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Quarter-symmetric metric connection in a P-Sasakian manifold

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Abstract. In this paper, we consider a quarter-symmetric metric connection in a P-Sasakian manifold. We investigate the curvature tensor and the Ricci tensor of a P-Sasakian manifold with respect to the quarter-symmetric metric connection. We consider semisymmetric P-Sasakian manifold with respect to the quarter-symmetric metric connection. Furthermore, we consider generalized recurrent P-Sasakian manifolds and prove the non-existence of recurrent and pseudosymmetric P-Sasakian manifolds with respect to the quarter-symmetric metric connection. Finally, we construct an example of a 5-dimensional P-Sasakian manifold admitting quarter-symmetric metric connection which verifies Theorem 4.1.

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1 Introduction

A linear connection $\widetilde{\nabla}$ in a Riemannian manifold M is said to be a quarter-symmetric connection [10] if the torsion tensor T of the connection $\widetilde{\nabla}$

$$T(X,Y) = \widetilde{\nabla}_X Y - \widetilde{\nabla}_Y X - [X,Y] \tag{1.1}$$

satisfies

$$T(X,Y) = \eta(Y)\phi X - \eta(X)\phi Y, \tag{1.2}$$

where η is a 1-form and ϕ is a $(1,1)$ tensor field. If moreover, a quarter-symmetric connection $\widetilde{\nabla}$ satisfies the condition

$$(\widetilde{\nabla}_X g)(Y,Z) = 0, \tag{1.3}$$

where $X, Y, Z \in \chi(M)$, then $\widetilde{\nabla}$ is said to be a quarter-symmetric metric connection, otherwise it is said to be a quarter-symmetric non-metric connection. If we put $\phi X = X$, then the quarter-symmetric metric connection reduces to a semi-symmetric metric connection [22]. Thus the notion of quarter-symmetric connection generalizes the idea of the semi-symmetric connection.

A non-flat n-dimensional Riemannian manifold $(n > 3)$ is called generalized recurrent [7] if its curvature tensor R satisfies the condition

$$(\nabla_X R)(Y,Z)W = \alpha(X)R(Y,Z)W + \beta(X)[g(Z,W)Y - g(Y,W)Z], \tag{1.4}$$

where ∇ is the Levi-Civita connection and α and β are two 1-forms, $\beta \neq 0$. If $\beta = 0$ and $\alpha \neq 0$, then M is called recurrent [21].

A non-flat n-dimensional Riemannian manifold M $(n > 3)$ is said to be pseudosymmetric [4] if there exists a non-zero 1-form α on M such that

$$\begin{aligned}(\nabla_X R)(Y,Z)W &= 2\alpha(X)R(Y,Z)W + \alpha(Y)R(X,Z)W \\ &\quad +\alpha(Z)R(Y,X)W + \alpha(W)R(Y,Z)X \\ &\quad +g(R(Y,Z)W,X)\rho,\end{aligned} \tag{1.5}$$

where $X, Y, Z, W \in \chi(M)$ and ρ is the corresponding vector field metrically equivalent to the 1-form α defined by

$$g(X,\rho) = \alpha(X), \tag{1.6}$$

for all $X \in \chi(M)$.

A Riemannian manifold (M, g) is called locally symmetric if its curvature tensor R is parallel, that is, $\nabla R = 0$. The notion of semisymmetric, a proper generalization of locally symmetric manifold, is defined by $R(X,Y) \cdot R = 0$, where $R(X,Y)$ acts on R as a derivation. A complete intrinsic classification of these manifolds was given by Szabó in [20].

Quarter-symmetric metric connection in a Riemannian manifold studied by several authors such as S.C. Rastogi ([15], [16]), Yano and Imai [23], Mukhopadhyay, Roy and Barua [13], Biswas and De [3], De and Mondal [5], Sular et al [19], Kumar et al [11] and many others.

