Local convergence radius for the Mann-type iteration

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Abstract. A procedure to estimate the local convergence radius for a Mann-type iteration is given in the setting of a finite dimensional space. In particular we obtain the estimation of radius for classical Newton method. Numerical experiments are presented showing the efficiency of the proposed procedure in comparison with other known methods. In some cases our procedure gives the maximum local convergence radius.

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1 Introduction

Let $F : C \rightarrow \mathbb{R}^m$ be a nonlinear mapping, where C is an open subset of the m -dimensional space $\mathbb{R}^m$. Consider the following Mann-type iteration for the solution of nonlinear equation $F(x) = 0$

$$x_{n+1} = x_n - D_n F(x_n), \quad (1.1)$$

where $\{D_n\}$ is a sequence of matrices; usually this sequence is defined as a function of x , $D : C \rightarrow \mathbb{R}^{m \times m}$ (we will use the notations $D_x = D(x)$ and $D_n = D(x_n)$), or it can be defined recursively as a mapping depending on x_n and D_{n-1} . Some well known iterative methods are particular cases of

(1.1). For example, if $D_x = I$ (the unitary matrix) then we obtain the Picard method; if F is Fréchet differentiable and $D_x = F'(x)^{-1}$ then (1.1) reduces to the classical Newton method; if $D_x = \alpha_x F'(x)^T$ we obtain the gradient method with steplength α_x (in particular, $\alpha_x = \|F(x)\|^2 / \|F'(x)^T F(x)\|^2$). If we take $F = I - T$ then (1.1) becomes a Mann-type iteration with generalized control sequence $D_n = D(x_n)$.

In this paper we are concerned with the estimation of local convergence radius of (1.1). Recall that the local convergence radius r is defined as the radius of a ball centred on the fixed point p , $B(p, r) = \{x \mid \|x - p\| \leq r\}$, such that the sequence generated by a certain iterative scheme starting at any point in $B(p, r)$ converges to p . Obviously, such a ball is entirely contained in the attraction basin of that iteration.

The estimation of local convergence radius has been of some interest over several decades and some efforts have been made to obtain improved values for this radius. However "... effective, computable estimates for convergence radii are rarely available" [9]. We present below some of the most known results on this topic for Newton method. The general assumptions are: F is Fréchet differentiable on C , p is an isolated zero of $F(x) = 0$ and there exists $F'(p)^{-1}$.

One of the first results in this direction was presented by Rall (1974) [7] for Newton method in the setting of Banach spaces; he proved that if the Fréchet derivative F' is k -Lipschitz continuous on some sphere centred on p and $\|F'(p)\| \leq \beta$, then the local convergence radius is given by $r = (2 - \sqrt{2}) / (2\beta k)$. In similar conditions and in finite dimensional spaces Rheinboldt (1975) [9] proposed the improved value $r = 2 / (3\beta k)$. A value still closer to the maximum local convergence radius was given by Traub and Wozniakowski (1979) [10]: let $A = A(r) = \sup_{x, y \in S(p, r)} \|F'(p)^{-1}(F'(x) - F'(y))\| / 2\|x - y\|$ then a positive number r satisfying $rA < q / (1 + 2q)$ where $0 < q < 1$ is a local convergence radius; the value $r < 1 / (3A)$ is also allowable. Smale (1997) [11] gave the following value: suppose that F has infinitely many derivatives at p and satisfies

$$\|F'(p)^{-1}F^{(k)}(p)\| \leq k!\gamma^{k-1}, \quad k = 2, 3, \dots,$$

then $r = (5 - \sqrt{17}) / (4\gamma)$. Wang (2000) [12] suggested a more general formula for convergence radius under the hypothesis that the derivative satisfies some kind of weak Lipschitz condition (named "radius Lipschitz condition with L average", L being a positive integrable function). Argyros (2005) [2] gave the following value: assume that the Fréchet derivative of F satisfies the following Höder and center-Höder conditions

