let $(G, \cdot)$ be a loop. Let

$$\begin{aligned} \psi : \mathcal{P}(G, \cdot) &\rightarrow N_\rho(G, \cdot) \uparrow \psi(U) = eU, \delta : \Lambda(G, \cdot) \rightarrow N_\lambda(G, \cdot) \uparrow \delta(U) = eU, \\ \varphi : \Phi(G, \cdot) &\rightarrow \Psi(G, \cdot) \uparrow \varphi(U) = U', \sigma : \Phi(G, \cdot) \rightarrow N_\mu(G, \cdot) \uparrow \\ \sigma(U) &= eU \text{ and } \beta : \Psi(G, \cdot) \rightarrow N_\mu(G, \cdot) \uparrow \beta(U') = eU' \end{aligned}$$

$$\text{Then, } \mathcal{P}(G, \cdot) \stackrel{\psi}{\cong} N_\rho(G, \cdot), \Lambda(G, \cdot) \stackrel{\delta}{\cong} N_\lambda(G, \cdot), \Phi(G, \cdot) \stackrel{\varphi}{\cong} \Psi(G, \cdot),$$

$$\Phi(G, \cdot) \stackrel{\sigma}{\cong} N_\mu(G, \cdot), \Psi(G, \cdot) \stackrel{\beta}{\cong} N_\mu(G, \cdot).$$

3 Main Results

Theorem 3.1. Let $(L, \cdot)$ be a loop and $A(L)$ be a group of automorphisms of $(L, \cdot)$. Then, the $A(L)$ -holomorph $(H, \circ)$ of $(L, \cdot)$ is an Osborn loop if and only if

$$x\alpha(yz \cdot x\phi^{-1}) = x\alpha(yx^\lambda \cdot x) \cdot zx\phi^{-1} \quad \forall x, y, z \in L \text{ and } \alpha, \phi \in A(L) \quad (3.1)$$

Proof. Suppose $A(L)$ -holomorph $(H, \circ)$ of $(L, \cdot)$ is an Osborn loop, then we have

$$\begin{aligned} (\alpha, x) \circ \{[(\beta, y) \circ (\gamma, z)] \circ (\alpha, x)\} &= (\alpha, x) \circ \{[(\beta, y) \circ (\alpha, x)^\lambda] \circ (\alpha, x)\} \{(\gamma, z) \circ (\alpha, x)\} \\ \Leftrightarrow (\alpha(\beta\gamma\alpha), x\beta\gamma\alpha[(y\gamma \cdot z)\alpha \cdot x]) &= (\alpha, x) \circ \{(\beta\alpha^{-1}, yx^\lambda \cdot x) \circ (\alpha, x)\} \circ (\gamma\alpha, z\alpha \cdot x) \\ \Leftrightarrow (\alpha(\beta\gamma\alpha), x\beta\gamma\alpha[(y\gamma \cdot z)\alpha \cdot x]) &= [(\alpha, x) \circ (\beta, yx^\lambda \cdot x)] \circ (\gamma\alpha, z\alpha \cdot x) \\ \Leftrightarrow (\alpha(\beta\gamma\alpha), x\beta\gamma\alpha[(y\gamma \cdot z)\alpha \cdot x]) &= (\alpha(\beta\gamma\alpha), (x\beta(yx^\lambda \cdot x))\gamma\alpha \cdot (z\alpha \cdot x)) \\ \Leftrightarrow x\beta\gamma\alpha[(y\gamma \cdot z)\alpha \cdot x] &= (x\beta \cdot (yx^\lambda \cdot x))\gamma\alpha \cdot (z\alpha \cdot x) \Leftrightarrow \\ x\beta\gamma\alpha[(y\gamma\alpha \cdot z\alpha) \cdot x] &= (x\beta\gamma\alpha((yx^\lambda)\gamma\alpha \cdot x\gamma\alpha))(z\alpha \cdot x) \quad \forall x, y, z \in L \end{aligned}$$

and $\alpha, \beta, \gamma \in A(L)$. Putting $\phi = \gamma\alpha$, we have

$$x\beta\phi[(y\phi \cdot z\alpha) \cdot x] = (x\beta\phi \cdot ((yx^\lambda)\phi \cdot x\phi))(z\alpha \cdot x) \quad \forall x, y, z \in L \text{ and } \alpha, \beta, \phi \in A(L).$$

Therefore,

$$x\beta[(y \cdot z\alpha\phi^{-1})x\phi^{-1}] = (x\beta \cdot (yx^\lambda \cdot x))(z\alpha\phi^{-1} \cdot x\phi^{-1}).$$

Letting $\bar{x} = x\phi^{-1}$ and $x = \bar{x}\phi$, $\bar{z} = z\alpha\phi^{-1}$, we obtain

$$\bar{x}\phi\beta(y\bar{z} \cdot \bar{x}) = \bar{x}\phi\beta(y(\bar{x}\phi)^\lambda \cdot \bar{x}\phi) \cdot \bar{z}\bar{x}$$

Again, since ϕ is an automorphism, then letting $\bar{x}\phi = x$ and $\bar{x} = x\phi^{-1}$, and replacing $\bar{z}$ with z and β with α , we obtain

$$x\alpha(yz \cdot x\phi^{-1}) = x\alpha(yx^\lambda \cdot x) \cdot zx\phi^{-1} \quad \forall x, y, z \in L \text{ and } \alpha, \phi \in A(L).$$

The converse is obtained by reversing the process. $\square$

Corollary 3.2. *Let $(L, \cdot)$ be a loop, and $A(L)$ be the group of all automorphisms of L . Then, the holomorph $(H, \circ)$ of $(L, \cdot)$ is an Osborn loop if and only if*

$$(R_{x^\lambda}R_xL_{x\alpha}, R_{x\phi^{-1}}, R_{x\phi^{-1}}L_{x\alpha})$$

is an autotopism of L for all $x \in L$ and all $\alpha, \phi \in A(L)$.

