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The Bandwidths of a Matrix. A Survey of Algorithms

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Abstract. The bandwidth, average bandwidth, envelope, profile and antibandwidth of the matrices have been the subjects of study for at least 45 years. These problems have generated considerable interest over the years because of them practical relevance in areas like: solving the system of equations, finite element methods, circuit design, hypertext layout, chemical kinetics, numerical geophysics etc. In this paper a brief description of these problems are made in terms of their definitions, followed by a comparative study of them, using both approaches: matrix geometry and graph theory. Time evolution of the corresponding algorithms as well as a short description of them are made. The work also contains concrete real applications for which a large part of presented algorithms were developed.

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1 Introduction

The bandwidth, average bandwidth, envelope, profile and antibandwidth optimization of matrices have been the subjects of study for at least 45 years.

These problems have generated considerable interest over the years because of them practical relevance for a significant range of global optimization applications.
A first approach to bandwidth problem was made by Harper in 1964 [58]. The same author introduces the concept of the lower boundary of the bandwidth in paper [59]. After that, a lot of approaches have been made of this issue, concretized in algorithms and their real life applications. Today, sophisticated approaches, known as *metaheuristics*, that combines techniques from artificial intelligence with powerful optimization methods from mathematics are more and more used. The bandwidths optimization algorithms are currently used in many real life applications, such as: preprocessing the coefficient matrix for solving the system of equations, finite element methods for approximating solutions of partial differential equations, largescale power transmission systems, circuit design, hypertext layout, chemical kinetics, numerical geophysics etc.
In the first part of this work a brief description of these problems are made in terms of their definitions, using both approaches: matriceal and graph theory. That is followed by a comparative study of them. Time evolution of the corresponding algorithms as well as a short description of them it is also made in other section of this paper. The last section contains concrete real applications, for which a large part of presented algorithms were developed.

2 Theoretical considerations for matrix bandwidths optimization

2.1 Matrices and graphs. Generalities

The connection between matrices and graphs is very well known: a graph can be represented by an adjacency matrix and a matrix can be easily translated into a graph. Thus, the terms symmetrical/unsymmetrical from matrices becomes undirected/directed to graphs.
The case of sparse matrices presents a particular interest due to real problems needed to be solved. Most of the references to sparse matrices define them as matrices which has very few nonzero elements relative to their size. This definition is somewhat vaguely, although most often a percentage of 5-10 % for the number of nonzero elements is given. We consider that a matrix can be called "sparse" if special techniques that exploits the small nonzeros/zeros ratio can be used to improve a computing process. A first application of sparsity consist in reducing memory space needed for such matrices, based

on the idea that null values should not be stored. And the examples could go on but we will mention here only the special numerical methods developed for solving large and sparse systems of equations.
Time and experience have proven that graph theory is a good choice to represent sparse matrices. Examples of this are: the preconditioning techniques used in solving equations systems and parallel models for solving systems of equations. An example of such transposing from matrix format in graph theory is the case of an equations system, outlined below.
Consider a system of n equations with n unknowns and its nxn associated sparse matrix. The adjacenty graph of this matrix is a graph $G(V, E)$ whose n vertices in V set represent the n unknowns of equations system. The edges of graph represent the binary relation established by the equations: an edge between i and j nodes exist if the element a_{ij} of the associated matrix is nonzero ($a_{ij} \neq 0$). Therefore, an edge represent this binary relation: equation i involves unknown j. A diagonal matrix involve that each unknown is independent, otherwise the unknown dependence of the other increases with sparsity decreasing (nonzeros/zeros ratio). Depending on the symmetry of the matrix, the adjacenty graph is an *undirected* graph if the matrix pattern is symetric and it is a *directed* graph otherwise.
In many application, permuting/reordering/interchanging the rows, columns or both of a sparse matrix is a common operation, such as in parallel solving of equations system by iterative or direct methods. A rows permuting in an associated matrix of an equations system represent a change in order in which equations are written. In a column permuting in associated matrix, the system unknowns are in effect relabeled/reordered. If the same permutation is applied to rows and columns we have a symmetric permutation and the results consist in renaming the unknowns and reodering the equations. From graph point of view, a symetric permutation it is equivalent to relabeling the vertices without altering the edges.

