Factorizations in preduals associated to ρ -contractions

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Abstract. We establish directly factorization results for classes of ρ -contractions, corresponding to those obtained for certain classes of contractions. As an example of result which is not an immediate consequence of the fact that any ρ -contraction is similar to an ordinary contraction we give an "optimal" characterization of absolutely continuous ρ -contractions whose associated functional calculus has Property $(\mathbf{A}_{\aleph_0})$. Along the way we show that a ρ -contraction is absolutely continuous if and only if it admits an absolutely continuous unitary ρ -dilation generalizing a well-known result for usual contractions.

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1 Introduction.

It is well-known that dual algebra techniques have led to interesting results in terms of existence of invariant subspaces and structure information (predual factorization, reflexivity ...) particularly for certain classes of (Hilbert space) contractions (cf. for example [3], for an early - up to 1985 - survey, and [2], [1], [5], [7], [4], [6] for later results). These results are easily seen to be "qualitatively" preserved by similarity.

Among operators similar to contractions are the ρ -contractions that is those admitting a ρ -unitary dilation (see definition in next section) for some $\rho > 0$. These operators (called also $\mathcal{C}_\rho$ -contractions) were introduced by Sz.-Nagy and Foias who also established their similarity to contractions (see [9]). Therefore the classical results obtained for contractions via dual algebra theory have corresponding ones for ρ -contractions, e.g., *any ρ -contraction whose spectrum contains the unit circle has nontrivial invariant subspaces, the functional calculus of any absolutely continuous ρ -contraction with essential spectrum dominating has Property $(\mathbf{A}_{\aleph_0})$* ; indeed this follows respectively from the results in [2], [3] Chap. V. In fact, looser hypothesis related to the so-called approximating sets (see definitions below) will also lead to the same last factorization result. However the similarity to contractions does not preserve certain constants appearing in these processes. In particular it does not seem possible to deduce the optimal characterizations of the analogues of the classes $\mathbf{A}_{\aleph_0}$ and $\mathbf{A}_{1,\aleph_0}$ from this similarity result.

These observations motivate our attempt to study directly (that is without using the above similarity result) these factorizations in the predual of dual algebras generated by $\mathcal{C}_\rho$ -contractions. As a first step in this program, in this note we do obtain the "best" characterization of the class of absolutely continuous ρ -contractions whose functional calculus has Property $(\mathbf{A}_{\aleph_0})$ (analogue of the class $\mathbf{A}_{\aleph_0}$) in terms of approximating sets (cf. our main result, Theorem 3.4. below).

The paper is organized as follows: in Section 2 after recalling basic definitions and terminology we characterize absolutely continuous ρ -contractions, give a property sort of saying what is preserved for general ρ -contractions of the existence for usual contractions (i.e. 1-contractions) of a coisometric extension. This is exploited in the last section to connect spectral properties with membership in approximating sets; this is a key ingredient in the proof of our main result.

2 Preliminaries

Let $\mathcal{H}$ be a complex, infinite dimensional, separable Hilbert space and let $L(\mathcal{H})$ denote the Banach algebra of (bounded linear) operators on $\mathcal{H}$. As usual, for $T \in L(\mathcal{H})$, $\sigma_{le}(T)$, $\sigma_{re}(T)$ and $\sigma_e(T)$ denote respectively the left essential, right essential and essential spectrum of T .

We denote by $A(\mathbb{D})$ the disc algebra, that is, the algebra of complex functions continuous on the closed unit disc and analytic in $\mathbb{D}$ and by $H^\infty = H^\infty(\mathbb{D})$ the algebra of bounded analytic functions in the open unit disc

(equipped with the sup norm ($\|h\|_\infty = \sup_{z \in \mathbb{D}} |h(z)|$). Alternatively $H^\infty = H^\infty(\mathbb{T})$ that is, as well-known, it can be viewed as the subalgebra of $L^\infty(\mathbb{T})$ consisting of the (classes of Lebesgue-measurable) functions whose Fourier coefficients of (strictly) negative index vanish. As such it is a weak*-closed subalgebra of $L^\infty(\mathbb{T})$ whose predual is the quotient space $\mathcal{Q} = L^1/H_0^1$. It is also well-known that H^∞ is the weak*-closure of $A(\mathbb{D})$ in $L^\infty(\mathbb{T})$. More precisely $A(\mathbb{D})$ is *strongly sequentially pointwise boundedly dense* in H^∞ that is, for any f in H^∞ there exist a sequence (g_n) in $A(\mathbb{D})$ converging pointwise (in $\mathbb{D}$) to f and satisfying $\|g_n\|_\infty \leq \|f\|_\infty$, $n \in \mathbb{N}$. Recall that for λ in $\mathbb{D}$ the linear functional (point evaluation at λ) $f \rightarrow f(\lambda)$ is weak* continuous on H^∞ and thus can be identified with an element E_λ in $\mathcal{Q}$. Hence the kernel $H_0^\infty (= \alpha H^\infty)$ (with α the position function) of the evaluation at 0 is a weak*-closed subspace of H^∞ and we have the (weak* as well as norm) topological direct sum decomposition $H^\infty = \mathbb{C}1 \oplus H_0^\infty$.