$$\begin{aligned} \|F'(p)^{-1}(F(x) - F(y))\| &\leq L\|x - y\|^\mu, \\ \|F'(p)^{-1}(F(x) - F(p))\| &\leq L_0\|x - p\|^\mu, \end{aligned}$$

for all $x, y \in B(x_0, r) \subset C$, where L, L_0 are some positive numbers and $0 < \mu \leq 1$. Define $q = [(1 + \mu)/(L + (1 + \mu)L_0)]^{1/\mu}$ and suppose that $q \leq r$. Then q is the local convergence radius for Newton method. More recently, Ferreira (2009) [3] considered as the main condition the following relaxed Lipschitz condition: there exists a real function f (with suitable properties) such that

$$\|F'(p)^{-1}(F'(x) - F'(p + \tau(x - p)))\| \leq f'(\|x - p\|) - f'(\tau\|x - p\|),$$

where $0 < \tau < 1$. The local convergence radius is then expressed in terms of this function. Relatively recent results (in the last decade) on these topics were communicated by Argyros [1], Ferreira [4, 5], Hernandez-Veron and Romero [6], Ren [8].

A specific goal of this paper is to give a procedure for estimating the local convergence radius of Newton method. Some number of numerical experiment are also presented showing the efficiency of proposed procedure in comparison with other results. It would be mentioned that in some cases our procedure gives the best convergence radius.

2 Preliminary lemmas

Let E^m denote m -dimensional Euclidean space endowed with the standard metric and let C be an open subset of E^m . Let $T : C \rightarrow E^m$ be a nonlinear mapping; we will assume throughout that the set of fixed points of T is nonempty, $\text{Fix}(T) \neq \emptyset$.

Lemma 1. *Suppose T is Fréchet differentiable on C and let $p \in C$ be a given point. Then there exists a linear mapping $R_{x,p}$ (which depends on x, p), such that*

- (i) $T(x) - T(p) = (T'(x) + R_{x,p})(x - p)$;
- (ii) *for any $\varepsilon > 0$ there exists $r_\varepsilon > 0$ such that if $x \in B(p, r_\varepsilon) = \{x \mid \|x - p\| \leq r_\varepsilon\}$ then $\|R_{x,p}\| \leq \varepsilon$.*

The proof is straightforward if we define

$$R_{x,p} = \frac{(T(x) - T(p) - T'(x)(x - p))(x - p)^T}{\|x - p\|^2}.$$

Lemma 2. *Suppose T is Fréchet differentiable on C , $I - T'(x)$ is invertible for every $x \in C$ and $\|(I - T'(x))^{-1}\| \leq M$, $\forall x \in C$. Then $\text{Fix}(T)$ is a set of isolated points.*

Proof. Let p be a fixed point of T . Let c be a real number such that $0 < c < M^{-1}$ and let r_c be defined in Lemma 1 for $\varepsilon = c$. From Banach lemma there exists $(I - T'(x) - R_{x,p})^{-1}$ and $\|(I - T'(x) - R_{x,p})^{-1}\| \leq M/(1 - cM)$, $\forall x \in B(p, r_c)$. Using Lemma 1 (i) we have

$$\begin{aligned} \|x - T(x)\| &= \|(x - p) - (T(x) - T(p))\| = \|(I - T'(x) - R_{x,p})(x - p)\| \\ &\geq \|(I - T'(x) - R_{x,p})^{-1}\|^{-1} \|x - p\|, \quad \forall x \in B(p, r_c). \end{aligned}$$

Therefore

$$\begin{aligned} \|x - p\| &\leq \|(I - T'(x) - R_{x,p})^{-1}\| \|x - T(x)\| \\ &\leq \frac{M}{1 - cM} \|x - T(x)\|, \quad \forall x \in B(p, r_c). \end{aligned}$$

□

Remark 2.1. For every $p \in \text{Fix}(T)$ the radius r_c has a specific value since this value is determined by the condition $\|R_{x,p}\| \leq \varepsilon$, $\forall x \in B(p, r_c)$