Proof. This is a consequence of Theorem 3.1. $\square$

Lemma 3.3. *Let $A(L)$ be an automorphism group of an Osborn loop $(L, \cdot)$. The holomorph $(H, \circ)$ of $(L, \cdot)$ is Osborn if and only if the triples*

$$(L_x^{-1}L_{x\alpha}, I, L_x^{-1}L_{x\alpha}) \text{ and } (I, R_x^{-1}R_{x\phi^{-1}}, L_x^{-1}R_x^{-1}R_{x\phi^{-1}}L_x)$$

are autotopisms of L for all $x \in L$ and all $\alpha \in A(L)$.

Proof. Let

$$A = (R_{x^\lambda}R_xL_{x\alpha}, R_{x\phi^{-1}}, R_{x\phi^{-1}}L_{x\alpha}) \text{ and } B = (R_{x^\lambda}R_xL_x, R_x, R_xL_x) \quad (3.2)$$

Since $(L, \cdot)$ is an Osborn loop, B is an autotopism of L for all $x \in L$. The holomorph of $(L, \cdot)$ is an Osborn if and only if A is autotopism of L (by Corollary 3.2). So, the triple

$$B^{-1} = (L_x^{-1}R_x^{-1}R_{x^\lambda}^{-1}, R_x^{-1}, L_x^{-1}R_x^{-1}) \quad (3.3)$$

is also an autotopism of L for all $x \in L$. Hence, $(H, \circ)$ is an Osborn loop if and only if

$$B^{-1}A = (L_x^{-1}L_{x\alpha}, R_x^{-1}R_{x\phi^{-1}}, L_x^{-1}R_x^{-1}R_{x\phi^{-1}}L_{x\alpha}) \in AUT(L, \cdot). \quad (3.4)$$

$$\text{Thus, } yL_x^{-1}L_{x\alpha} \cdot zR_x^{-1}R_{x\phi^{-1}} = (yz)L_x^{-1}R_x^{-1}R_{x\phi^{-1}}L_{x\alpha}$$

$$\Leftrightarrow [(x\alpha) \cdot (x \setminus y)] \cdot [(z/x) \cdot x\phi^{-1}] = (x\alpha) \cdot \{[x \setminus (yz)]/x \cdot x\phi^{-1}\}. \quad (3.5)$$

Put $\phi = I$ into equation (3.5) to get

$$[(x\alpha) \cdot (x \setminus y)] \cdot z = [(x\alpha) \cdot (x \setminus (yz))] \quad (3.6)$$

$$\Leftrightarrow yL_x^{-1}L_{x\alpha} \cdot z = (yz)L_x^{-1}L_{x\alpha} \Leftrightarrow (L_x^{-1}L_{x\alpha}, I, L_x^{-1}L_{x\alpha}) \in AUT(L, \cdot).$$

Now, putting $\alpha = I$ into equation (3.5), we obtain

$$\begin{aligned} [x \cdot (x \setminus y)] \cdot [(z/x) \cdot x\phi^{-1}] &= x \cdot \{[x \setminus (yz)]/x \cdot x\phi^{-1}\} \Leftrightarrow y \cdot zR_x^{-1}R_{x\phi^{-1}} = \\ &\{[x \setminus (yz)]/x \cdot x\phi^{-1}\}L_x \Leftrightarrow (I, R_x^{-1}R_{x\phi^{-1}}, L_x^{-1}R_x^{-1}R_{x\phi^{-1}}L_x) \in AUT(L, \cdot). \end{aligned}$$

The converse follows from Theorem 3.1 and Corollary 3.2. $\square$

Theorem 3.4. *Let $A(L)$ be an automorphism group of a loop $(L, \cdot)$. The holomorph $(H, \circ)$ of $(L, \cdot)$ is an Osborn loop if and only if:*

- (i) $(L, \cdot)$ is an Osborn loop,
- (ii) $x\alpha \cdot x^\rho, x^\lambda \cdot x\phi \in N(L, \cdot)$,
- (iii) $(x\alpha \cdot x^\rho)x = x\alpha$,
- (iv) $x(x^\lambda \cdot x\phi^{-1}) = x\phi^{-1}$,

for every $x, y \in L$ and $\alpha, \phi \in A(L)$.

Proof. (i) Suppose $(H, \circ)$ is an Osborn loop. $(K, \circ)$ is a subloop of $(H, \circ)$ given by $K = \{(I, x) : x \in L\}$. Therefore, $(L, \cdot) \cong (K, \circ)$. Since $(K, \circ)$ is an Osborn loop, it follows that $(L, \cdot)$ is an Osborn loop.