For $\rho > 0$, an operator $T \in L(\mathcal{H})$ is said to be of *class C_ρ* (or a ρ -contraction), if there is a Hilbert space $\mathcal{K} \supset \mathcal{H}$ and a unitary operator $U \in L(\mathcal{K})$ such that

$$T^n x = \rho P_{\mathcal{H}} U^n x, \quad n \in \mathbb{N}, \quad x \in \mathcal{H},$$

where $P_{\mathcal{H}}$ is the orthogonal projection of $\mathcal{K}$ on $\mathcal{H}$. The unitary operator U is called a ρ -unitary dilation of T and, provided it is *minimal* in the sense that $\mathcal{K} = \bigvee_{n \in \mathbb{Z}} U^n \mathcal{H}$, it is uniquely determined up to unitary equivalence. The adjoint of a ρ -contraction is also a ρ -contraction and the spectral radius $r(T)$ of a ρ -contraction satisfies $r(T) \leq 1$ and therefore its spectrum is included in the closure of the unit disk, $\mathbb{D} \subset \mathbb{C}$. We recall that a ρ -contraction is polynomially bounded, that is, T being a ρ -contraction, there exists a constant M such that for any polynomial p we have

$$\|p(T)\| \leq M \|p\|_\infty,$$

where

$$\|p\|_\infty = \sup_{z \in \mathbb{D}} |p(z)| = \sup_{z \in \mathbb{T}} |p(z)|.$$

In fact (cf. [10]), we can take $M = \rho + |\rho - 1|$.

As well-known, an operator $T \in \mathcal{L}(\mathcal{H})$ is polynomially bounded if and only if it admits a (unital, norm continuous) representation Φ_T of the disc algebra $A(\mathbb{D})$ into $\mathcal{L}(\mathcal{H})$, extending the polynomial calculus associated to T . Recall that a polynomially bounded operator T is said to be *absolutely continuous* if the representation Φ_T admits a weak*-continuous extension to

H^∞ (which we still denote Φ_T and write as usual $h \rightarrow h(T)$). Here the weak*-topology of $\mathcal{L}(\mathcal{H})$ is that coming from the classical duality $\mathcal{L}(\mathcal{H}) = \mathcal{C}_1(\mathcal{H})^*$ where $\mathcal{C}_1(\mathcal{H})$ is the Banach space of trace class operators.

The following proposition gives convenient characterizations of absolute continuity for polynomially bounded operators.

PROPOSITION 2.1. *Let $T \in \mathcal{L}(\mathcal{H})$ be a polynomially bounded operator and Φ the associated representation of $A(\mathbb{D})$ into $\mathcal{L}(\mathcal{H})$; then the following assertions are equivalent:*

- a) *T is absolutely continuous,*
- b) *Whenever a sequence $(p_n)_n$ in $A(\mathbb{D})$ converges pointwise boundedly to 0 the sequence $(p_n(T))_n$ converges weak* to 0 in $\mathcal{L}(\mathcal{H})$,*
- c) *For any x, y in $\mathcal{H}$ the linear functional $p \rightarrow (p(T)x, y)$ is weak*-continuous ($A(\mathbb{D})$ being equipped with the relative weak*-topology).*

Proof. In view of the characterization of sequential weak*-convergence in $H^\infty(\mathbb{D})$ the implication $a) \Rightarrow b)$ is immediate.

Since, in $\mathcal{L}(\mathcal{H})$, weak*-convergence and weak-operator convergence coincide for bounded sequences we have the equivalence $b) \Leftrightarrow c)$.

The remaining implication " $b) \Rightarrow c)$ " can be proved along the lines of [8], p.152 (cf. also [6] or [11]). For the sake of completeness we sketch below a somewhat more elementary argument.