Lemma 3. Let p be a fixed point of T . Suppose T is Fréchet differentiable on C , $I - T'(x)$ is invertible for every $x \in C$ and $\|(I - T'(x))^{-1}\| \leq M$, $\forall x \in C$. Then for some $r > 0$, p is the unique fixed point in $B(p, r)$ and T satisfies the following condition of demicontractive-type

$$\langle D_x(x - T(x))(x - p), x - p \rangle \geq \lambda \|D_x(x - T(x))\|^2, \quad \forall x \in B(p, r), \quad (2.1)$$

where $D_x = (x - T'(x))^{-1}$ and $\lambda > 0.5$.

Proof. Let η be a positive number such that $\eta < \sqrt{5} - 2$, take $\varepsilon = \eta M^{-1}$ in Lemma 1 and let r_ε defined in this Lemma. Let r_c defined in Lemma 2 and then define $r = \min\{r_c, r_\varepsilon\}$; it is obvious that we can apply both Lemma 1 and Lemma 2 on the ball $B(p, r)$.

The uniqueness of p in $B(p, r)$ results from Lemma 2.

Observe that if $\eta < \sqrt{5} - 2$ then $(1 - \eta)/(1 + \eta)^2 > 0.5$ and we can take λ such that $0.5 < \lambda \leq (1 - \eta)/(1 + \eta)^2$. Using Lemma 1 with $\varepsilon = \eta M^{-1}$ we have

$$\begin{aligned} D_x(x - T(x)) &= D_x[(x - p) - (T(x) - T(p))] \\ &= (I - (I - T'(x))^{-1} R_{x,p})(x - p), \quad x \in S(p, r), \end{aligned}$$

and $\|R_{x,p}\| \leq \eta M^{-1}$. With the notation $\Delta_x = (I - T'(x))^{-1} R_{x,p}$, (2.1) becomes

$$\langle (I - \Delta_x)(x - p), x - p \rangle > \lambda \|(I - \Delta_x)(x - p)\|^2, \quad x \in S(p, r). \quad (2.2)$$

Consider now the quadratic polynomial:

$$P(t, \lambda) = \lambda t^2 + (2\lambda + 1)t - 1 + \lambda.$$

The largest solution of P is $s(\lambda) = (-2\lambda - 1 + \sqrt{8\lambda + 1})/(2\lambda)$ which is a decreasing function for $\lambda > 0.5$. We have

$$\|\Delta_x\| = \|(I - T'(x))^{-1}R_{x,p}\| \leq M\|R_{x,p}\| \leq \eta < s(\lambda), \quad \forall x \in S(p, r).$$

Therefore $P(\|\Delta_x\|, \lambda) < 0$ which implies $1 - \|\Delta_x\| > \lambda(1 + \|\Delta_x\|)^2$. For any $y \in \mathcal{H}$, $\|y\| = 1$ we have

$$\begin{aligned} \langle (I - \Delta_x)y, y \rangle &= 1 - \langle \Delta_x y, y \rangle \geq 1 - \|\Delta_x\| > \lambda(1 + \|\Delta_x\|)^2 \\ &\geq \lambda\|y - \Delta_x y\|^2 = \lambda\|(I - \Delta_x)y\|^2. \end{aligned}$$

Taking $y = (x - p)/\|x - p\|$ we obtain (2.2). $\square$

3 Local convergence

Theorem 3.1. *Let p_0 be a fixed point of T and $r > 0$ such that $B(p_0, r) \subset C$. Suppose T and D satisfy following conditions:*

- (i) $I - T$ is demiclosed at zero on C ;
 - (ii) There exists $\lambda > 0.5$ such that $\langle D_x(x - T(x), x - p) \rangle \geq \lambda\|D_x(x - T(x))\|^2$, $\forall x \in B(p_0, r)$, $p \in \text{Fix}(T)$;
 - (iii) D_x is invertible and $\|D_x^{-1}\| \leq M$, $\forall x \in B(p_0, r)$.
- Then the sequence given by (1.1) converges to a fixed point of T for any starting point in $B(p_0, r)$.