(ii) Since $(H, \circ)$ is Osborn, then by Lemma 3.3, we have

$$(L_x^{-1}L_{x\alpha}, I, L_x^{-1}L_{x\alpha}) \in AUT(L, \cdot)$$

$$\Leftrightarrow yL_x^{-1}L_{x\alpha} \cdot zI = (yz)L_x^{-1}L_{x\alpha} \text{ for all } y, z \in (L, \cdot). \quad (3.7)$$

Putting $y = e$ in equation (3.7) gives: $eL_x^{-1}L_{x\alpha} \cdot zI = (ez)L_x^{-1}L_{x\alpha} \Rightarrow (x\alpha \cdot x^\rho)z = (x\alpha)(x \setminus z)$

$$\Rightarrow L_{x\alpha \cdot x^\rho} = L_x^{-1}L_{x\alpha}. \quad (3.8)$$

Again, since $(I, R_x^{-1}R_{x\phi^{-1}}, L_x^{-1}R_x^{-1}R_{x\phi^{-1}}L_x) \in AUT(L, \cdot)$, we have

$$yI \cdot zR_x^{-1}R_{x\phi^{-1}} = (yz)L_x^{-1}R_x^{-1}R_{x\phi^{-1}}L_x, \quad (3.9)$$

putting $z = e$, we have

$$y \cdot eR_x^{-1}R_{x\phi^{-1}} = (ye)L_x^{-1}R_x^{-1}R_{x\phi^{-1}}L_x \Rightarrow y \cdot (x^\lambda \cdot x\phi^{-1}) =$$

$$yL_x^{-1}R_x^{-1}R_{x\phi^{-1}}L_x \Rightarrow R_{x^\lambda \cdot x\phi^{-1}} = L_x^{-1}R_x^{-1}R_{x\phi^{-1}}L_x. \quad (3.10)$$

Also, substituting $y = e$ in equation (3.9), we get

$$R_x^{-1}R_{x\phi^{-1}} = L_x^{-1}R_x^{-1}R_{x\phi^{-1}}L_x \quad (3.11)$$

$$\text{so, } R_{x^\lambda \cdot x\phi^{-1}} = R_x^{-1}R_{x\phi^{-1}} = L_x^{-1}R_x^{-1}R_{x\phi^{-1}}L_x. \quad (3.12)$$

So, $x\alpha \cdot x^\rho \in N_\lambda(L, \cdot)$ and $x^\lambda \cdot x\phi \in N_\rho(L, \cdot)$, hence,

$x^\lambda \cdot x\phi, x\alpha \cdot x^\rho \in N(L, \cdot)$ for all $x \in L$ and all $\alpha, \phi \in A(L)$.

(iii) From equation (3.8),

$$L_x L_{x\alpha \cdot x^\rho} = L_{x\alpha} \Leftrightarrow (x\alpha \cdot x^\rho) \cdot xy = x\alpha \cdot y.$$

Since $(x\alpha \cdot x^\rho) \in N_\lambda(L, \cdot)$, then

$$(x\alpha \cdot x^\rho)x = x\alpha \text{ for all } x \in L, \alpha, \phi \in A(L). \quad (3.13)$$

(iv) From equation (3.12),

$$R_x L_x R_{x^\lambda \cdot x\phi^{-1}} = R_{x\phi^{-1}} L_x \Leftrightarrow (x \cdot yx)(x^\lambda \cdot x\phi^{-1}) = x(y \cdot x\phi^{-1}).$$

Since $x^\lambda \cdot x\phi^{-1} \in N_\rho(L, \cdot)$, $x \cdot (yx)(x^\lambda \cdot x\phi^{-1}) = x(y \cdot x\phi^{-1}) \Rightarrow$

$$x(x^\lambda \cdot x\phi^{-1}) = x\phi^{-1}. \quad (3.14)$$

The converse: suppose $(L, \cdot)$ is an Osborn loop such that (ii), (iii) and (iv) hold. We need to show that $(H, \circ)$ is an Osborn loop.

Already, $(x\alpha \cdot x^\rho)x = x\alpha$, thence $(x\alpha \cdot x^\rho)x \cdot y = x\alpha \cdot y$.

Since $x\alpha \cdot x^\rho \in N_\lambda(L, \cdot)$, $(x\alpha \cdot x^\rho) \cdot xy = (x\alpha \cdot x^\rho)x \cdot y = x\alpha \cdot y \Rightarrow L_{x\alpha \cdot x^\rho} = L_x^{-1} L_{x\alpha}$. Next, since $(x^\lambda \cdot x\phi^{-1}) \in N_\rho(L, \cdot)$, then

$$\begin{aligned} x(x^\lambda \cdot x\phi^{-1}) &= x\phi^{-1} \Rightarrow (yx)(x^\lambda \cdot x\phi^{-1}) = (y \cdot x\phi^{-1}) \\ \Rightarrow (x \cdot yx)(x^\lambda \cdot x\phi^{-1}) &= x(y \cdot x\phi^{-1}) \Rightarrow R_{x^\lambda} \cdot x\phi^{-1} = L_x^{-1} R_x^{-1} R_{x\phi^{-1}} L_x. \end{aligned}$$

Already, $x(x^\lambda \cdot x\phi^{-1}) = x\phi^{-1}$. Since, $x^\lambda \cdot x\phi^{-1} \in N(L, \cdot)$, then $y \cdot x(x^\lambda \cdot x\phi^{-1}) = y \cdot x\phi^{-1} \Rightarrow yx \cdot (x^\lambda \cdot x\phi^{-1}) = y \cdot x\phi^{-1} \Rightarrow$

$$R_{x^\lambda \cdot x\phi^{-1}} = R_x^{-1} R_{x\phi^{-1}} \quad (3.15)$$

Since $x\alpha \cdot x^\rho \in N_\lambda(L, \cdot)$, then $(L_x^{-1} L_{x\alpha}, I, L_x^{-1} L_{x\alpha}) \in AUT(L, \cdot)$.