Motivated by the above studies in the present paper, we study quarter-symmetric metric connection in a P-Sasakian manifold. The paper is organized as follows: In Section 2, we first give a brief account of P-Sasakian manifolds. In Section 3, we obtain the expressions of the curvature tensor and the Ricci tensor of a P-Sasakian manifold with respect to the quarter-symmetric metric connection. Section 4 is devoted to study semisymmetric P-sasakian manifolds with respect to the quarter-symmetric metric connection and in this case we have shown that such manifolds are Einstein manifolds with respect to the quarter-symmetric metric connection. Section 5 deals with generalized recurrent P-Sasakian manifold with respect to the quarter-symmetric metric connection. In Section 6, we consider pseudosymmetric P-Sasakian manifold with respect to the quarter-symmetric metric connection and we obtain the non-existence of these type of manifolds. Finally, we construct an example of a 5-dimensional P-Sasakian manifold admitting quarter-symmetric metric connection which verifies Theorem 4.1.

2 P-Sasakian manifolds

Let M be an n-dimensional differentiable manifold of class C^∞ in which there are given a $(1,1)$-type tensor field ϕ, a characteristic vector field ξ and a 1-form η such that

$$\phi^2 X = X - \eta(X)\xi,\ \phi\xi = 0,\ \eta(\xi) = 1,\ \eta(\phi X) = 0. \tag{2.1}$$

Then (ϕ, ξ, η) is called an almost paracontact structure and M an almost paracontact manifold. Moreover, if M admits a Riemannian metric g such that

$$g(\xi, X) = \eta(X),\ g(\phi X, \phi Y) = g(X,Y) - \eta(X)\eta(Y), \tag{2.2}$$

then (ϕ, ξ, η, g) is called an almost paracontact metric structure and M an almost paracontact metric manifold [17]. If (ϕ, ξ, η, g) satisfy the following equations:

$$\begin{aligned} &d\eta = 0,\ \nabla_X \xi = \phi X, \\ &(\nabla_X \phi)Y = -g(X,Y)\xi - \eta(Y)X + 2\eta(X)\eta(Y)\xi, \end{aligned} \tag{2.3}$$

then M is called a para-Sasakian manifold or briefly a P-Sasakian manifold [1]. Especially, a P-Sasakian manifold M is called a special para-Sasakian manifold or briefly a SP-Sasakian manifold [18] if M admits a 1-form η satisfying

$$(\nabla_X \eta)(Y) = -g(X,Y) + \eta(X)\eta(Y). \tag{2.4}$$

Also in a P-Sasakian manifold the following relations hold [1], [6], [14]:

$$S(X,\xi) = -(n-1)\eta(X),\ Q\xi = -(n-1)\xi, \tag{2.5}$$

$$\eta(R(X,Y)Z) = g(X,Z)\eta(Y) - g(Y,Z)\eta(X), \tag{2.6}$$

$$R(X,\xi)Y = g(X,Y)\xi - \eta(Y)X, \tag{2.7}$$

$$R(X,Y)\xi = \eta(X)Y - \eta(Y)X, \tag{2.8}$$

$$S(\phi X,\phi Y) = S(X,Y) + (n-1)\eta(X)\eta(Y), \tag{2.9}$$

$$\eta(R(X,Y)\xi) = 0, \tag{2.10}$$

for any vector fields $X, Y, Z \in \chi(M)$, where R is the Riemannian curvature tensor, S is the Ricci tensor and Q is the Ricci operator defined by

$$g(QX,Y) = S(X,Y).$$

P-Sasakian manifolds have been studied by several authors such as De et al [8], Yildiz et al [24], Deshmukh and Ahmed [9], Matsumoto, Ianus and Mihai [12], Özgür [14], Adati and Miyazawa [2] and many others.

An almost paracontact Riemannian manifold M is said to be an η-Einstein manifold if the Ricci tensor S satisfies the condition

$$S(X,Y) = ag(X,Y) + b\eta(X)\eta(Y),$$

where a and b are smooth functions on the manifold. In particular, if $b = 0$, then M is an Einstein manifold.