Proof. Suppose that $x_n \in B(p_0, r)$. For any $p \in \text{Fix}(T)$, using (1.1) and (ii) we have

$$\begin{aligned} \|x_{n+1} - p\|^2 &= \|x_n - p - D_n(x_n - T(x_n))\|^2 \\ &\leq \|x - p\|^2 - (2\lambda - 1)\|D_n(x_n - T(x_n))\|^2 < \|x_n - p\|^2. \end{aligned}$$

Therefore $\{x_n\} \subset B(p_0, r)$ and $\|x_n - p\| \rightarrow \varrho_p$, $n \rightarrow \infty$. From (iii) we have

$$\begin{aligned} M^{-2}\|x_n - T(x_n)\|^2 &\leq \|D_n^{-1}\|^{-2}\|x_n - T(x_n)\|^2 \leq \|D_n(x_n - T(x_n))\|^2 \\ &\leq (2\lambda - 1)^{-1}(\|x_n - p\|^2 - \|x_{n+1} - p\|^2) \rightarrow 0. \end{aligned}$$

As the sequence $\{x_n\}$ is bounded, there exists a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ which converge to an element $q \in B(p_0, r)$. It is obvious that $\|x_{n_j} - T(x_{n_j})\| \rightarrow 0$ which, together with (i), gives $q \in \text{Fix}(T)$. Now, (ii) being valid for any $p \in C$, it results $\|x_{n+1} - q\| \leq \|x_n - q\|$ and finally $\|x_n - q\| \rightarrow 0$. $\square$

The following Corollary is a straightforward consequence of Lemma 2 and Theorem 1.

Corollary 1. *Suppose T is Fréchet differentiable on C , $I - T'(x)$ is invertible and $\|(I - T'(x))^{-1}\| \leq M$, $\forall x \in C$. Let p be a fixed point of T . Then, for r sufficiently small, p is the unique fixed point of T in $B(p, r)$ and the sequence $\{x_n\}$ defined by (1.1) with $D_x = (I - T'(x))^{-1}$ converges to p for any starting point $x_0 \in B(p, r)$.*

Remark 3.1. The Mann-type iteration (1.1) for $D_n = (I - T'(x_n))^{-1}$ is just the classical Newton method for the equation $x - T(x) = 0$ and the Corollary 1 gives conditions for local convergence of this method.

We can obtain the following conditions for the superlinear convergence of the Mann-type iteration (1.1).

Corollary 2. *Suppose that T is Fréchet differentiable in the fixed point p and that $I - T'(p)$ is invertible. Suppose in addition that $D_n \rightarrow (I - T'(p))^{-1}$. Then the sequence $\{x_n\}$ converges superlinear to p .*

Proof. We have

$$\begin{aligned} \frac{x_{n+1} - p}{\|x_n - p\|} &= \frac{(x_n - p) - D_n(x_n - T(x_n))}{\|x_n - p\|} \\ &= \frac{(x_n - p) - D_n[(x_n - p) - (T(x_n) - T(p))]}{\|x_n - p\|} \\ &= (I - D_n) \frac{x_n - p}{\|x_n - p\|} + D_n \frac{T(x_n) - T(p)}{\|x_n - p\|} \\ &= [I - D_n(I - T'(p))] \frac{x_n - p}{\|x_n - p\|} + D_n \frac{T(x_n) - T(p) - T'(p)(x_n - p)}{\|x_n - p\|}. \end{aligned}$$

Thus

$$\frac{\|x_{n+1} - p\|}{\|x_n - p\|} \leq \|I - D_n(I - T'(p))\| + M \frac{\|T(x_n) - T(p) - T'(p)(x_n - p)\|}{\|x_n - p\|} \rightarrow 0.$$