And also, since $x^\lambda \cdot x\phi^{-1} \in N_\rho(L, \cdot)$, then $(I, R_x^{-1} R_{x\phi^{-1}}, L_x^{-1} R_x^{-1} R_{x\phi^{-1}} L_x) \in AUT(L, \cdot)$. Hence, by Lemma 3.3, the holomorph $(H, \circ)$ of $(L, \cdot)$ is an Osborn loop. $\square$

Lemma 3.5. *Let $A(L)$ be an automorphism group of a loop $(L, \cdot)$. If the holomorph $(H, \circ)$ of $(L, \cdot)$ is an Osborn loop, then the following identities hold:*

- (1) $(x\alpha \cdot x^\rho) \cdot xy = x\alpha \cdot y; x \cdot (x\alpha)^\rho = (x\alpha \cdot x^\rho)^\rho, (x\alpha \cdot x^\rho)x = x\alpha,$
- (2) $(x \cdot yx)(x^\lambda \cdot x\phi^{-1}) = x(y \cdot x\phi^{-1}); (x\phi^{-1})^\lambda \cdot x = (x^\lambda \cdot x\phi^{-1})^\lambda,$
- (3) $yx \cdot (x^\lambda \cdot x\phi^{-1}) = y \cdot x\phi^{-1}; x(x^\lambda \cdot x\phi^{-1}) = x\phi^{-1},$

$$(4) \quad x(y/x^\lambda \cdot x\phi^{-1}) = (xy)/x \cdot x\phi^{-1}; \quad x(x^\rho/x^\lambda \cdot x\phi^{-1}) = x^\lambda \cdot x\phi^{-1},$$

for all $x, y \in L$ and $\alpha, \phi \in A(L)$.

Proof. From Theorem 3.4, we have

$$(1) \quad L_{x\alpha \cdot x^\rho} = L_x^{-1} L_{x\alpha} \Rightarrow (x\alpha \cdot x^\rho) \cdot xy = x\alpha \cdot y \quad (3.16)$$

Put $y = (x\alpha)^\rho$ in (3.16) to get $(x\alpha \cdot x^\rho)^\rho = x \cdot (x^\alpha)^\rho$. Putting $y = e$ in (3.16), then $(x\alpha \cdot x^\rho)x = x\alpha$.

$$(2) \quad R_{x^\lambda \cdot x\phi^{-1}} = L_x^{-1} R_x^{-1} R_{x\phi^{-1}} L_x \Rightarrow (x \cdot yx)(x^\lambda \cdot x\phi^{-1}) = x(y \cdot x\phi^{-1}) \quad (3.17)$$

Put $y = (x\phi^{-1})^\lambda$ in (3.17) to get $(x^\lambda \cdot x\phi^{-1})^\lambda = (x\phi^{-1})^\lambda x$.

$$(3) \quad R_{x^\lambda \cdot x\phi^{-1}} = R_x^{-1} R_{x\phi^{-1}} \Rightarrow yx \cdot (x^\lambda \cdot x\phi^{-1}) = y \cdot x\phi^{-1} \quad (3.18)$$

Put $y = e$ in (3.17) to get $x(x^\lambda \cdot x\phi^{-1}) = x\phi^{-1}$.

$$(4) \quad R_x^{-1} R_{x\phi^{-1}} = L_x^{-1} R_x^{-1} R_{x\phi^{-1}} L_x \Rightarrow L_x R_x^{-1} R_{x\phi^{-1}} = R_x^{-1} R_{x\phi^{-1}} L_x \Rightarrow x(y/x^\lambda \cdot x\phi^{-1}) = (xy)/x \cdot x\phi^{-1}. \text{ Put } y = x^\rho, \text{ then } x(x^\rho/x^\lambda \cdot x\phi^{-1}) = x^\lambda \cdot x\phi^{-1}.$$

The proof is complete. $\square$

Lemma 3.6. *Let $(L, \cdot)$ be a loop. If the A -holomorph $H(L)$ of L is an Osborn loop, then for all $x \in L$ and $\alpha, \phi \in A(L)$.*

- (a) $(L_x^{-1} L_{x\alpha}, I, L_x^{-1} L_{x\alpha}) \in \text{AUT}(L, \cdot)$.
- (b) $(L_{x\alpha \cdot x^\rho}, I, L_{x\alpha \cdot x^\rho}) \in \text{AUT}(L, \cdot)$.
- (c) $(I, R_x^{-1} R_{x\phi^{-1}}, R_x^{-1} R_{x\phi^{-1}}) \in \text{AUT}(L, \cdot)$.
- (d) $(I, L_x^{-1} R_x^{-1} R_{x\phi^{-1}} L_x, L_x^{-1} R_x^{-1} R_{x\phi^{-1}} L_x) \in \text{AUT}(L, \cdot)$.
- (e) $(I, R_{x^\lambda \cdot x\phi^{-1}}, R_{x^\lambda \cdot x\phi^{-1}}) \in \text{AUT}(L, \cdot)$.
- (f) $(R_{x\alpha \cdot x^\rho}, L_{x\alpha \cdot x^\rho}^{-1}, I), (R_{x^\lambda \cdot x\phi^{-1}}, L_{x^\lambda \cdot x\phi^{-1}}^{-1}, I) \in \text{AUT}(L, \cdot)$.

Proof. (a) Following the steps in the proof of Lemma 3.3, we obtain (a).

(b) Use Theorem 3.4 and the fact that $L_{x\alpha \cdot x^\rho} = L_x^{-1} L_{x\alpha}$.

(c) Follow the steps in Lemma 3.3 and Theorem 3.4.