Recall that whenever the Banach space X is separable there exists a translation-invariant metric d on X^* such that on any closed ball in X^* the weak*-topology coincides with that induced by the metric d (indeed take a countable dense set $\{x_n; n \geq 1\}$ in the unit ball of X , then $d(f, g) := \sum_n 2^{-n} | \langle x_n, g - f \rangle |$ will do). We choose then d and d' two such metrics on respectively $H^\infty = (\mathcal{Q})^*$ and $\mathcal{L}(\mathcal{H}) = (\mathcal{C}_1(\mathcal{H}))^*$. Let B (resp. B') the closed unit balls in H^∞ (resp. $\mathcal{L}(\mathcal{H})$); given $r > 0$, the continuity of Φ at 0 together with the invariance by translation of the metrics d_1, d_2 yields easily the uniform continuity of $\Phi_r := \Phi|_{rB \cap A(\mathbb{D})}$. The map Φ_r , uniformly continuous on the dense subset $rB \cap A(\mathbb{D})$ and valued in the complete metric space $r'B'$ (with $r' = r\|\Phi\|$), has a unique (uniformly) continuous extension $\tilde{\Phi}_r$ to rB . Since $\tilde{\Phi}_{r'}$ is clearly an extension of $\tilde{\Phi}_r$ whenever $r' \geq r$, setting $\tilde{\Phi}(f) = \tilde{\Phi}_r(f)$ for f in H^∞ and $r \geq \|f\|$ we obtain a well-defined extension $\tilde{\Phi}$ of Φ to all of H^∞ satisfying $\|\tilde{\Phi}(f)\| \leq \|\Phi\| \|f\|$. The linearity of $\tilde{\Phi}$ is deduced from that of Φ via a routine argument (for scalars λ, μ and f, g in H^∞ choose $r \geq \max\{\|f\|, \|g\|, |\lambda| + \|f\| + |\mu| + \|g\|\}$ and sequences $(u_n)_n, (v_n)_n$ in $A(\mathbb{D})$ converging pointwise to, respectively, f, g and satisfying

$\|u_n\| \leq \|f\|, \|v_n\| \leq \|g\|$ for all n etc..). Thus $\tilde{\Phi}$ is a bounded linear map from H^∞ into $\mathcal{L}(\mathcal{H})$ which, by construction, is sequentially weak*-continuous, hence weak*-continuous by a standard argument. The multiplicativity of $\tilde{\Phi}$ is proved by the usual 2 steps argument (starting with $\Phi(u_nv_n) = \Phi(u_n)\Phi(v_n)$ for u_n, v_n in $A(\mathbb{D})$ we obtain $\tilde{\Phi}(fv_n) = \tilde{\Phi}(f)\Phi(v_n)$ for f in H^∞ , v_n in $A(\mathbb{D})$ and next $\tilde{\Phi}(fg) = \tilde{\Phi}(f)\tilde{\Phi}(g)$ for f, g in H^∞). $\square$

From now on, when T is an absolutely continuous polynomially bounded operator with Φ_T the associated weak*-continuous representation, for x, y in $\mathcal{H}$ we denote by $x \square_T y$ (just $x \square y$ if there is no ambiguity on the operator under consideration) the element of $\mathcal{Q}$ defined by

$$\langle x \square_T y, f \rangle = (f(T)x, y),$$

$f \in H^\infty$.

Before discussing absolute continuity for ρ -contractions we gather some easy (and well-known) facts about this sesquilinear map $\square_T$. Recall that a semi-invariant subspace for an operator T is the orthogonal difference of two nested invariant subspaces. If $\mathcal{J} = \mathcal{M} \ominus \mathcal{N}$ ($\mathcal{M}, \mathcal{N} \in \text{Lat} T$, $\mathcal{N} \subset \mathcal{M}$) is such a subspace we write $T_{\mathcal{J}}$ for the compression of T to $\mathcal{J}$, that is, the operator defined on $\mathcal{J}$ by $T_{\mathcal{J}}(x) = P_{\mathcal{J}}Tx$, $x \in \mathcal{J}$.

PROPOSITION 2.2. *Let $T \in \mathcal{L}(\mathcal{H})$ be an absolutely continuous polynomially bounded operator.*

- a) *For any $x, y \in \mathcal{H}$ and $g \in H^\infty$ we have $g(T)x \square y = x \square (g(T))^*y$.*
- b) *Let $\mathcal{M} \in \text{Lat} T$, $T_0 = T|_{\mathcal{M}}$, $P_{\mathcal{M}}$ the orthogonal projection on $\mathcal{M}$, $x \in \mathcal{M}$ and $y \in \mathcal{H}$, then*

$$x \square_T y = x \square_T P_{\mathcal{M}}y = x \square_{T_0}(P_{\mathcal{M}}y).$$

Similarly, let $\mathcal{F} \in \text{Lat} T^$, T_1 the compression of T to $\mathcal{F}$ (in this case $(T_1)^* = (T^*)|_{\mathcal{F}}$), $x \in \mathcal{H}$ and $y \in \mathcal{F}$; then $x \square_T y = (P_{\mathcal{F}}x) \square_{T_1} y = (P_{\mathcal{F}}x) \square_{T_1} y$.*

- c) *If $\mathcal{J}$ is a semi-invariant subspace for T , then for $x, y \in \mathcal{J}$, $x \square_{T_{\mathcal{J}}} y = x \square_T y$.*
- d) *Let $(x_i)_{i \in J}$ be a sequence of vectors ($J \subset \mathbb{N}$, finite or not); the map $y \rightarrow (x_i \square y)_{i \in J}$ from $\mathcal{H}$ into $J^{\mathcal{Q}}$ (the space of $\mathcal{Q}$ valued sequences indexed by J) has kernel $\mathcal{M}^\perp$ where $\mathcal{M}$ is the invariant subspace for T generated by the x_i 's.*

The following proposition generalizes to ρ -contractions the corresponding (well-known) result for usual contractions.