2.2 The minimization of the bandwidth of a matrix

The *matrix bandwidth minimization problem* (MBMP) for matrices or *graph bandwidth minimization problem* (GBMP) for graphs are the same and represents old concerns, originates from sparse matrix computation. The MBMP has been the subject of study for at least 45 years, beginning with the Cuthill - McKee algorithm in 1969 [28]. As we said, the bandwidth minimization problem can be viewed in the context of both graphs and matrices. From graph point of view the problem consists of finding a special labeling of vertices which minimizes the maximum absolute difference between the labels of adjacent vertices. From matrix point of view, the minimization problem

consists of finding a rows/columns permutation that keeps most nonzero elements in a band that is as close as possible to the main diagonal of matrix. These two points of view can be transformed interchangeably by associating a matrix to a given graph (incidence matrix) or by associating a graph to a given matrix, as was shown in the previous subsection.
Should be noted that there is the dual problem of the MBMP, named *Matrix Antibandwidth Maximization Problem* (MABMP) [76]. In this case the optimization consist in maximize the minimum distance of entries of a sparse symmetric matrix from the main diagonal.

a) Matrix perspective

For a given nxn sparse, positive and symmetric matrix, A=$(a_{ij})_{,1\leq i,j\leq n}$, the bandwidth is:

$$bw(A) = max_{a_{ij}\neq 0}|i-j| \tag{1}$$

i.e. maximum distance of a nonzero element from the main diagonal.
Optimization consist in searching rows/columns permutation that minimize the bandwidth for the new matrix, i.e. all nonzero entries should be located as close as possible to the main diagonal, as shown in the example from Figure 1 a).

b) Graph perspective

For a given G(V,E) an undirected and connected graph, and σ a bijection between E and $\{1, 2, \cdots |E|\}$, the bandwidth minimization problem is:

$$bw(G) = min_{\sigma}(max_{u,v\in E}(|\sigma(u) - \sigma(v)|)) \tag{2}$$

and consist in finding a linear ordering of graph vertices (a labeling of vertices) that minimizes the maximum absolute difference between the labels of adjacent vertices, i.e. those which are connected by an edge, Figure 1 b).

That is, if the nonzero values of a square and symmetric matrix are identified with the edges of an undirected graph and the permutations of rows/columns are identified with the relabeling of the vertices, then the bandwidth of the matrix is the same with the bandwidth of the graph. In brief, the graph labeling corresponds to a permutation of rows and columns in matrix. The unsymmetrical case is treated in [116], where we have the left and the right half bandwidth.
The Bandwidth Minimization Problem (BMP) is NP-complete [101], that means that it is highly unlikely that there exists an algorithm which finds the minimum bandwidth in polynomial time. Only in a few special cases it is possible to find the optimal ordering in polynomial time [43, 74].
Because the bandwidth reduction process can be seen as a graph partitioning problem, there are many linear algebra methods that have been parallelized, the keyword being *partitioning*.

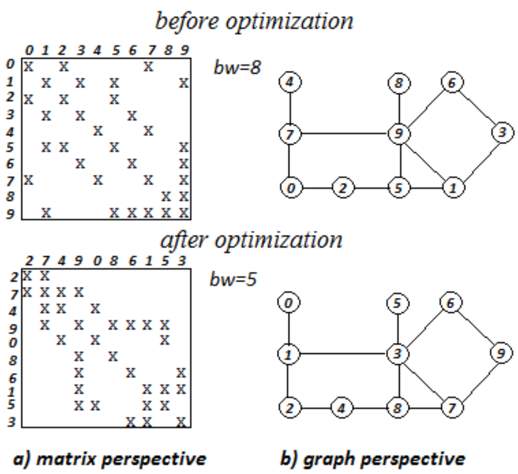

Figure 1: The analogy Matrix bandwidth - Graph bandwidth

2.3 The minimization of the envelope of a matrix

The *envelope* or *variable band* or *skyline* concept was introduced by Jennings in 1966 [62], who proposed a simple but very efficient scheme for storing big and sparse matrices.

a) Matrix perspective

For a given nxn sparse, positive and symmetric matrix, A=$(a_{ij})_{,1\leq i,j\leq n}$, and for each row define the *row-bandwidth*:

$$bw_i = i - j_{min}(i), a_{ij} \neq 0 \tag{3}$$

$\Rightarrow$

$$bw = max_{1\leq i\leq n}(bw_i). \tag{4}$$

The *envelope* is:

$$Env(A) = \{a_{ij}, i \neq j, 0 < i - j \leq bw_i\} \tag{5}$$

That is, the envelope of a matrix is the set of column indices that lie between the first nonzero column index and the diagonal in each row in part.

b) Graph perspective

For a given G(V,E) an undirected graph and:

-$adj(u)$ the set of vertices distinct from u that are joined to a vertex u by an edge;

-nbr(u)={u} the set of neighbors of u;

-σ an ordering of V,

the *envelope* is [9]:

$$Env(G) = max\{\sigma(u) - \sigma(v) : y \in nbr(u), \sigma(v) \leq \sigma(u)\} \tag{6}$$

Additional measures derived from envelope indicator are *envelope size*, *envelope work*, $1-sum$ and $2-sum$ [9]:

$$Esize = \sum_{i=1,n;j\in row(i)} max(i-j) \tag{7}$$

$$\sigma_1(A) = \sum_{i=1,n} \sum_{j\in row(i)} (i-j) \tag{8}$$

$$Wbound(A) = \sum_{i=1,n;j\in row(i)} max(i-j)^2 \tag{9}$$

$$\sigma_2^2(A) = \sum_{i=1,n} \sum_{j\in row(i)} (i-j)^2 \tag{10}$$

In Figure 2 a graphical representation of *Envelope* indicator in terms of matrix and graph theory can be seen. From graph theory point of view the envelope reduction can be seen as a transformation of an undirected graph into a directed graph.