3 Curvature tensor of a P-Sasakian manifold with respect to the quarter-symmetric metric connection

Let $\widetilde{\nabla}$ be a linear connection and ∇ be the Levi-Civita connection of a P-Sasakian manifold M such that

$$\widetilde{\nabla}_X Y = \nabla_X Y + U(X,Y), \tag{3.1}$$

where U is a $(1,1)$-type tensor. For $\widetilde{\nabla}$ to be a quarter-symmetric metric connection in M, we have [10],

$$U(X,Y) = \frac{1}{2}[T(X,Y) + T'(X,Y) + T'(Y,X)], \tag{3.2}$$

where

$$g(T'(X,Y),Z) = g(T(Z,X),Y). \tag{3.3}$$

From (1.2) and (3.3) we get

$$T'(X,Y) = \eta(X)\phi Y - g(\phi X,Y)\xi. \tag{3.4}$$

Using (1.2) and (3.4) in (3.2), we have

$$U(X,Y) = \eta(Y)\phi X - g(\phi X,Y)\xi. \tag{3.5}$$

Therefore a quarter-symmetric metric connection $\widetilde{\nabla}$ in a P-Sasakian manifold is given by

$$\widetilde{\nabla}_X Y = \nabla_X Y + \eta(Y)\phi X - g(\phi X,Y)\xi. \tag{3.6}$$

Let $\widetilde{R}$ and R be the curvature tensors with respect to the quarter-symmetric metric connection $\widetilde{\nabla}$ and the Levi-Civita connection ∇ respectively. Then we have from (3.6),

$$\begin{aligned}\widetilde{R}(X,Y)U \quad = \quad & R(X,Y)U + 3g(\phi X,U)\phi Y - 3g(\phi Y,U)\phi X \\ & +\eta(U)[\eta(X)Y - \eta(Y)X] \\ & -[\eta(X)g(Y,U) - \eta(Y)g(X,U)]\xi,\end{aligned} \tag{3.7}$$

where

$$\widetilde{R}(X,Y)U = \widetilde{\nabla}_X\widetilde{\nabla}_Y U - \widetilde{\nabla}_Y\widetilde{\nabla}_X U - \widetilde{\nabla}_{[X,Y]}U$$

and $X,Y,Z \in \chi(M)$. By suitable contraction we have from (3.7)

$$\widetilde{S}(Y,U) = S(Y,U) + 2g(Y,U) - (n+1)\eta(Y)\eta(U) - 3\ \mathrm{trace}\phi\ g(\phi Y,U), \tag{3.8}$$

where $\widetilde{S}$ and S are the Ricci tensors of the connections $\widetilde{\nabla}$ and ∇, respectively. Hence we have the following theorem:

Theorem 3.1. *For a P-Sasakian manifold (M,g) with respect to the quarter-symmetric metric connection $\widetilde{\nabla}$*

(a) The curvature tensor $\widetilde{R}$ is given by (3.7),

(b) The Ricci tensor $\widetilde{S}$ is symmetric,

(c) $\widetilde{R}(X,Y,Z,W) + \widetilde{R}(X,Y,W,Z) = 0$,

(d) $\widetilde{R}(X, Y, Z, W) + \widetilde{R}(Y, X, Z, W) = 0$,

(e) $\widetilde{R}(X, Y, Z, W) = \widetilde{R}(Z, W, X, Y)$,

(f) $\widetilde{S}(Y, \xi) = -2(n-1)\eta(Y)$,

where $X, Y, Z, W \in \chi(M)$.

With the help of $(2.7), (2.8)$ and (2.1) in (3.7) we obtain

$$\widetilde{R}(\xi, Y)U = 2[\eta(U)Y - g(U, Y)\xi] \tag{3.9}$$

and

$$\widetilde{R}(X, Y)\xi = 2[\eta(X)Y - \eta(Y)X], \tag{3.10}$$

where $X, Y \in \chi(M)$.

4 Semisymmetric P-Sasakian manifolds with respect to the quarter-symmetric metric connection

In this section we characterize semisymmetric P-Sasakian manifolds with respect to the quarter-symmetric metric connection, that is, the curvature tensor satisfies the condition

$$(\widetilde{R}(\xi, Y) \cdot \widetilde{R})(U, V)W = 0.$$

This implies

$$\begin{aligned} &\widetilde{R}(\xi, Y)\widetilde{R}(U, V)W - \widetilde{R}(\widetilde{R}(\xi, Y)U, V)W \\ &-\widetilde{R}(U, \widetilde{R}(\xi, Y)V)W - \widetilde{R}(U, V)\widetilde{R}(\xi, Y)W = 0. \end{aligned} \tag{4.1}$$