PROPOSITION 2.3. *Let $T \in \mathcal{L}(\mathcal{H})$ be a ρ -contraction; then T is absolutely continuous iff it admits an absolutely continuous unitary ρ -dilation.*

Proof. Let $U \in \mathcal{L}(\mathcal{K})$ a unitary ρ -dilation of $T \in \mathcal{L}(\mathcal{H})$; for any polynomial p and any $x \in \mathcal{H}$ we have

$$p(T)x = p(0)x + \rho P_{\mathcal{H}}q(U)x,$$

where $q = p - p(0)$. This equality extends by continuity to any $p \in A(\mathbb{D})$ and the implication " U absolutely continuous $\Rightarrow T$ absolutely continuous" follows easily.

Suppose, to prove the converse, that the ρ -contraction T (with ρ -unitary dilation U) is absolutely continuous; we may and do assume that U is minimal, that is, $\mathcal{K} = \bigvee_{j \in \mathbb{Z}} U^j \mathcal{H}$. Let then $(h_n)_n$ a sequence in $A(\mathbb{D})$ converging weak* to 0 and let

$$\mathcal{G} := \{(u, v) \in \mathcal{K} \times \mathcal{K} \mid (h_n(U)u, v) \rightarrow 0\}.$$

To avoid confusion between ordered pairs and inner products, down to the end of this proof, we write $\langle x, y \rangle$ for the inner product.

We wish to show that $\mathcal{G} = \mathcal{K} \times \mathcal{K}$. Obviously $\mathcal{G}$ is a closed linear subspace (just use the norm-boundedness of $(h_n(U))_n$); thus, by the minimality hypothesis, it suffices to show that any given element (u, v) of the form $(u, v) = (U^j x, U^k y)$ ($j, k \in \mathbb{Z}, x, y \in \mathcal{H}$) is in $\mathcal{G}$.

a) for $j > k$

$$\langle h_n(U)u, v \rangle = \langle h_n(U)U^{j-k}x, y \rangle = \frac{1}{\rho} \langle h_n(T)T^{j-k}x, y \rangle$$

that is,

$$\langle h_n(U)u, v \rangle = \frac{1}{\rho} \langle T^{j-k}x \square y, h_n \rangle$$

and hence $(U^j x, U^k y) \in \mathcal{G}$ in this case;

b) for $j \leq k$ we write

$$h_n = p_n + \alpha^{k-j+1} g_n$$

where $p_n := \sum_{i \leq k-j} \frac{1}{i!} h_n^{(i)}(0) \alpha^i$ is the Taylor polynomial of degree $k-j$ of h_n at the origin; thus

$$\langle h_n(U)u, v \rangle = \langle p_n(U)U^j x, U^k y \rangle + \langle U^{k-j+1} g_n(U)U^j x, U^k y \rangle;$$

the Taylor coefficients are weak*-continuous, therefore the sequence (p_n) converges to 0 in norm which yields first that (g_n) converges weak* to 0 and second that the first term on the right hand side of the above equality tends to 0; as to second term we have

$$\langle U g_n(U)x, y \rangle = \frac{1}{\rho} \langle T g_n(T)x, y \rangle = \frac{1}{\rho} \langle T x \square y, g_n \rangle \rightarrow 0.$$

(We remark that Case *b*) could be deduced from *a*) using the fact that U^* is a unitary ρ -dilation of T^* and treating apart -via the decomposition $h_n = h_n(0) + \alpha g_n$ - the special case $j = k$.) $\square$

We now review the factorization properties.

DEFINITION 2.4. *Given cardinal numbers, $0 < m, n \leq \aleph_0$, we say that the (absolutely continuous) representation Φ_T has Property $(\mathbb{A}_{m,n})$ if given an $m \times n$ matrix with entries in $\mathcal{Q}$,*

$$\begin{aligned} (L_{i,j}) \quad & 0 \leq i < m \\ & 0 \leq j < n \end{aligned}$$

there exist vectors $(x_i)_{0 \leq i < m}$ and $(y_j)_{0 \leq j < n}$ such that

$$L_{i,j} = x_i \square_T y_j, \quad 0 \leq i < m, \quad 0 \leq j < n.$$

We write $(\mathbb{A}_n)$ for $(\mathbb{A}_{n,n})$.