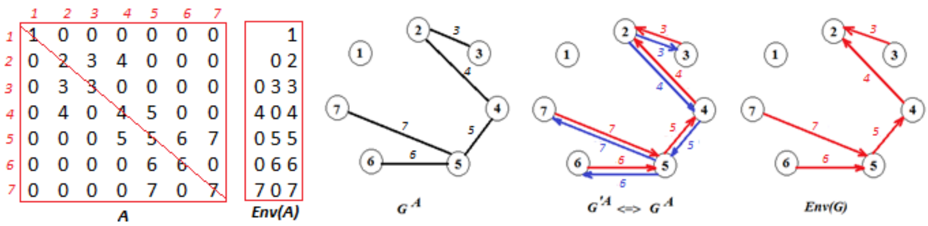

Figure 2: The envelope: matrix perspective and graph perspective

2.4 The minimization of the profile of a matrix

The matrix *profile* indicator was proposed as a way to reduce the storage space of sparse and large matrices in computer memory [127]. Shortly thereafter, another indicator for matrices was proposed in [31] named *SumCut*. Note that the *SumCut problem* is equivalent to the *Profile problem*, the differences consist in the numbering of the nodes [115].

a) Matrix perspective

For a given nxn sparse, positive and symmetric matrix, A=$(a_{ij})_{1\le i,j\le n}$, based on relation (3) for calculating the bandwidth for each line of the matrix in part, the matrix *profile* is the sum of these for all lines:

$$Pr(A) = \sum_{i=1,n} bw_i \tag{11}$$

b) Graph perspective
For a given G(V,E) an undirected and connected graph, σ a given ordering, the *profile of vertex* i is:

$$\delta_{\sigma(i)} = i - f_{\sigma(i)} \tag{12}$$

where $f_{\sigma(i)}$ is the position of the left-most vertex adjacent to $\sigma(i)$.
The *profile of graph* G is the sum of the profiles of all its vertices:

$$Pr(G) = \sum_{i=1,n} \delta_{\sigma(i)} \tag{13}$$

The *Profile Minimization Problem* consists in finding an ordering σ of graph G that minimize $Pr(G)$. That is equivalent to the interval graph completion problem [131].

2.5 The minimization of the average bandwidth of a matrix

Because the matrix bandwidth does not provide too much information about the grouping pattern of nonzero elements along and around the main diagonal, a new indicator of the non-zero compactness , called *Average Bandwidth* (*mbw*), was proposed in [86] and described by equation:

$$mbw(A) = \frac{1}{m} \sum_{i,j=1,n,a_{ij}\neq 0} |i - j| \tag{14}$$

where A is a nxn sparse matrix, m denotes the number of nonzero elements, and i, j represents row/column indices of nonzero elements. To note that the matrix can be symmetric or asymmetric.
The average bandwidth indicator was designed to be helpful in parallelization of methods for solving linear equations systems, namely to ensure a better arrangement of nonzero elements in the matrix. This fact lead to a better balancing of processors and to a decrease in communication processes, advantages mentioned in papers [87, 90, 91]. This indicator represents the average value of the distances of all nonzero elements to the main diagonal of the matrix, as can be seen in Figure 3. At the same time, if we consider that each nonzero element creates an imaginary band - bordered by the main diagonal and semidiagonal that passing through the "point" a_{ij} - the average bandwidth indicator is an imaginary band that has a width equal to the average of all these individual bands, as can be seen in Figure 3. It can be

seen that this is an approach through direct processing of the matrix that do not use concepts from graph theory. According to Figure 3, the relation (14) can be written as:

$$mbw(A) = \frac{1}{m} \sum_{i,j=1,n,a_{ij} \neq 0} d_{ij} \tag{15}$$

where A is a nxn matrix, m denotes the number of nonzero elements, and d_{ij} is the distance measured horizontally (or vertically) from a nonzero element to the main diagonal. It was considered the distance between any two elements equal to 1.