Using (3.9) and (4.1) yields

$$\begin{aligned} &2\eta(\widetilde{R}(U, V)W)Y - 2g(Y, \widetilde{R}(U, V)W)\xi - 2\eta(U)\widetilde{R}(Y, V)W \\ &+2g(Y, U)\widetilde{R}(\xi, V)W - 2\eta(V)\widetilde{R}(U, Y)W + 2g(V, Y)\widetilde{R}(U, \xi)W \\ &-2\eta(W)\widetilde{R}(U, V)Y + 2g(Y, W)\widetilde{R}(U, V)\xi = 0. \end{aligned} \tag{4.2}$$

Taking inner product of (4.2) with ξ, we obtain

$$\begin{aligned} &2\eta(\widetilde{R}(U, V)W)\eta(Y) - 2g(Y, \widetilde{R}(U, V)W) - 2\eta(U)\eta(\widetilde{R}(Y, V)W) \\ &+2g(Y, U)\eta(\widetilde{R}(\xi, V)W) - 2\eta(V)\eta(\widetilde{R}(U, Y)W) + 2g(V, Y)\eta(\widetilde{R}(U, \xi)W) \\ &-2\eta(W)\eta(\widetilde{R}(U, V)Y) + 2g(Y, W)\eta(\widetilde{R}(U, V)\xi) = 0. \end{aligned} \tag{4.3}$$

With the help of (3.7), (3.9), (3.10) we get from (4.3)

$$\begin{aligned}
&4\eta(Y)[g(U,W)\eta(V)-g(V,W)\eta(U)]-2\{g(R(U,V)W,Y)\\
&+3g(\phi U,W)g(\phi V,Y)-3g(\phi V,W)g(\phi U,Y)+\eta(W)[\eta(U)g(V,Y)\\
&-\eta(V)g(U,Y)]-\eta(Y)[\eta(U)g(V,W)-\eta(V)g(U,W)]\}\\
&-4\eta(U)[g(Y,W)\eta(V)-g(V,W)\eta(Y)]+4g(Y,U)[\eta(V)\eta(W)\\
&-g(V,W)]-4\eta(V)[g(U,W)\eta(Y)-g(Y,W)\eta(U)]\\
&+4g(V,Y)[g(U,W)-\eta(W)\eta(U)]\\
&-4\eta(W)[g(U,Y)\eta(V)-g(V,Y)\eta(U)]=0. \qquad (4.4)
\end{aligned}$$

Equation (4.4) implies

$$\begin{aligned}
&g(R(U,V)W,Y)+3g(\phi U,W)g(\phi V,Y)-3g(\phi V,W)g(\phi U,Y)\\
&+g(V,Y)\eta(U)\eta(W)-g(U,Y)\eta(V)\eta(W)-g(V,W)\eta(U)\eta(Y)\\
&+g(U,W)\eta(V)\eta(Y)+2g(Y,U)g(V,W)-2g(U,W)g(V,Y)=0. \qquad (4.5)
\end{aligned}$$

Now putting $V=W=e_i$ in (4.5), where $\{e_i\}$ is an orthonormal basis of the tangent space at each point of the manifold, and taking summation over $i=1,2,...,n$, we get

$$S(U,Y)=-2ng(U,Y)+(n+1)\eta(U)\eta(Y)+3\mathrm{trace}\phi\ g(\phi U,Y). \qquad (4.6)$$

Again from (3.8) and (4.6) we have

$$\widetilde{S}(U,Y)=-2(n-1)g(U,Y). \qquad (4.7)$$

By making contraction of (4.7) we obtain

$$\widetilde{r}=-2n(n-1). \qquad (4.8)$$

This leads to the following theorem:

Theorem 4.1. *If a P-Sasakian manifold is semisymmetric with respect to the quarter-symmetric metric connection, then the manifold is an Einstein manifold with respect to the quarter-symmetric metric connection and the scalar curvature with respect to the quarter-symmetric metric connection is a negative constant.*