- (d) Follow the steps in Lemma 3.3 and Theorem 3.4.
- (e) Follow the steps in Lemma 3.3 and Theorem 3.4.
- (f) Since $x\alpha \cdot x^\rho \in N(L, \cdot)$, obviously, it is in $N_\mu(L, \cdot)$. Then:

$$x(x\alpha \cdot x^\rho) \cdot y = x \cdot (x\alpha \cdot x^\rho)y \Rightarrow xR_{x\alpha \cdot x^\rho} \cdot yL_{x\alpha \cdot x^\rho}^{-1} = xy \quad (3.19)$$

which implies that

$$(R_{x\alpha \cdot x^\rho}, L_{x\alpha \cdot x^\rho}^{-1}, I) \in AUT(L, \cdot).$$

Since $x^\lambda \cdot x\phi^{-1} \in N(L, \cdot) \Rightarrow x^\lambda \cdot x\phi^{-1} \in N_\mu(L, \cdot)$, then by definition,

$$x(x^\lambda \cdot x\phi^{-1}) \cdot y = x \cdot (x^\lambda \cdot x\phi^{-1})y \Rightarrow xR_{x^\lambda \cdot x\phi^{-1}} \cdot y = x \cdot yL_{x^\lambda \cdot x\phi^{-1}} \Rightarrow$$

$$(R_{x^\lambda \cdot x\phi^{-1}}, L_{x^\lambda \cdot x\phi^{-1}}, I) \in AUT(L, \cdot).$$

That completes the proof. $\square$

Corollary 3.7. *Let $(L, \cdot)$ be a loop. If the A -holomorph $H(L)$ of L is an Osborn loop, then for all $x \in L$ and $\alpha, \phi \in A(L)$.*

- (a) $L_x^{-1}L_{x\alpha}, L_{x\alpha \cdot x^\rho} \in \Lambda(L, \cdot); L_{x\alpha} \in L_x\Lambda(L, \cdot)$.
- (b) $R_x^{-1}R_{x\phi^{-1}}, L_x^{-1}R_x^{-1}R_{x\phi^{-1}}L_x, R_{x^\lambda \cdot x\phi^{-1}} \in \mathcal{P}(L, \cdot);$
 $R_{x\phi^{-1}} \in R_x\mathcal{P}(L, \cdot), R_{x\phi^{-1}}L_x \in R_xL_x\mathcal{P}(L, \cdot)$.
- (c) $R_{x\alpha \cdot x^\rho}, R_{x^\lambda \cdot x\phi^{-1}} \in \Phi(L, \cdot), L_{x\alpha \cdot x^\rho}, L_{x^\lambda \cdot x\phi^{-1}} \in \Psi(L, \cdot)$.

Proof. Use Lemma 3.6. $\square$

Theorem 3.8. *Let L be a loop and $H(L)$ its A -holomorph. If $H(L)$ is an Osborn loop, then $A(L, \cdot) = \mathcal{P}(L, \cdot) \cap \Lambda(L, \cdot) \cap \Phi(L, \cdot) \cap \Psi(L, \cdot)$ and for any $\alpha \in A(L)$, $\alpha = L_{e\pi} = R_{e\rho}^{-1}$ for some $\pi \in \Phi(L, \cdot)$ and some $\rho \in \Psi(L, \cdot)$*

Proof. From Corollary 3.7, $L_{x\alpha} \in L_x\Lambda(L, \cdot) \Rightarrow L_{x\alpha} = L_x\lambda$ for some $\lambda \in \Lambda(L, \cdot)$. So

$$x\alpha \cdot y = (xy)\lambda \quad (3.20)$$

implies that, $(\alpha, I, \lambda) \in AUT(L, \cdot) \Rightarrow \alpha = \lambda \Rightarrow \alpha \in \Lambda(L, \cdot)$. Also, $L_{x\alpha \cdot x^\rho} = \lambda$.

$R_{x\phi^{-1}} \in R_x\mathcal{P}(L, \cdot)$ implies that $R_{x\phi^{-1}} = R_{x\rho} \Rightarrow y \cdot x\phi^{-1} = (yx)\rho \Rightarrow (I, \phi^{-1}, \rho) \in AUT(L, \cdot) \Rightarrow \phi^{-1} = \rho \Rightarrow \phi \in \mathcal{P}(L, \cdot)$.

Next, $R_{x\alpha \cdot x^\rho} \in \Phi(L, \cdot) \Rightarrow yR_{x\alpha \cdot x^\rho} = y\pi$ for some $\pi \in \Phi(L, \cdot) \Rightarrow y(x\alpha \cdot x^\rho) \cdot xy = y\pi \cdot xy \Rightarrow y(x\alpha \cdot y) = y\pi \cdot xy$. Putting $y = e$, $e \cdot (x\alpha \cdot e) = e\pi \cdot xe \Rightarrow \alpha = L_{e\pi}$.

Next, $L_{x\alpha \cdot x^\rho} \in \Psi(L, \cdot) \Rightarrow yL_{x\alpha \cdot x^\rho} = y\rho$ for some $\rho \in \Psi(L, \cdot) \Rightarrow (x\alpha \cdot x^\rho) \cdot xy = (xy)\rho \Rightarrow x\alpha \cdot y = (xy)\rho$.

Thus, $(\alpha, I, \rho) \in AUT(L, \cdot) \Rightarrow \alpha = \rho = \lambda$. Also, $R_{x^\lambda \cdot x\phi^{-1}} \in \Phi(L, \cdot) \Rightarrow R_{x^\lambda \cdot x\phi^{-1}} = \pi \Rightarrow y(x^\lambda \cdot x\phi^{-1}) = y\pi \Rightarrow (yx)(x^\lambda \cdot x\phi^{-1}) = (yx)\pi(y \cdot x\phi^{-1} = (yx)\pi \Rightarrow (I, \phi^{-1}, \pi) \in AUT(L, \cdot) \Rightarrow \phi^{-1} = \pi = \rho$. So, $\phi \in \Phi(L, \cdot)$.