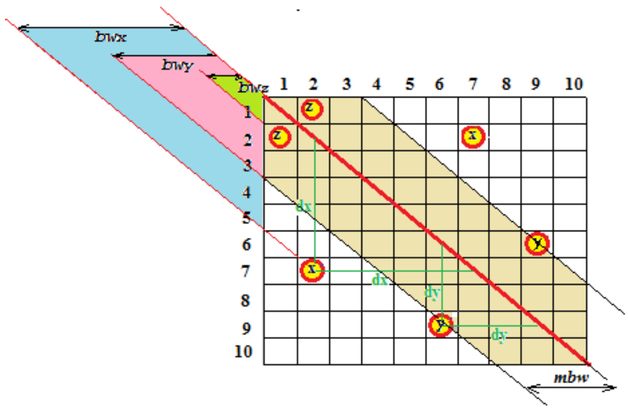

Figure 3: The geometric significance of average bandwidth (mbw)

Minimizing the *average bandwidth* (mbw) instead of *bandwidth* (bw) has the advantage of leading to a more uniform distribution of non-zero elements around the main diagonal. Moreover, the average bandwidth (mbw) is more sensitive than the standard maximal bandwidth (bw) to the presence around the main diagonal of the so-called "holes" (compact regions of zero values) but it is less sensitive to the presence of some isolated nonzero elements far from the main diagonal, as can be seen in Figure 4. This means that a matrix which minimizes the average bandwidth will have most non-zero elements very close to the main diagonal and very few non-zero elements away from it, which is advantageous in the case when the bandwidth reduction is used in the context of parallel solving systems of equations [7].

a) Matrix perspective

For a given matrix A=$(a_{ij})_{1 \leq i,j \leq n}$, square, sparse, symmetric or non-symmetric, with m nonzero elements, we define the *bandwidth of an element*:

$$bw_{a_{ij}} = |i - j| \tag{16}$$

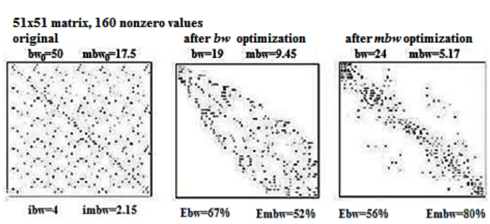

Figure 4: The average bandwidth relevance

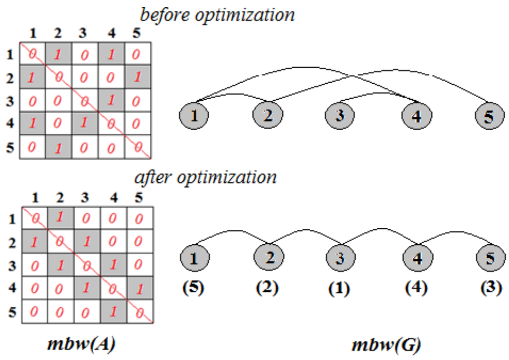

Figure 5: The average bandwidth: matrix and graph perspectives

and the *average bandwidth*:

$$mbw(A) = \frac{1}{m} \sum_{i,j=1,n,a_{ij}\neq 0} bw_{a_{ij}} = \frac{1}{m} \sum_{i,j=1,n,a_{ij}\neq 0} |i-j| \tag{17}$$

The optimization (Figure 5) consists in minimizing the *mbw* value by row/column permutations, with the main objective to approach nonzero elements to the main diagonal and ensuring their uniform distribution along and around the main diagonal.

b) Graph perspective

For a given G(V,E) an undirected/directed and connected/non-connected graph, with $|V| = n$ and $|E| = m$, σ a labeling of its vertices, $\sigma : V \to \{1, ..., n\}$, the *bandwidth of a vertex* is:

$$bw_i = |\sigma(u) - \sigma(v)| \tag{18}$$

and the *average bandwidth* of graph G:

$$mbw(G) = \frac{1}{m} \sum_{u,v\in V} |\sigma(u) - \sigma(v)| \tag{19}$$

The optimization problem consists in finding an ordering σ of graph G that minimize $mbw(G)$, ie obtaining a more linear arrangement of the vertices as can been seen in Figure 5. The *Matrix Average Bandwidth Problem* seems to be similar with *Minimum Linear Arrangement Problem*, the differences being shown in an upcoming section.

2.6 The maximization of the antibandwidth of a matrix

The Matrix Antibandwidth Maximization Problem (MABMP) was introduced by Leung in 1984 [76] and can be regarded as the dual of the Matrix Bandwidth Minimization Problem (MBMP). The name *antibandwidth* was proposed in paper [114]. MABMP it is also known as the *separation problem* [76] or the *dual bandwidth problem* [137]. The antibandwidth maximization problem is also NP-Complete [76].

a) Matrix perspective

In MABMP the optimization consist in maximize the minimum distance of nonzero values of a sparse matrix from the main diagonal. Thus, for a given matrix A=$(a_{ij})_{,1\leq i,j\leq n}$, square, sparse and symmetric, and consider the relation for MBMP:

$$bw(A) = min_i max_j |i - j|, i \neq j, a_{ij} \neq 0 \tag{20}$$

we have the relation for the *antibandwidth problem*:

$$abw(A) = max_i min_j |i - j|, i \neq j, a_{ij} \neq 0 \tag{21}$$

So, after row/column permutations all nonzero entries should be located as far as possible to the main diagonal, as shown in the example from Figure 6.