□

4 Local convergence radius

We propose now the following procedure to estimate the local convergence radius for the Mann-type iteration (1.1). Suppose that T and D satisfies the

conditions (i) and (iii) of Theorem 1 respectively. Let p be a fixed point of T and suppose that p is an isolated point in a ball $B(p, r_0)$. In these conditions it is sufficient to find a ball $B(p, r) \subset B(p, r_0)$ on which the condition (ii) is satisfied. Therefore the procedure consist in finding the largest value of r for which the following condition

$$\frac{\langle D_x(x - T(x)), x - p \rangle}{\|D_x(x - T(x))\|^2} \geq 0.5, \quad \forall x \in B(p, r).$$

is satisfied. Note that in the case of Newton method both conditions (i) and (iii) are ensured by the requirements of Corollary 1.

Several numerical experiments in one and two dimensions were performed to validate this method. It is worthwhile to emphasize that the values obtained by our procedure are, in some extent, larger than those given by the methods presented in Section 1 and, in some cases, it gives the maximum local convergence radius.

Remark 4.1. Several numerical experiments with mappings in one or two variables show that for those mappings for which the immediate basin of attraction is a ball centred in the fixed point, the proposed procedure gives the maximum local convergence radius corresponding to Newton method.

We can do the following

Presumption

Let $T : C \rightarrow \mathbb{R}^m$ be a nonlinear mapping and p an isolated fixed point of T . If the immediate attraction basin of the Newton method applied to the function $F(x) = x - T(x)$ corresponding to p is a ball center at p , then the proposed procedure gives the maximum local convergence radius.

5 Numerical experiences

This section is devoted to numerical experiments in order to evaluate the efficiency of the proposed procedure. Both this procedure and the procedures described in section 1, involve the following main processing:

1. Apply a search line algorithm (for example of the type half-step algorithm) on the positive real axis to find the largest value for r ;
2. At every step of 1 solve some constraint optimization problems and verify the conditions required by the considered procedure.

The convergence radius of the various examples considered in this experiment were computed with the help of the *maximize/minimize* function of

different mathematical software systems. We also *numerically* evaluated the maximum convergence radius for every cases. The values of these radii were computed by direct checking the convergence of iteration process starting from all points of a given net of points; of course, these values generally have an orientative character. However, these numerical investigations give significant information on the convergence ball. For example, finding a point outside of a convergence ball and close to or on its border, shows that the considered sphere is a good approximation of the maximum convergence ball.

In the first experiment we apply the proposed procedure and the procedures described in section 1 to several real functions. For the following four of them: Ex. 1: $f(x) = 2x^3 - 3x^2 + x + 0.5$, $p = 0.5$; Ex. 2: $f(x) = x^5/5 - x^2 + x + 0.8$, $p = 0.5$; Ex. 3: $f(x) = 0.5x^2 + \cos(x)$, $p = 1.0485$; Ex.4: $(2/3)x^{3/2} - x$, $p = 9/4$, the results are presented below (the last row contains the maximum local convergence radius). The values of different radius are given with four decimal digits.

	<i>Ex.1</i>	<i>Ex.2</i>	<i>Ex.3</i>	<i>Ex.4</i>
=====	=====	=====	=====	=====
<i>Rall</i>	$r = 0.1913$	0.0914	0.3053	0.3985
<i>Rheinboldt</i>	$r = 0.2886$	0.1666	0.5373	0.8021
<i>Traub, Wozniakowski</i>	$r = 0.2357$	0.1581	0.5373	0.8021
<i>Smale</i>	$r = 0.1898$	0.1550	0.5319	0.6583
<i>Argyros</i>	$r = 0.3535$	0.1811	0.6130	0.8552
<i>Proposed procedure</i>	$r = 0.3873$	0.1924	0.6481	0.8782
=====	=====	=====	=====	=====
<i>The maximum radius</i>	$r = 0.3873.$	0.2282	0.8860	1.2500

In all the examples considered in this experiment the proposed procedure give the best local convergence radius and in some cases (Ex. 1) it gives the maximum local convergence radius.