5 Generalized recurrent P-Sasakian manifolds with respect to the quarter-symmetric metric connection

In this section we consider generalized recurrent P-Sasakian manifolds with respect to the quarter-symmetric metric connection $\widetilde{\nabla}$. Let us assume that there exists a generalized recurrent P-Sasakian manifold M with respect to the quarter-symmetric metric connection $\widetilde{\nabla}$. Then from (1.4), we have

$$(\widetilde{\nabla}_X \widetilde{R})(Y,Z)W = \alpha(X)\widetilde{R}(Y,Z)W + \beta(X)[g(Z,W)Y - g(Y,W)Z], \quad (5.1)$$

for all vector fields $X, Y, Z, W \in \chi(M)$. Substituting $Y = W = \xi$ in (5.1) we obtain

$$(\widetilde{\nabla}_X \widetilde{R})(\xi,Z)\xi = \alpha(X)\widetilde{R}(\xi,Z)\xi + \beta(X)[\eta(Z)\xi - Z]. \quad (5.2)$$

We have from (3.10)

$$(\widetilde{\nabla}_X \widetilde{R})(Y,Z)\xi = 2[((\widetilde{\nabla}_X \eta)Y)Z - ((\widetilde{\nabla}_X \eta)Z)Y]. \quad (5.3)$$

On the other hand using (3.6), (2.3) and (1.3) we get

$$(\widetilde{\nabla}_X \eta)Y = 2g(Y, \phi X). \quad (5.4)$$

So by the use of (5.4) in (5.3) we have

$$(\widetilde{\nabla}_X \widetilde{R})(Y,Z)\xi = 4[g(Y,\phi X)Z - g(Z,\phi X)Y]. \quad (5.5)$$

Putting $Y = \xi$ in the above equation yields

$$(\widetilde{\nabla}_X \widetilde{R})(\xi,Z)\xi = -4g(Z,\phi X)\xi. \quad (5.6)$$

Again from (3.10) we have

$$\widetilde{R}(\xi,Z)\xi = 2[Z - \eta(Z)\xi]. \quad (5.7)$$

Thus we get from (5.2) and (5.7)

$$(\widetilde{\nabla}_X \widetilde{R})(\xi,Z)\xi = 2\alpha(X)[Z - \eta(Z)\xi] + \beta(X)[\eta(Z)\xi - Z]. \quad (5.8)$$

Hence comparing the right hand sides of the equations (5.6) and (5.8) we obtain

$$-4g(Z,\phi X)\xi = 2\alpha(X)[Z - \eta(Z)\xi] - \beta(X)[Z - \eta(Z)\xi]. \quad (5.9)$$

Operating ϕ both sides of (5.9) and using (2.1), we get

$$\beta(X) = 2\alpha(X). \quad (5.10)$$

This leads to the following theorem:

Theorem 5.1. *If a P-Sasakian manifold is generalized recurrent with respect to the quarter-symmetric metric connection $\widetilde{\nabla}$, then the associated 1-forms are related by $\beta = 2\alpha$.*

Next, we consider recurrent P-Sasakian manifold M with respect to the quarter-symmetric metric connection $\widetilde{\nabla}$. Then from (1.4), we have

$$(\widetilde{\nabla}_X \widetilde{R})(Y,Z)W = \alpha(X)\widetilde{R}(Y,Z)W, \tag{5.11}$$

for all vector fields $X, Y, Z, W \in \chi(M)$. From Theorem 5.1. we obtain

$$\alpha(X) = 0. \tag{5.12}$$

Hence we have the following corollary:

Corollary 5.2. *There is no recurrent P-Sasakian manifold with respect to the quarter-symmetric metric connection $\widetilde{\nabla}$.*

6 Pseudosymmetric P-Sasakian manifolds with respect to the quarter-symmetric metric connection

This section is devoted to study of pseudosymmetric P-Sasakian manifolds with respect to the quarter-symmetric metric connection. We prove the following theorem:

Theorem 6.1. *There is no pseudosymmetric P-Sasakian manifold with respect to the quarter-symmetric metric connection $\widetilde{\nabla}$.*

Proof. Let us assume that there exists a pseudosymmetric P-Sasakian manifold with respect to the quarter-symmetric metric connection $\widetilde{\nabla}$. Then we get from (1.5)

$$\begin{aligned}(\widetilde{\nabla}_X \widetilde{R})(Y,Z)W &= 2\alpha(X)\widetilde{R}(Y,Z)W + \alpha(Y)\widetilde{R}(X,Z)W \\ &\quad +\alpha(Z)\widetilde{R}(Y,X)W + \alpha(W)\widetilde{R}(Y,Z)X \\ &\quad +g(\widetilde{R}(Y,Z)W,X)\rho. \end{aligned}\tag{6.1}$$