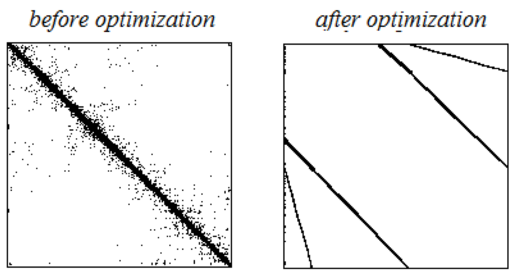

Figure 6: An example for the antibandwidth optimization

b) Graph perspective

Consider G=(V, E) an undirected graph, where V denotes the set of vertices and E the set of edges, with n = $|V|$, m = $|E|$ and σ a labeling of the vertices of G. In solving some real life problem, from graph theory point of view, it is necessary to maximize the minimum difference between adjacent vertices. That is, the *antibandwidth problem* from graph point of view consists in maximizing the value of abw(G) over all possible labelings. The *antibandwidth* of the graph G is defined as:

$$abw_{\sigma(G)} = min\{abw_{\sigma(v)} : v \in V\} \tag{22}$$

where $abw_{\sigma(v)} = min\{|\sigma(u) - \sigma(v)| : u \in S(v)\}$ is the antibandwidth of the vertex v with $S(v)$ the set of its adjacent vertices.

2.7 Proposed indicators for measuring optimization by bandwidths reduction

Some useful indicators to measure the effectiveness of the optimization of the bandwidth and the average bandwidth are proposed in paper [86].

2.7.1 Ideal Bandwidth and Ideal Average Bandwidth

The indicators Ideal Bandwidth (*ibw*) and Ideal Average Bandwidth (*imbw*) represent the ideal (perfect) values for bandwidth and average bandwidth and require a compact arrangement of nonzero elements around the main diagonal, ie without "holes". "Holes" are compact groups of zeros around the main diagonal. The values of these indicators can be determined by a trivial algorithm. These indicators can be used in bandwidth or average bandwidth optimizations as criteria for the end of processes. In practice, it is desirable to obtain the nonzero elements bands around the main diagonal with fewer holes, and sizes of these holes as small. At the same time an uniform arrangement of holes along the main diagonal, it is also in favor of an efficient parallelization process in solving systems of equations.

2.7.2 Bandwidth Reduction Efficiency and Average Bandwidth Reduction Efficiency

The purpose of Bandwidth Reduction Efficiency (*Ebw*) and Average Bandwidth Reduction Efficiency (*Embw*) indicators is to be measures of bandwidth or average bandwidth optimization processes. These indicators can be used to compare the performance of these optimization algorithms. The relations for calculating the values of these two indicators are:

$$Ebw = \frac{bw_0 - bw}{bw_0 - ibw} * 100 \tag{23}$$

$$Embw = \frac{mbw_0 - mbw}{mbw_0 - imbw} * 100 \qquad (24)$$

where the index 0 indicates the initial values (before optimization process). These relations show how much was achieved, from what could be theoretically, in percents, as can be seen in Figure 4.

3 Algorithms for matrix bandwidths optimization

Due to the links between matrix bandwidths (bw, Env, Pr, mbw and abw) and a wide range of scientific and engineering problems, many methods have been proposed for bandwidths optimization.
Mainly, there are two types of algorithms for these problems: exact and heuristic. At the same time we can speak about a splitting of last in heuristics and metaheuristics. This because we consider that metaheuristics are heuristics about heuristics, i.e. strategies that guide the search process. So, a metaheuristic is more than an heuristic and this is the reason why in the rest of this work we distinguish between heuristics and metaheuristics. The field of metaheuristics is between simple local search procedures to processes from artificial intelligence. A few representatives techniques that falling under this term are: tabu search, evolutionary algorithms, simulated annealing, ant colony optimization, bee colony optimization etc. A new class of metaheuristics was developed in last years, called *hybrid metaheuristics* [14] This new class combine algorithmic ideas from advanced math optimization method and artificial intelligence. More, in accordance with the paper [14], hybrid metaheuristics can be subdivided into five different categories, namely, the hybridization of metaheuristics with metaheuristics, constraint programming, tree search methods, problem relaxations and dynamic programming.
Exact algorithms guarantee always to discover the optimal even if it take a long time for this. In general, exact methods are useful for small matrices/graphs.
The heuristic and metaheuristic algorithms discover a good solution in a short time, but they do not guarantee the optimality of this. That is, these algorithms have a running time which is polynomial in the matrix size, but they are not guaranteed always to find the optimum.
Further, separately for each problem in part according to the above mentioned categories will be remember the most important algorithms proposed along the time.

3.1 Types of algorithms for matrix bandwidth reduction

3.1.1 Exact Algorithms for Matrix Bandwidth Optimization

In general, this category of algorithms uses mainly:
- analysis of all $n!$ possible permutations of lines/columns, methods that are good only for small matrices;
- branch and bound methods, good for small and medium matrices.