In this note we will be mostly interested in the strongest of these properties, namely Property $(\mathbb{A}_{\aleph_0})$. Recall that one essential approach to obtain this property is via the so-called *approximation set* $\mathfrak{X}_\circ (= \mathfrak{X}_\circ(T))$ defined as the set of those elements $L \in \mathcal{Q}$ for which there exist sequences $(x_n)_n$ and $(y_n)_n$ in the closed unity ball of $\mathcal{H}$ such that

$$\lim_{n \rightarrow \infty} \|L - x_n \square_T y_n\| = 0,$$

and

$$\lim_{n \rightarrow \infty} (\|w \square_T y_n\| + \|x_n \square_T w\|) = 0.$$

The importance of this set lies in the following proposition (cf. [5]).

PROPOSITION 2.5. *The set $\mathfrak{X}_\circ$ is absolutely convex and closed. If it contains a ball of (strictly) positive radius then Φ_T does have Property $(\mathbb{A}_{\aleph_0})$.*

We conclude this preliminary section with a property better expressed in the setting of coisometric ρ -dilation. Given $T \in L(\mathcal{H})$ a ρ -contraction, we consider a unitary ρ -dilation $U \in L(\mathcal{K})$ of T^* . Set $\mathcal{K}^+ = \bigvee_{n \geq 0} U^n \mathcal{H}$, $U^+ = U|_{\mathcal{K}^+}$, and $B = (U^+)^*$. The B operator is a coisometry called *the minimal coisometrical ρ -dilation of T* . In accordance with Wold's Theorem we will have $\mathcal{K}^+ = \mathcal{S} \oplus \mathcal{R}$, with $\mathcal{S}, \mathcal{R} \in Lat B$ and $B = S^* \oplus R$, where S is a shift operator on $\mathcal{S}$, and R is unitary on $\mathcal{R}$. Moreover

$$\mathcal{S} = \bigvee_{n \geq 1} Ker B^n = \bigvee_{\mu \in \mathbb{D}} Ker (B - \mu).$$

The first part of the following proposition, besides its usefulness later on, presents some independant interest in the sense that it shows what is preserved in general of the special $\rho = 1$ case (where B is an extension of T). The second part is just an easy but useful observation with a convenient reformulation based on the decomposition

$$H^\infty = \mathbb{C}1 \oplus H_0^\infty$$

and the notations $\widehat{L} = L|_{H_0^\infty}$, $x \otimes_T y := \widehat{x \square_T y}$.

PROPOSITION 2.6. *In the above context a), for $x \in \mathcal{H}$ and $n \in \mathbb{N}$:*

a) $Bx - Tx \in Ker B$

b) For $x, y \in \mathcal{H}$

$$Tx \square_T y = \rho Bx \square_B y,$$

and consequently, $Tx \otimes_T y = \rho Bx \otimes_B y$.

c) *Given an arbitrary sequence $(L_n)_n$ in $\mathcal{Q}$ the following two assertions are equivalent:*

(i) $L_n \rightarrow 0$

(ii) $\widehat{L_n} \rightarrow 0$ and $\langle 1, L_n \rangle \rightarrow 0$.

Proof. For $x, y \in \mathcal{H}$ and $n \in \mathbb{N}$ we will have

$$\begin{aligned} \langle B(Bx - Tx), B^{*n}y \rangle &= \langle x, U^{n+2}y \rangle - \langle Tx, U(U^n y) \rangle = \\ &= \frac{1}{\rho} \langle x, (T^*)^{n+2}y \rangle - \frac{1}{\rho} \langle Tx, (T^*)^{n+1}y \rangle = 0 \end{aligned}$$

Therefore,

$$Bx - Tx \in \text{Ker } B \quad (2.1)$$

This proves "a)".

We have also, for h in H^∞

$$\langle Tx \square_T y, h \rangle = (h(T)Tx, y) = \rho(h(B)x, y) = \langle \rho Bx \square_B y, h \rangle$$

that is "b)".

Finally "c)" results from the obvious fact that $\mathcal{Q}$ is isomorphic to the direct sum $\mathbb{C} \oplus \mathcal{Q}_0$ where $\mathcal{Q}_0$ is the predual of αH^∞ . $\square$

3 The main result

We begin with a lemma which establishes an essential connection between spectral properties and approximation set. We follow the usual notation for weak convergence of the sequence $(x_n)_n$ toward x : $x_n \rightharpoonup x$.