So contracting Y in (6.1), we have

$$\begin{aligned}(\widetilde{\nabla}_X \widetilde{S})(Z,W) &= 2\alpha(X)\widetilde{S}(Z,W) + \alpha(\widetilde{R}(X,Z)W) \\ &\quad +\alpha(Z)\widetilde{S}(X,W) + \alpha(W)\widetilde{S}(Z,X) \\ &\quad +g(\widetilde{R}(\rho,Z)W,X). \end{aligned}\tag{6.2}$$

Substituting $W = \xi$ in (6.2) we get

$$\begin{aligned}(\widetilde{\nabla}_X \widetilde{S})(Z,\xi) &= 2\alpha(X)\widetilde{S}(Z,\xi) + \alpha(\widetilde{R}(X,Z)\xi) \\ &\quad +\alpha(Z)\widetilde{S}(X,\xi) + \alpha(\xi)\widetilde{S}(Z,X) \\ &\quad +g(\widetilde{R}(\rho,Z)\xi,X). \end{aligned}\tag{6.3}$$

From Theorem 3.1. we get

$$\widetilde{S}(Z,\xi) = -2(n-1)\eta(Z).$$

Hence using (5.4) it follows that

$$(\widetilde{\nabla}_X\widetilde{S})(Z,\xi) = -4(n-1)g(Z,\phi X). \tag{6.4}$$

On the other hand, in view of (2.1) and Theorem 3.1. we obtain

$$\begin{aligned}(\widetilde{\nabla}_X\widetilde{S})(Z,\xi) &= -4n\alpha(X)\eta(Z) + 2\eta(X)\alpha(Z)\\ &\quad -2(n-1)\alpha(Z)\eta(X) + 2\alpha(\xi)g(X,Z)\\ &\quad +\alpha(\xi)\widetilde{S}(X,Z).\end{aligned} \tag{6.5}$$

From (6.4) and (6.5), we get

$$\begin{aligned}-4(n-1)g(Z,\phi X) &= -4n\alpha(X)\eta(Z) + 2\eta(X)\alpha(Z)\\ &\quad -2(n-1)\alpha(Z)\eta(X) + 2\alpha(\xi)g(X,Z)\\ &\quad +\alpha(\xi)\widetilde{S}(X,Z).\end{aligned} \tag{6.6}$$

Taking $X=\xi$ in the above equation gives

$$\begin{aligned}-4(n-1)g(Z,\phi\xi) &= -4n\alpha(\xi)\eta(Z) + 2\eta(\xi)\alpha(Z)\\ &\quad -2(n-1)\alpha(Z)\eta(\xi) + 2\alpha(\xi)\eta(Z)\\ &\quad +\alpha(\xi)\widetilde{S}(\xi,Z).\end{aligned} \tag{6.7}$$

By making use of (2.1), (2.2), (1.6) and Theorem 3.1. in (6.7) yields

$$(2-3n)\alpha(\xi)\eta(Z) + (2-n)\alpha(Z) = 0. \tag{6.8}$$

Replacing Z with ξ in (6.8), we have (since $n > 3$)

$$\alpha(\xi) = 0. \tag{6.9}$$

Now using (6.9) it follows from (6.8) that

$$\alpha(Z) = 0,$$

for every vector field Z on M, which implies that $\alpha = 0$ on M. This contradicts to our assumption.
Thus the proof of our theorem is completed. □

7 Example of a 5-dimensional P-Sasakian manifold admitting quarter-symmetric metric connection

Example 7.1. We consider the 5-dimensional manifold $\{(x, y, z, u, v) \in \mathbb{R}^5\}$, where (x, y, z, u, v) are the standard coordinates in $\mathbb{R}^5$.
We choose the vector fields

$$e_1 = \frac{\partial}{\partial x},\ e_2 = e^{-x}\frac{\partial}{\partial y},\ e_3 = e^{-x}\frac{\partial}{\partial z},\ e_4 = e^{-x}\frac{\partial}{\partial u},\ e_5 = e^{-x}\frac{\partial}{\partial v},$$

which are linearly independent at each point of M.
Let g be the Riemannian metric defined by

$$g(e_i, e_j) = \begin{cases} 1 & \text{if} \quad i = j \\ 0 & \text{if} \quad i \neq j;\ i, j = 1, 2, 3, 4, 5. \end{cases}$$