The method proposed in [2] is based on analysis of all $n!$ possible permutations of lines/columns of a matrix. This method is prohibitive in cases of large matrices, from runtime point of view. This approach is based on a direct processing of the matrix, without using concepts from graph theory. To complete the process in a reasonable time, special conditions are imposed to reject those unfavorable permutations: selection process of the permutation that ensures a lower value for the bandwidth.

One of the first highlighted the problem of relabelling vertices of a graph - in fact lines/columns permutation in the associated matrix of graph- was Harary in work [57].

In 1968, Rosen proposes an iterative method to reduce the bandwidth in sparse matrices [117], based on the lines/columns interchanging. In the initial phase are performed those interchanges that lead to the objective stated earlier. Thereafter different interchanges are possible, even if not reduces bandwidth by structural changes made to the matrix, but may lead in the future to the decrease of bandwidth.

Using dynamic programming, the authors propose in paper [55] an algorithm to reduce the matrix bandwidth based on the algorithm proposed in [119].

Corso and Manzini in paper [25] propose two exact branch and bound methods based on an enumeration scheme for bandwidth minimization problem. One of these is based on a depth first search strategy, while the second use a perimeter strategy to improve the depth-first search that provides a significant gain of runtime. The proposed methods are able to solve small instances, with matrix dimension beetween 40 and 100. At the same time, the proposed method works very well for dense matrices.

In paper [20] Caprara and Sallazar, extends Corso-Manzini method by introducing the *tighter lower bounds* in each node of the search tree, which can be computed in a very efficient way. The proposed method it is also a branch and bound method, in which vertices are labeled with the first or last available labels. The authors prove that the lower band can be computed in $O(nm)$ time.

In paper [94] the authors propose two exact algorithms based on the branch and bound methodology. The authors extend Caprara and Salazar theoretical results presented in [20] and offer new proofs and expressions for lower bounds. They use the GRASP [102] solution as the initial upper bound of the proposed branch and bound method. So, it is used the heuristic principle quality solutions can exist in the neighborhood of good solutions in according with GRASP method, i.e. a vertices renumbering according to the GRASP solution and then perform the enumeration following this ordering in a depth first search strategy. In one of proposed versions, the perimeter strategy was included but experimentally was shown that this strategy can not perform large instances due to memory limitations.

3.1.2 Heuristic Algorithms for Matrix Bandwidth Optimization

In general, this category of algorithms uses mainly breadth-first search method from graph theory, node-centroid methodology, hill-climbing technique, spectral algorithms or direct matrix processing.
CutHill-McKee (CM) was the first heuristic algorithm proposed for bandwidth reduction [28]. The algorithm consist in relabelling the vertices of the graph to reduce the bandwidth of the adjacency matrix, this being a variant of the breadth-first search of a graph. It is one of the most famous bandwidth reduction algorithms for sparse matrices and therefore it is referred in many other works that treat this problem. The first step is to choose a peripheral vertex of graph, an idea that will be used in many other bandwidth reduction algorithms. In summary, the algorithm consists in choosing a peripheral vertex (grade 1) and generate levels for all other vertices. Then, the vertices are selected depending on their degrees, i.e. ascending. This is the difference from breadth-first search. The algorithm is suited to matrices with a symmetric zero-nonzero structure. Note that the final result depend on the starting vertex, even if more vertices have same degree.
An improved version of CM algorithm, known as Reverse Cuthill-McKee (RCM) algorithm is proposed by A. George [44]. RCM version differs from CM by order in which vertices are processed, i.e. in reverse order of their degrees. This artifice has resulted in the reduction of processing time and memory space used as it is mentioned in paper [133]. Another important remark is that if the graph is not connected, the RCM algorithm must be applied to each component in part. Also, a comparative analysis of these two algorithms and rigorous mathematical demonstrations can be found in [133] where it is shown that the change proposed by A. George is equivalent, but the main conclusion after experiments is that RCM is often much better than the CM algorithm. In paper [22] is proposed a linear time implementation of the Reverse Cuthill-McKee algorithm. Because RCM algorithm is one of the most widely used bandwidth and profile reduction algorithms, there are

many analyzes of it, some critical, such as those highlighted in the paper [47]:
- the exhaustive search needed to find rooted level structures of minimal width, that will be very exhaustive in the case when all vertices of graph have the same degree;
- the graph is relabeling and the bandwidth recomputed for every level structure found of minimal bandwidth;
- the bandwidth obtained by a Cuthill-McKee algorithm can never be less than the width of rooted level structure used.

In paper [111] the authors propose an automatic node resequencing algorithm for matrix bandwidth reduction, useful in the constraint problems from the fluid-solid interaction analysis.