LEMMA 3.1. *Let $T \in L(\mathcal{H})$ be an absolutely continuous ρ -contraction and let $\lambda \in \sigma_{le}(T) \cap \mathbb{D}$, then $E_\lambda \in \mathfrak{X}_0(= \mathfrak{X}_0(T))$ via any sequences $(x_n)_n, (y_n)_n$ such that $x_n \rightharpoonup 0$, $\|x_n\| \rightarrow 1$ and $\|(T - \lambda)x_n\| \rightarrow 0$. The same result holds if $\lambda \in \sigma_{re}(T)$ and the sequence $(x_n)_n$ satisfies $x_n \rightharpoonup 0$, $\|x_n\| \rightarrow 1$ and $\|(T - \lambda)^*x_n\| \rightarrow 0$.*

Proof. It is straightforward to check that, for any $x, y \in \mathcal{H}$, $\lambda \in \mathbb{D}$, we have $\|x \square_{T^*} y\| = \|y \square_T x\|$ and $\|E_\lambda - x \square_T y\| = \|E_{\bar{\lambda}} - y \square_{T^*} x\|$. From these equalities combined with the fact that $\lambda \in \sigma_{re}(T)$ if and only if $\bar{\lambda} \in \sigma_{le}(T^*)$ we see that if $E_{\bar{\lambda}}$ belongs to $\mathfrak{X}_0(T^*)$ via sequences $(x_n), (y_n)$ then E_λ belongs to $\mathfrak{X}_0(T)$ via the sequences $(y_n), (x_n)$. It follows from these considerations that we just need to treat the case $\lambda \in \sigma_{le}(T)$. (Note that, by the classical geometric characterization of the left essential spectrum, among the sequences $(x_n)_n$ satisfying the above conditions, there are orthonormal ones.) To show that E_λ belongs to the approximation set $\mathfrak{X}_0$ via the sequences $(x_n), (x_n)$ we have three assertions to prove:

- (i) (the approximation of E_λ by the $x_n \square x_n$) $\lim_{n \rightarrow \infty} a_n = 0$ where $a_n := \|x_n \square x_n - E_\lambda\|$;
- (ii) (the vanishing condition on the right) given $w \in \mathcal{H}$, $\lim_{n \rightarrow \infty} b_n = 0$ where $b_n := \|x_n \square w\|$;

(iii) (the vanishing condition on the left) given $w \in \mathcal{H}$, $\lim_{n \rightarrow \infty} c_n = 0$ where $c_n := \|w \square x_n\|$.

Among those three, the first two hold (with of course the same spectral hypothesis) for any absolutely continuous polynomially bounded operator. This is rather well known and we give the proof only for the sake of completeness.

Proof of (i): By the Hahn-Banach theorem, for each n , there exists h_n of norm 1 in H^∞ such that

$$a_n = \langle x_n \square x_n - E_\lambda, h_n \rangle = (h_n(T)x_n, x_n) - h_n(\lambda).$$

Writing

$$h_n = h_n(\lambda) + (\alpha - \lambda)g_n$$

with g_n in H^∞ and (via the maximum modulus principle) $\|g_n\| \leq \frac{2}{1 - |\lambda|}$ we obtain (using also the fact that $\|x_n\| = 1$)

$$a_n = (g_n(T)(T - \lambda)x_n, x_n),$$

thus

$$a_n (\leq 2(1 - |\lambda|)^{-1} \|(T - \lambda)x_n\|) \rightarrow 0.$$

The proof of (ii) is similar: there exists a sequence (h_n) of functions of norm 1 in H^∞ such that (applying the same decomposition of h_n as in (i))

$$b_n = \langle x_n \square w, h_n \rangle = (h_n(T)x_n, w) = h_n(\lambda)(x_n, w) + (g_n(T)(T - \lambda)x_n, w),$$

thus, using the fact that (x_n) tends weakly to 0)

$$b_n (\leq |(x_n, w)| + 2(1 - |\lambda|)^{-1} \|(T - \lambda)x_n\| \|w\|) \rightarrow 0.$$

As to Assertion (iii), since

$$\langle w \square_T x_n, 1 \rangle = (w, x_n) \rightarrow 0$$

in view of Prop.2.5. we just need to show that $w \oslash_T x_n \rightarrow 0$ or, equivalently, $w \oslash_B x_n \rightarrow 0$. To show this last convergence it obviously suffices to prove that $w \square_B x_n \rightarrow 0$.

But with the notations introduced about the Wold decomposition of B it is easy to check that

$$w \square_B x_n = Qw \square_{S^*} Qx_n + Aw \square_R Ax_n,$$

(recall that Q, A are orthogonal projections on the subspaces $\mathcal{S}$, respectively $\mathcal{R}$ of this Wold decomposition).

Again here it is well-known that, since (Qx_n) tends weakly to 0, the first term in the right hand side of the above equality will tend to 0.