Let η be the 1-form defined by

$$\eta(Z) = g(Z, e_1),$$

for any $Z \in \chi(M)$.
Let ϕ be the $(1, 1)$-tensor field defined by

$$\phi(e_1) = 0,\ \phi(e_2) = e_2,\ \phi(e_3) = e_3,\ \phi(e_4) = e_4,\ \phi(e_5) = e_5.$$

Using the linearity of ϕ and g, we have

$$\eta(e_1) = 1,\ \phi^2 Z = Z - \eta(Z)e_1$$

and

$$g(\phi Z, \phi U) = g(Z, U) - \eta(Z)\eta(U),$$

for any vector fields $Z, U \in \chi(M)$. Thus for $e_1 = \xi$, the structure (ϕ, ξ, η, g) defines an almost paracontact metric structure on M.
Then we have

$$[e_1, e_2] = -e_2, [e_1, e_3] = -e_3, [e_1, e_4] = -e_4, [e_1, e_5] = -e_5,$$
$$[e_2, e_3] = [e_2, e_4] = [e_2, e_5] = [e_3, e_4] = [e_3, e_5] = [e_4, e_5] = 0.$$

The Levi-Civita connection ∇ of the metric tensor g is given by Koszul's formula which is given by

$$\begin{aligned} 2g(\nabla_X Y, Z) &= Xg(Y, Z) + Yg(Z, X) - Zg(X, Y) \\ &\quad - g(X, [Y, Z]) - g(Y, [X, Z]) + g(Z, [X, Y]). \end{aligned} \tag{7.1}$$

Taking $e_1 = \xi$ and using (7.1), we get the following:

$$\begin{aligned}
&\nabla_{e_1} e_1 = 0,\ \nabla_{e_1} e_2 = 0,\ \nabla_{e_1} e_3 = 0,\ \nabla_{e_1} e_4 = 0,\ \nabla_{e_1} e_5 = 0,\\
&\nabla_{e_2} e_1 = e_2,\ \nabla_{e_2} e_2 = -e_1,\ \nabla_{e_2} e_3 = 0,\ \nabla_{e_2} e_4 = 0,\ \nabla_{e_2} e_5 = 0,\\
&\nabla_{e_3} e_1 = e_3,\ \nabla_{e_3} e_2 = 0,\ \nabla_{e_3} e_3 = -e_1,\ \nabla_{e_3} e_4 = 0,\ \nabla_{e_3} e_5 = 0,\\
&\nabla_{e_4} e_1 = e_4,\ \nabla_{e_4} e_2 = 0,\ \nabla_{e_4} e_3 = 0,\ \nabla_{e_4} e_4 = -e_1,\ \nabla_{e_4} e_5 = 0,\\
&\nabla_{e_5} e_1 = e_5,\ \nabla_{e_5} e_2 = 0,\ \nabla_{e_5} e_3 = 0,\ \nabla_{e_5} e_4 = 0,\ \nabla_{e_5} e_5 = -e_1.
\end{aligned}$$

Using the above equations in (3.6) yields

$$\begin{aligned}
&\widetilde{\nabla}_{e_1} e_1 = 0,\ \widetilde{\nabla}_{e_1} e_2 = 0,\ \widetilde{\nabla}_{e_1} e_3 = 0,\ \widetilde{\nabla}_{e_1} e_4 = 0,\ \widetilde{\nabla}_{e_1} e_5 = 0,\\
&\widetilde{\nabla}_{e_2} e_1 = 2e_2,\ \widetilde{\nabla}_{e_2} e_2 = -2e_1,\ \widetilde{\nabla}_{e_2} e_3 = 0,\ \widetilde{\nabla}_{e_2} e_4 = 0,\ \widetilde{\nabla}_{e_2} e_5 = 0,\\
&\widetilde{\nabla}_{e_3} e_1 = 2e_3,\ \widetilde{\nabla}_{e_3} e_2 = 0,\ \widetilde{\nabla}_{e_3} e_3 = -2e_1,\ \widetilde{\nabla}_{e_3} e_4 = 0,\ \widetilde{\nabla}_{e_3} e_5 = 0,\\
&\widetilde{\nabla}_{e_4} e_1 = 2e_4,\ \widetilde{\nabla}_{e_4} e_2 = 0,\ \widetilde{\nabla}_{e_4} e_3 = 0,\ \widetilde{\nabla}_{e_4} e_4 = -2e_1,\ \widetilde{\nabla}_{e_4} e_5 = 0,\\
&\widetilde{\nabla}_{e_5} e_1 = 2e_5,\ \widetilde{\nabla}_{e_5} e_2 = 0,\ \widetilde{\nabla}_{e_5} e_3 = 0,\ \widetilde{\nabla}_{e_5} e_4 = 0,\ \widetilde{\nabla}_{e_5} e_5 = -2e_1.
\end{aligned}$$