In paper [37] a few parallel heuristics methods for bandwidth reduction based on combinatorial optimization are proposed. The main advantage consists in obtaining a smaller execution times.

In paper [4] a graph approach is used to minimize the matrix bandwidth. Mainly, the method consist in a nodes separation into disjoint levels followed by renumbering them if a better value for bandwidth was obtained. To mention that in paper [5] the same author proposes an efficient method for finding the pseudo-peripheral nodes, useful in some bandwidth reducing algorithms.

An algorithm based on graph theory is proposed in [121]. The main operations performed by the algorithm are: obtaining the corresponding graph of working matrix, determination a spanning tree of maximum length, modification of the spanning tree into a free level structure of small width and numbering the level structure level by level. The last phase corresponds to a renumbering of the rows and columns in original matrix.

In 1976 one of the most popular algorithms for bandwidth reduction was proposed by Gibbs et. all in paper [47] (GPS), an algorithm based on pseudo-diameter concept, that leads to a decrease in search space (pseudo-diameter is a pair of vertices that are at nearly maximal distance apart). The GPS algorithm is about eight times faster than the Cuthill-McKee algorithm but with no major differences in terms of the quality of the solution. More, GPS algorithm is more complex to implement than RCM algorithm.

Ilona Arany in paper [6] propose an exact definition for pseudo-peripheral nodes from GPS method, definition that is based on the nodes properties. The main conclusion of this work is that the GPS method always converges to a pseudo-peripheral node in a finite number of steps, ie the GPS method is not an heuristic method.

Luo [85] developed an algorithm to reduce the bandwidth and profile of a matrix based on some refinements of the GPS algorithm. The proposed algorithm works well only for special structured graphs.

A particular bandwidth reduction algorithm used only in Triangular Finite Element Mesh was proposed in paper [124]. Its major particularity consists in that the proposed method doesn't require finding the starting vertex like in other methods based on level structure of associated graph [28]. This leads to a reduction in execution time .
In paper [122] improved variants are proposed for GPSs [47] and Arany's [4] algorithms. Thus, the original algorithms (called Minimum Degree GPS and Minimum Degree Arany) has been improved by adding a heuristic for finding a vertex with minimum degree.
An improved GPS algorithm to optimize the FEM (Finite Element Method) mesh nodes coding for reducing the bandwidth and profile in case of stiffness matrices is proposed in paper [134]. The authors propose replacing the parameter pseudo-perimeter from GPS algorithm with another heuristic parameter: width-depth report in aim of providing the correct vertices pseudo-perimeter. Experimental results show an improvement in the quality of solutions, that are in some cases as high as 30% approximately.
In paper [50] is proposed a new heuristic parameter for bandwidth reduction algorithms, called $width - depth\ ratio$, useful to find suitable pseudo-peripheral vertices. An improved GPS algorithm with this included parameter is proposed by authors and the results were superior than those obtained with GPS algorithm: bandwidth improvement up to 33%.
An approach based on Node-Centroid is proposed in [80]. Centroid term refers the idea of equilibrium point or center of mass of the objects. For a more detailed approach and an algorithm for calculating the centroid can be consulted the paper [35].
In paper [38] the authors propose a $Wonder\ Bandwidth\ Reduction\ Algorithm$ (WBRA) and a $Tabu\ Search$ (TS) algorithm based on level structures. First seems to be a little higher than GPS and RCM algorithms on average. The second, based on TS technique gave low quality solutions but higher execution time.
In paper [112] is proposed a bandwidth reduction method for matrices based on the form of associated equations system, inspired from decomposition in the finite element method (FEM). Mainly two cases are treated separately: row-dominant or column-dominant matrices. The main idea in both cases consists in putting the dominant row/column in a more central position. The proposed method is independent on the problem, i.e. it can be applied to any linear system of equations.
A spectral method was proposed in [9] for envelope reduction of sparse matrices. Even if the main purpose of the proposed method is to reduce the envelope of a matrix, the experimental results show the effectiveness of the proposed method in terms of reducing the bandwidth in some cases. An

analysis of the spectral ordering technique used as a basis for preconditioning by bandwidth reducing is contained in [24].
In paper [51] is proposed a bandwidth reduction algorithm aimed at encapsulating as many of nonzero elements into o diagonal narrow band. The algorithm uses a weighted spectral reordering (WSO), in order to apply a banded solver after this preconditioning. Spectral technique proposed for bandwidth reduction by authors is a generalization of spectral reordering proposed in [9].
In paper [26] another spectral technique for bandwidth reduction is proposed. The basic ideas are derived from a precision low enough computing and able to reapply the spectral technique on an reordered matrix with a modified Laplacian. The main purpose is to reduce the number of nonzero elements located far from the main diagonal. One application of the spectral technique was to reduce the envelope-size of a sparse symmetric matrix in paper [9], that is in fact the basis for the methods proposed in [26]. An interesting idea proposed by the authors in [26] is to apply first the RCM algorithm (like a preconditioning) and then their spectral method, idea that lead to improving the quality of the solution.
Four heuristics for bandwidth reduction based on directly processing of the matrix are presented in paper [54]. The first step of these algorithms consist in renumbering the matrix rows/columns by computing the eigenvector values.
A heuristic that uses the arrangement of nonzero elements in relation to zero elements, separate from one side and the other side of the main diagonal of the matrix, is proposed in paper [86].
In paper [88] several heuristics which exploit the matrix symmetry are proposed. Variants that are characterized by the usage of two optimization criteria(bandwidth and average bandwidth) were tested.