(Sketch of a quick proof - with the "boxes" here referring exclusively to the operator S^* -: The set

$$\mathcal{V} := \{z \in \mathcal{S}; z \square Qx_n \rightarrow 0\}$$

is a closed subspace of $\mathcal{S}$; for $\mu \in \mathbb{D}$ and $(S^* - \mu)z = 0$ we have, for some h_n of norm 1,

$$\|z \square Qx_n\| = (h_n(S^*)z, Qx_n) = h_n(\lambda)(z, Qx_n) \rightarrow 0.$$

This shows that $\mathcal{V} \supset \text{Span}_{\mu \in \mathbb{D}} \ker(S^* - \mu)$, that is $\mathcal{V} = \mathcal{S}$.)

We finally use Part a) of Proposition 2.2 to show that, like in the case of ordinary contractions, we have $Ax_n \rightarrow 0$. This will finish the proof of the fact that $w \square_B x_n \rightarrow 0$, hence, that of (iii) and complete the proof of the lemma.

Indeed from $Bx_n - Tx_n \in \text{Ker } B \subset \mathcal{S}$ it results that $ABx_n = ATx_n$. But $ABx_n = RAx_n$ and then

$$(1 - |\lambda|)\|Ax_n\| \leq \|(R - \lambda)Ax_n\| = \|A(B - \lambda)x_n\| = \|A(T - \lambda)x_n\| \leq \|(T - \lambda)x_n\|,$$

yielding the desired conclusion. $\square$

A first consequence of this fundamental lemma is the ("direct") extension to absolutely continuous ρ -contractions of a classical sufficient spectral condition for the associated representation Φ_T to enjoy the $(\mathbb{A}_{\aleph_0})$ property with the extra bonus (not immediately available via similarity to a contraction) that the approximation set is "as big as it can be". We recall that a subset $\Lambda \subset \mathbb{D}$ is said to be dominating if for any h in H^∞ we have $\|h\|_\infty = \sup_{\lambda \in \Lambda} |h(\lambda)|$. A straightforward application of the Hahn-Banach Theorem shows that this is equivalent to the fact the absolutely closed convex hull of $\{E_\lambda; \lambda \in \Lambda\}$ is the closed unit ball $\mathcal{Q}_1$ of $\mathcal{Q}$. Since the approximation set $\mathfrak{X}_0$ is absolutely convex and closed we obtain the following result.

THEOREM 3.2. *Let T be an absolutely continuous ρ -contraction such that $\sigma_e(T) \cap \mathbb{D}$ is dominant in $\mathbb{D}$ then $\mathfrak{X}_0(T) = \mathcal{Q}_1$ and therefore Φ_T has the property $(\mathbb{A}_{\aleph_0})$.*

Our main goal is to obtain an optimal converse of this result. To that end we first develop some more or less known consequences of the property $(\mathbb{A}_{\mathbb{N}_0})$.

PROPOSITION 3.3. *Let T be an absolutely continuous polynomially bounded operator such that the associated representation Φ_T has property $(\mathbb{A}_{\mathbb{N}_0})$ and let $\lambda \in \mathbb{D}$; then there exists an infinite dimensional semi-invariant subspace $\mathcal{J}$ for T such that $T_{\mathcal{J}}$, the compression of T to $\mathcal{J}$, is $\lambda I_{\mathcal{J}}$.*

Moreover if, in addition, T is a ρ -contraction then $E_{\lambda} \in \mathfrak{X}_0(T)$ via sequences $(e_n)_n$, $(e_n)_n$ where $(e_n)_n$ is any sequence of unit vectors in $\mathcal{J}$ converging weakly to 0 and consequently $\mathfrak{X}_0(T) = \mathcal{Q}_1$.

Proof. We apply the definition of Property $(\mathbb{A}_{\mathbb{N}_0})$ to the matrix $(\delta_{i,j} E_{\lambda})_{i,j \in \mathbb{N}}$ and get sequences of vectors (x_i) , (y_i) such that

$$\delta_{i,j} E_{\lambda} = x_i \square y_j, \quad i, j \in \mathbb{N}.$$

Observe that both sequences of vectors $(x_i)_i$ and $(y_j)_j$ are (algebraically) linearly independant (indeed, if some x_k is a finite linear combination of the other x_i 's, say $x_k = \sum_{i \neq k} \beta_i x_i$, we have an immediate contradiction upon expanding $x_k \square y_k$, namely $E_{\lambda} = 0$; similarly for the y_j 's).

Using Prop. 2.2. we note first that we may (and do) assume that the y_j 's belong to the invariant subspace $\mathcal{M}$ for T generated by the x_i 's, next that

$$0 = (T - \lambda)x_i \square_T y_j = x_i \square_T (T - \lambda)^* y_j = x_i \square_{T|_{\mathcal{M}}} (T|_{\mathcal{M}} - \lambda)^* y_j,$$

for all $i, j \in \mathbb{N}$ and finally (via c) of that proposition) that

$$(T|_{\mathcal{M}} - \lambda)^* y_j = 0$$

for all j .