Finally, $L_{x^\lambda \cdot x\phi^{-1}} \in \Psi(L, \cdot) \Rightarrow yL_{x^\lambda \cdot x\phi^{-1}} = y\rho \Rightarrow (yx)(x^\lambda \cdot x\phi^{-1})y = yx \cdot y\rho \Rightarrow (y \cdot x\phi^{-1})y = yx \cdot y\rho$.

With $y = e$, then $(e \cdot x\phi^{-1})e = ex \cdot e\rho \Rightarrow \phi^{-1} = R_{e\rho} \Rightarrow \phi = R_{e\rho}^{-1}$.

Since $\alpha = \rho = \lambda$ and $\phi^{-1} = \pi = \rho$ and α and ϕ are arbitrary elements from $A(L)$, then $\alpha \in \Lambda(L, \cdot)$, $\alpha \in \mathcal{P}(L, \cdot)$, $\alpha \in \Phi(L, \cdot)$ and $\alpha \in \Psi(L, \cdot)$. Hence,

$$A(L) = \mathcal{P}(L, \cdot) \cap \Lambda(L, \cdot) \cap \Phi(L, \cdot) \cap \Psi(L, \cdot)$$

For any $\alpha \in A(L, \cdot)$, $\alpha = L_{e\Phi} = R_{e\Psi}^{-1}$ for some $\pi \in \Phi(L, \cdot)$ and some $\rho \in \Psi(L, \cdot)$. $\square$

Theorem 3.9. *Let L be a loop with an A -holomorph Osborn loop $H(L)$. Then for all $x \in L$ and $\alpha, \phi \in A(L)$.*

$$(a) \ L_x^{-1}L_{x\alpha} = L_{x\alpha \cdot x^\rho} \stackrel{\delta, \beta}{\cong} x\alpha \cdot x^\rho.$$

$$(b) \ R_x^{-1}R_{x\phi^{-1}} = L_x^{-1}R_x^{-1}R_{x\phi^{-1}}L_x = R_{x^\lambda \cdot x\phi^{-1}} \stackrel{\psi, \sigma}{\cong} x^\lambda \cdot x\phi^{-1} = x(x^\rho/x \cdot x\phi^{-1}).$$

$$(c) \ R_x^{-1}R_{x\phi^{-1}} = R_{x\alpha \cdot x^\rho} \stackrel{\sigma}{\cong} x\alpha \cdot x^\rho, L_{x^\lambda \cdot x\phi^{-1}} \stackrel{\beta}{\cong} x^\lambda \cdot x\phi^{-1}.$$

$$(d) \ R_x^{-1}R_{x\phi^{-1}} = R_{x\alpha \cdot x^\rho} \stackrel{\varphi}{\cong} L_{x\alpha \cdot x^\rho}, R_{x^\lambda \cdot x\phi^{-1}} \stackrel{\varphi}{\cong} L_{x^\lambda \cdot x\phi^{-1}}.$$

Proof. Let $U = L_x^{-1}L_{x\alpha} \in \Lambda(L, \cdot)$, so $\delta(U) = eU = eL_x^{-1}L_{x\alpha} = x^\rho L_{x\alpha} = x\alpha \cdot x^\rho \in N_\lambda(L, \cdot)$. Thus, $L_x^{-1}L_{x\alpha} \stackrel{\delta}{\cong} x\alpha \cdot x^\rho \forall x \in L, \alpha \in A(L)$.

Let $U = L_{x\alpha \cdot x^\rho} \in \Lambda(L, \cdot)$, then, $\delta(U) = eU = eL_{x\alpha \cdot x^\rho} = x\alpha \cdot x^\rho \in N_\lambda(L, \cdot)$.

Thus, $L_{x\alpha \cdot x^\rho} \stackrel{\delta}{\cong} x\alpha \cdot x^\rho \forall x \in L, \alpha \in A(L)$.

Let $U = R_x^{-1}R_{x\phi^{-1}} \in \mathcal{P}(L, \cdot)$, then, $\psi(U) = eU = eR_x^{-1}R_{x\phi^{-1}} = x^\lambda R_{x\phi^{-1}} = x^\lambda \cdot x\phi^{-1} \in N_\rho(L, \cdot)$. Thus, $R_x^{-1}R_{x\phi^{-1}} \stackrel{\psi}{\cong} x^\lambda \cdot x\phi^{-1} \forall x \in L, \phi \in A(L)$.

Let $U = L_x^{-1}R_x^{-1}R_{x\phi^{-1}}L_x \in \mathcal{P}(L, \cdot)$, so $\psi(U) = eU = eL_x^{-1}R_x^{-1}R_{x\phi^{-1}}L_x = x(x^\rho/x \cdot x\phi^{-1}) \in N_\rho(L, \cdot)$. Thus, $L_x^{-1}R_x^{-1}R_{x\phi^{-1}}L_x \stackrel{\psi}{\cong} x(x^\rho/x \cdot x\phi^{-1})$.

Let $U = R_{x^\lambda \cdot x\phi^{-1}} \in \mathcal{P}(L, \cdot)$. So, $\psi(U) = eU = eR_{x^\lambda \cdot x\phi^{-1}} = x^\lambda \cdot x\phi^{-1} \in N_\rho(L)$.

Thus $R_{x^\lambda \cdot x\phi^{-1}} \stackrel{\psi}{\cong} x^\lambda \cdot x\phi^{-1}$.

Let $U = R_{x\alpha \cdot x^\rho} \in \Phi(L, \cdot)$, so $\sigma(U) = eR_{x\alpha \cdot x^\rho} = x\alpha \cdot x^\rho \in N_\mu(L, \cdot)$. Thus, $R_{x\alpha \cdot x^\rho} \stackrel{\sigma}{\cong} x\alpha \cdot x^\rho$.