By the above results, we can easily obtain the non-vanishing components of the curvature tensors as follows:

$$\begin{aligned}
&R(e_1,e_2)e_1 = e_2,\ R(e_1,e_2)e_2 = -e_1,\ R(e_1,e_3)e_1 = e_3,\ R(e_1,e_3)e_3 = -e_1,\\
&R(e_1,e_4)e_1 = e_4,\ R(e_1,e_4)e_4 = -e_1,\ R(e_1,e_5)e_1 = e_5,\ R(e_1,e_5)e_5 = -e_1,\\
&R(e_2,e_3)e_2 = e_3,\ R(e_2,e_3)e_3 = -e_2,\ R(e_2,e_4)e_2 = e_4,\ R(e_2,e_4)e_4 = -e_2,\\
&R(e_2,e_5)e_2 = e_5,\ R(e_2,e_5)e_5 = -e_2,\ R(e_3,e_4)e_3 = e_4,\ R(e_3,e_4)e_4 = -e_3,\\
&R(e_3,e_5)e_3 = e_5,\ R(e_3,e_5)e_5 = -e_3,\ R(e_4,e_5)e_4 = e_5,\ R(e_4,e_5)e_5 = -e_4,
\end{aligned}$$

$$\begin{aligned}
&\widetilde{R}(e_1,e_2)e_1 = 2e_2,\ \widetilde{R}(e_1,e_2)e_2 = -2e_1,\ \widetilde{R}(e_1,e_3)e_1 = 2e_3,\\
&\widetilde{R}(e_1,e_3)e_3 = -2e_1, \widetilde{R}(e_1,e_4)e_1 = 2e_4,\ \widetilde{R}(e_1,e_4)e_4 = -2e_1,\\
&\widetilde{R}(e_1,e_5)e_1 = 2e_5,\ \widetilde{R}(e_1,e_5)e_5 = -2e_1, \widetilde{R}(e_2,e_3)e_2 = 2e_3,\\
&\widetilde{R}(e_2,e_3)e_3 = -2e_2,\ \widetilde{R}(e_2,e_4)e_2 = 2e_4,\ \widetilde{R}(e_2,e_4)e_4 = -2e_2,\\
&\widetilde{R}(e_2,e_5)e_2 = 2e_5,\ \widetilde{R}(e_2,e_5)e_5 = -2e_2,\ \widetilde{R}(e_3,e_4)e_3 = 2e_4,\\
&\widetilde{R}(e_3,e_4)e_4 = -2e_3, \widetilde{R}(e_3,e_5)e_3 = 2e_5,\ \widetilde{R}(e_3,e_5)e_5 = -2e_3,\\
&\widetilde{R}(e_4,e_5)e_4 = 2e_5,\ \widetilde{R}(e_4,e_5)e_5 = -2e_4.
\end{aligned}$$

From the expressions of the curvature tensor it follows that the manifold is a manifold of constant curvature -2 with respect to the quarter-symmetric metric connection. Hence the manifold is semisymmetric with respect to the quarter-symmetric metric connection. Using the above expressions of the curvature tensor we get

$$\widetilde{S}(e_1, e_1) = \widetilde{S}(e_2, e_2) = \widetilde{S}(e_3, e_3) = \widetilde{S}(e_4, e_4) = \widetilde{S}(e_5, e_5) = -8.$$

Hence the scalar curvature

$$\widetilde{r} = -40.$$

It can be easily verified that the manifold is an Einstein manifold.
Thus Theorem 4.1. is verified.

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