In the second experiment some higher dimensional cases were considered. The solution of the constrained optimization problem involved in considered procedures (except the procedure of Smale) should be global on some ball. Because the cost function can have several local extremal points, to solve this problem is usually a difficult task. The example below illustrates this fact.

Let $T : \mathcal{R}^2 \rightarrow \mathcal{R}^2$ be a function defined by

$$T(x_1, x_2) = \begin{pmatrix} 3x_1^2 - x_1x_2^3 + 3x_2 \\ 2x_1 + 2x_2^3 - 0.2x_2 - 1.2 \end{pmatrix}, \quad (5.1)$$

having the fixed point $p = (1, -1)^T$. The cost function in most methods is

$$c(x, y) = \frac{\|F'(p)(F'(x) - F'(y))\|}{2\|x - y\|},$$

where $F(x) = x - T(x)$ and $x = (x_1, x_2)^T$, $y = (y_1, y_2)^T$. In fact c is a function of four variables, x_1, x_2, y_1, y_2 , and the considered procedures involves the computation of the global maximum of this cost function on some ball, i.e., to find $\max c(x, y)$ subject to $\|x - p\| \leq r$, $\|y - p\| \leq r$. If $r = 0.2$ and the initial starting point is $(0.1, 0.2, 0.3, 0.5)$ the function *Maximize* (from MathCad, Professional 2007) gives 1.554; for the same value of r and the initial starting point $(0.1, 0.2, 0.3, 0.3)$ gives 2.217. The explanation of these two relatively very different values is the fact that c has several different local maximum points and the function *Maximize* is a local method. In Figure 1a is represented a "section" of the graph of c obtained as the intersection of the graph by a hyperplane passing through the fixed point.

Figure 1: The shape of cost function:
(a) Traub-Wosniakowski procedure; (b) Proposed procedure.

It can be seen that around the fixed point c has many extremal points and so a local optimization algorithm (like Conjugate Gradient, Levenberg Marquardt, quasi-Newton, etc., as the various mathematical softwares provide) gives only a local maximizer, close to the initial point. Finding the global extremal point is far more difficult and the use of specific global method often leads to very hard challenges.

The cost function in proposed procedure is

$$c(x) = \frac{\langle D(x)(x - f(x)), x - p \rangle}{\|D(x)(x - f(x))\|^2},$$

where $D(x) = (I - f'(x))^{-1}$ and $x = (x_1, x_2)^T$. The cost function depends on two variables and the procedure consists in finding the global minimum of c on some ball at every step of the line-search algorithm (this minimum should be greater than 0.5). In Figure 1b is represented the graph of this function around the fixed point. The graph shows a relatively smooth function around p in this example, and the global minimizer is located on the boundary of the considered circle. Of course to find the global minimizer is much easier in this case.

Remark 5.1. For functions in several variables (between 2 and 4), similar with (5.1), a number of numerical experiments show the same smooth graph of the cost function in the case of our procedure.

Figure 2: Local convergence radius estimates by proposed procedure.

Figure 2 illustrates the local convergence radius given by the proposed procedure for Newton method and for a function of two variables. The function

$$T(x) = \begin{pmatrix} 0.2x_1 - \cos(x_1) + x_2^2 + 1 \\ x_1^3 + 0.2x_2 \end{pmatrix}$$

was the test system and $p = (0, 0)$ the fixed point.

The reunion of black regions, including isolated points, is the basin of attraction; the connected black region around p is the immediate attraction basin. Inside of the immediate attraction basin are drawn the maximum convergence circle and its estimation with the proposed procedure and with Traub-Wozniakowski method (the white circles centred in p). In this example the maximum radius is $r = 0.43$, the radius of proposed estimate is $r = 0.23$ and the radius of Traub-Wozniakowski estimate is $r = 0.067$.

The validity of the Presumption for functions in two variables was also checked in this experiment. The results are promising, the Presumption is verified for the cases in which the immediate attraction basin is a ball centred in the fixed point.

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