Let $U = R_{x^\lambda \cdot x\phi^{-1}} \in \Phi(L, \cdot)$, so, $\sigma(U) = eR_{x^\lambda \cdot x\phi^{-1}}^{-1} = x^\lambda \cdot x\phi \in N_\mu(L, \cdot)$. Thus,

$$R_{x^\lambda \cdot x\phi^{-1}} \stackrel{\sigma}{\cong} x^\lambda \cdot x\phi^{-1}.$$

Let $U = L_{x\alpha \cdot x\rho} \in \Psi(L, \cdot)$, so, $\beta(U) = eL_{x\alpha \cdot x\rho} = x\alpha \cdot x\rho \in N_\mu(L, \cdot)$. Thus,

$$L_{x\alpha \cdot x\rho} \stackrel{\beta}{\cong} x\alpha \cdot x\rho.$$

Let $U = L_{x^\lambda \cdot x\phi^{-1}} \in \Psi(L, \cdot)$, so, $\beta(U) = eL_{x^\lambda \cdot x\phi^{-1}} = x^\lambda \cdot x\phi^{-1} \in N_\mu(L, \cdot)$.

$$\text{Thus, } L_{x^\lambda \cdot x\phi^{-1}} \stackrel{\beta}{\cong} x^\lambda \cdot x\phi^{-1}.$$

Let $U = R_{x\alpha \cdot x\rho} \in \Phi(L, \cdot)$. So, $\varphi(U) = U' = L_{x\alpha \cdot x\rho}$. Thus, $R_{x\alpha \cdot x\rho} \stackrel{\varphi}{\cong} L_{x\alpha \cdot x\rho} \in \Psi(L, \cdot)$.

Let $U = R_{x^\lambda \cdot x\phi^{-1}} \in \Phi(L, \cdot)$. So, $\varphi(U) = U' = L_{x^\lambda \cdot x\phi^{-1}}$. Thus, $R_{x^\lambda \cdot x\phi^{-1}} \stackrel{\varphi}{\cong} L_{x^\lambda \cdot x\phi^{-1}} \in \Psi(L, \cdot)$. $\square$

Theorem 3.10. Let $(L, \cdot)$ be loop with an A -holomorph Osborn loop $H(L)$. Then,

(a)

$$\begin{array}{ccc} & x\alpha \cdot x\rho & \\ \sigma \nearrow & \uparrow \delta, \beta & \\ R_{x\alpha \cdot x\rho} & \xrightarrow[\text{isomorphism}]{\varphi} & L_{x\alpha \cdot x\rho} \end{array} \in \begin{array}{ccc} & N_\mu(L, \cdot), N_\lambda(L, \cdot) & \\ \sigma \nearrow & \uparrow \delta, \beta \text{ isomorphism} & \\ \Phi(L, \cdot) & \xrightarrow[\text{isomorphism}]{\varphi} & \Lambda(L, \cdot), \Psi(L, \cdot) \end{array} \quad (3.21)$$

for all $x \in L, \alpha \in A(L)$, i.e. $\sigma = \varphi\delta$ and $\sigma = \varphi\beta$.

(b)

$$\begin{array}{ccc} & x^\lambda \cdot x\phi^{-1} & \\ \psi, \sigma \nearrow & \uparrow \beta & \\ R_{x^\lambda \cdot x\phi^{-1}} & \xrightarrow[\text{isomorphism}]{\varphi} & L_{x^\lambda \cdot x\phi^{-1}} \end{array} \in \begin{array}{ccc} & N_\rho(L, \cdot), N_\mu(L, \cdot) & \\ \psi, \sigma \nearrow & \uparrow \beta \text{ isomorphism} & \\ \mathcal{P}(L, \cdot), \Phi(L, \cdot) & \xrightarrow[\text{isomorphism}]{\varphi} & \Psi(L, \cdot) \end{array} \quad (3.22)$$

for all $x \in L, \phi \in A(L)$, i.e. $\sigma = \varphi\beta$ and $\psi = \varphi\beta$.

Proof. The proof follows from Theorem 3.9. $\square$

Theorem 3.11. Let $(L, \cdot)$ be a loop with an A -holomorph Osborn loop $H(L)$.

(a) The commutative diagram

$$\begin{array}{ccc} \Lambda(L, \cdot) & \xrightarrow{\delta} & N(L, \cdot) \\ \vdots & \nearrow \sigma & \uparrow \beta \text{ isomorphism} \\ \delta_1 \downarrow & & \\ \Phi(L, \cdot) & \xrightarrow[\text{isomorphism}]{\varphi} & \Psi(L, \cdot) \end{array} \quad (3.23)$$

is true and so, $\delta = \delta_1\sigma = \delta_1\varphi\beta$, $L_{x\alpha \cdot x^\rho} \stackrel{\delta_1}{\cong} R_{x\alpha \cdot x^\rho}$ and $x\alpha \cdot x^\rho \in Z(L, \cdot)$ for all $x \in L, \alpha \in A(L)$.

(b) The commutative diagram

$$\begin{array}{ccc}
 \Lambda(L, \cdot) & \xrightarrow{\delta} & N(L, \cdot) \\
 \delta_2 \uparrow & \nearrow \sigma & \uparrow \beta \text{ isomorphism} \\
 \Phi(L, \cdot) & \xrightarrow[\text{isomorphism}]{\varphi} & \Psi(L, \cdot)
 \end{array} \quad (3.24)$$

is true and so, $\sigma = \delta_2\delta$ and $\delta_2\delta = \varphi\beta$ and $R_{x\alpha \cdot x^\rho} \stackrel{\delta_1}{\cong} L_{x\alpha \cdot x^\rho}$.