The subspace $\mathcal{J} := \ker(T|_{\mathcal{M}} - \lambda)^* = \mathcal{M} \ominus \mathcal{N}$ with $\mathcal{N} = ((T - \lambda)\mathcal{M})^{\perp}$ is semi-invariant for T , infinite dimensional (it contains the y_j 's) and $T_{\mathcal{J}} (= (((T|_{\mathcal{M}})^*)|_{\mathcal{J}})^*) = \lambda I_{\mathcal{J}}$: it does the required job (note that so would do the a priori smaller subspace $\vee_j y_j$).

As to the last assertion, observe that $\mathcal{J} \subset \ker(T|_{\mathcal{M}} - \lambda)^* \cap \ker(T_{\mathcal{F}} - \lambda)$ where $\mathcal{F} := \mathcal{N}^{\perp}$ is invariant for T^* ; (this is easily seen from the 3×3 triangular block matrix decomposition of T relative to $\mathcal{H} = \mathcal{N} \oplus \mathcal{J} \oplus \mathcal{G} (= \mathcal{M} \oplus \mathcal{G} = \mathcal{N} \oplus \mathcal{F})$ with $\mathcal{G} = \mathcal{M}^{\perp}$

$$T = \begin{pmatrix} T|_{\mathcal{N}} & * & * \\ 0 & \lambda & * \\ 0 & 0 & T_{\mathcal{G}} \end{pmatrix}$$

Now it is a matter of combining Lemma 3.1. with a systematic use of Prop. 2.2. Let thus $(e_n)_n$ a sequence of unit vectors in $\mathcal{J}$ converging weakly to 0. Since $(T_{\mathcal{J}} - \lambda)e_n = 0$ and $(e_n, e_n) = 1$, we have $e_n \square_{T_{\mathcal{J}}} e_n = E_{\lambda}$; thus (via Prop. 2.2. c)) $e_n \square_{T_{\mathcal{J}}} e_n = E_{\lambda}$.

For $w \in \mathcal{H}$, we have (via Prop. 2.2. b)) $w \square_T e_n = P_{\mathcal{J}} w \square_{T_{\mathcal{J}}} e_n$ and this term tends to 0 by Lemma 3.1. applied to $T_{\mathcal{J}}$.

Similarly, $e_n \square_T w = e_n \square_{T|_{\mathcal{M}}} P_{\mathcal{M}} w \rightarrow 0$ (via Prop. 2.2. b) for the equality and by Lemma 3.1. applied to $T|_{\mathcal{M}}$ for the limit).

Thus we do have $E_{\lambda} \in \mathfrak{X}_0(T)$ via the sequences $(x_n)_n, (x_n)_n$.

This being true for any λ in $\mathbb{D}$ we obtain (via the absolute convexity and closedness of $\mathfrak{X}_0(T)$) the desired equality: $\mathfrak{X}_0(T) = \mathcal{Q}_1$. $\square$

We can now state and prove our main result.

THEOREM 3.4. *Let T be an absolutely continuous ρ -contraction. The following assertions are equivalent:*

- a) *The set $\mathfrak{X}_0(T)$ contains a ball centered at the origin.*
- b) *The associated functional calculus Φ_T has Property $(\mathbb{A}_{\mathbb{N}_0})$.*
- c) *For each $\lambda \in \mathbb{D}$ there exists a semi-invariant subspace $\mathcal{J}$ for T such that $T_{\mathcal{J}} = \lambda I_{\mathcal{J}}$.*
- d) *There exists a semi-invariant subspace $\mathcal{L}$ for T such that $\sigma_e(T_{\mathcal{L}}) \cap \mathbb{D}$ is dominating in $\mathbb{D}$.*
- e) *The approximation set $\mathfrak{X}_0(T)$ is maximal, i.e. coincides with the closed unit ball of $\mathcal{Q}$.*

Proof. We first point out the implications which hold for any absolutely continuous polynomially bounded operator: $e) \Rightarrow a) \Rightarrow b) \Rightarrow c)$ and $b) \Rightarrow d)$.

Among these, the first one is obvious, the second one ($a) \Rightarrow b)$) is the content of Prop. 2.5., the 3rd one is the first part of Prop. 3.3. and the last one can be proved along the same lines (cf. [3], Prop. 5.4.), whose proof given there for contractions works as well for polynomially bounded operators).

Thus to finish the proof it suffices to prove the implications $d) \Rightarrow b)$ and $c) \Rightarrow e)$. The latter follows from the second part of Prop. 3.3. while for $d) \Rightarrow b)$, under the dominatingness of $\sigma_e(T_{\mathcal{L}}) \cap \mathbb{D}$ we obtain (via Theorem 3.2.) property $(\mathbb{A}_{\mathbb{N}_0})$ for $\Phi_{T_{\mathcal{J}}}$ and consequently (via c) of Prop. 2.6.) also for Φ_T . $\square$

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