(c) The commutative diagram

$$\begin{array}{ccc}
 \mathcal{P}(L, \cdot) & \xrightarrow{\psi} & N(L, \cdot) \\
 \psi_1 \downarrow & \nearrow \sigma & \uparrow \beta \text{ isomorphism} \\
 \Phi(L, \cdot) & \xrightarrow[\text{isomorphism}]{\varphi} & \Psi(L, \cdot)
 \end{array} \quad (3.25)$$

is true and so, $\psi = \psi_1\sigma = \psi_1\varphi\beta$ and $R_{x^\lambda \cdot x\phi^{-1}} \stackrel{\psi_1}{\cong} R_{x^\lambda \cdot x\phi^{-1}}$.

(d) The commutative diagram

$$\begin{array}{ccc}
 \mathcal{P}(L, \cdot) & \xrightarrow{\psi} & N(L, \cdot) \\
 \psi_2 \uparrow & \nearrow \sigma & \uparrow \beta \text{ isomorphism} \\
 \Phi(L, \cdot) & \xrightarrow[\text{isomorphism}]{\varphi} & \Psi(L, \cdot)
 \end{array} \quad (3.26)$$

is true and so, $\sigma = \psi_2\psi$ and $\psi_2\psi = \varphi\beta$ and $R_{x^\lambda \cdot x\phi^{-1}} \stackrel{\psi_2}{\cong} R_{x^\lambda \cdot x\phi^{-1}}$.

(e) The commutative diagram

$$\begin{array}{ccc}
 & & \mathcal{P}(L, \cdot) \\
 & \searrow \omega_1 & \nearrow \psi \\
 & \Lambda(L, \cdot) & \xrightarrow{\delta} N(L, \cdot) \\
 \psi_1 \downarrow & \nearrow \sigma & \uparrow \beta \\
 \Phi(L, \cdot) & \xrightarrow[\text{isomorphism}]{\varphi} & \Psi(L, \cdot)
 \end{array} \quad (3.27)$$

is true and so, $\psi_1 = \omega_1\delta_1$ and $\psi = \omega_1\delta$ and $R_{x^\lambda \cdot x\phi^{-1}} \stackrel{\omega_1}{\cong} L_{x^\lambda \cdot x\phi^{-1}}$.

(f) The commutative diagram

$$\begin{array}{ccccc}
 & \mathcal{P}(L, \cdot) & & & \\
 & \searrow \psi & & & \\
 & & \Lambda(L, \cdot) & \xrightarrow{\delta} & N(L, \cdot) \\
 \psi_2 \nearrow & & \uparrow \delta_2 & \nearrow \sigma & \uparrow \beta \\
 & \Phi(L, \cdot) & \xrightarrow{\varphi} & \Psi(L, \cdot) &
 \end{array}
 \quad (3.28)$$

is true and so, $\psi_2 = \delta_2 \omega_2$ and $\delta = \omega_2 \psi$, and $L_{x^\lambda \cdot x\phi^{-1}} \stackrel{\omega_2}{\cong} R_{x^\lambda \cdot x\phi^{-1}}$.

Proof. The proof is a consequence of Theorem 3.9. $\square$

Corollary 3.12. Let L be a loop with an A -holomorph Osborn loop $H(L)$.

(a) The commutative diagram

$$\begin{array}{ccc}
 \mathcal{P}(L, \cdot) & \xrightarrow{w_1 = \psi\delta^{-1}} & \Lambda(L, \cdot) \\
 \psi_1 = \psi\beta^{-1}\varphi^{-1} = \psi\sigma^{-1} \uparrow & & \epsilon_1 = \beta\delta^{-1} \uparrow \text{isomorphism} \\
 \Phi(L, \cdot) & \xrightarrow[\text{isomorphism}]{\varphi} & \Psi(L, \cdot)
 \end{array}
 \quad (3.29)$$

is true and $L_{x^\lambda \cdot x\phi^{-1}} \stackrel{\epsilon_1}{\cong} L_{x^\lambda \cdot x\phi^{-1}}$.

(b) The commutative diagram

$$\begin{array}{ccc}
 \mathcal{P}(L, \cdot) & \xrightarrow{w_2 = \delta\psi^{-1}} & \Lambda(L, \cdot) \\
 \psi_2 = \varphi\beta\psi^{-1} = \sigma\psi^{-1} \downarrow & & \epsilon_2 = \delta\beta^{-1} \downarrow \text{isomorphism} \\
 \Phi(L, \cdot) & \xrightarrow[\text{isomorphism}]{\varphi} & \Psi(L, \cdot)
 \end{array}
 \quad (3.30)$$

is true and $L_{x^\lambda \cdot x\phi^{-1}} \stackrel{\epsilon_2}{\cong} L_{x^\lambda \cdot x\phi^{-1}}$.

Proof. The proof follows from Theorem 3.9 and Theorem 3.10. $\psi = \psi_1\varphi\beta \Rightarrow \psi_1 = \psi\beta^{-1}\varphi^{-1}$. $\beta = \epsilon_1\delta \Rightarrow \epsilon_1 = \beta\delta^{-1}$. $w_1 = \psi_1\varphi\epsilon = \psi\delta^{-1}$. $w_2\psi = \delta \Rightarrow w_2 = \delta\psi^{-1}$. $\psi_2\varphi = \varphi\beta \Rightarrow \psi_2 = \varphi\beta\psi^{-1}$. $\square$

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