3.1.3 Metaheuristic Algorithms for Matrix Bandwidth Optimization

In the last years approaches based on various metaheuristics have been proposed: Tabu Search (TS), Variable Neighborhood Search (VNS), Genetic Algorithms (GA), Genetic Programming (GP), Greedy Randomized Adaptive Search Procedure (GRASP), Simulating Annealing (SA), Ant Colony Optimization (ACO), Particle Swarm Optimization (PSO), Charged System Search (CSS), etc. An extensive description of these techniques can be found in [135].
In paper [82] is proposed a genetic algorithm that improve the solution quality reported to GPS, Tabu-Search and GRASP algorithms. The proposed

genetic algorithm is combined with Hill Climbing (HC). It was observed that the bandwidth decreased rapidly up to a number of generations (30 in experiments) and after that value, there is a small decrease of bandwidth value.
Paper [70] presents an algorithm based in a standard Evolutionary Algorithms (EA). The authors propose a new fitness function and a new mutation operator. The chromosomes on EA are encoded using a randomly generated permutations of nodes. To note that the proposed fitness function is based on a combination of two indicators: bandwidth and profile. The proposed mutation operator it is inspired by RCM algorithm.
An improved genetic algorithm for bandwidth reduction is proposed in [110]. The most important features of the proposed method are: a new representation of the candidate solution and the multitude of genetic operators used. The method based on a cost function is developed for a large class matrix optimization problems: bandwidth (MBMP), antibandwidth (ABMP) and linear ordering (LOP).
In paper [102] Pinana et al. used a *Greedy Randomized Adaptive Search Procedure* (GRASP) with Path Relinking method for bandwidth minimization problem. In their approach, a GRASP iteration has two phases: construction and improvement. The experimental results given in the paper show a solution quality comparable with TS algorithm but a lower execution time than GPS algorithm.
In paper [129] is proposed a *Simulating Annealing* (SA) method for bandwidth reduction, based on a new proposed measure which enhances major the results obtained with SA method and overcome the problems of arithmetic precision when working with large matrices.
Lim et. all propose in paper [81] an algorithm to reduce the sparse matrix bandwidth by hybridizing a bio-inspired metaheuristic with a probabilistic technique, ie *Ant Colony Optimization* (ACO) and $Hill - Climbing$ (HC). The proposed algorithm uses ACO technique to generate a relabeling of vertices in graph followed by refining the solution using HC at every step. The authors reported that the execution times become prohibitive for big instances. Also, in paper [81] the authors propose an algorithm to randomly generate a matrix with known optimal bandwidth in order to test the optimality of solution found by their bandwidth reduction algorithm.
Another hybrid heuristic method for the Matrix Bandwidth Minimization Problem (MBMP), based on *Ant Colony Optimization* method and several local-search mechanisms was proposed in paper [27].
Dueck et al. [33] used a *Simulated Annealing* algorithm for bandwidth reduction, an algorithm with good performance on binary trees reported to the GPS algorithm.
In paper [130] a simulated annealing algorithm is presented for the bandwidth

minimization problem. The proposed method is based on three distinguished features: an original internal representation of solutions, a highly discriminating evaluation function and an effective neighborhood. The last is in fact a new neighborhood definition.

In paper [29] two hybrid algorithms was proposed: the first is a genetic algorithm and the second is based on *Ant Colony Optimization*. Also, a theoretical $Reinforcement\ Learning$ (RL) model for improve the bandwidth reduction process is proposed in this paper.

Another hybridization of ACO technique with local search mechanisms for bandwidth minimization is proposed in paper [?Pintea2010].

Marti et al. proposed a $Tabu\ Search$ (TS) method in paper [93], a method in which a candidate list strategy was used to accelerate the selection of moves in a neighborhood. Extensive experimentation showed that proposed tabu search outperformed other best-results algorithms both in terms of quality and speed. An interesting experiment is also presented: if first run GPS algorithm and then use the solution as the starting point for TS, highest quality solutions are obtained. A complete description of TS method can be found in [48].

Another TS method for bandwidth matrix reduction is proposed in paper [19]. Also in this paper is presented an algorithm based on the *Scatter Search* (SS) method. Scatter search [49] is a metaheuristic, often associated with evolutionary computation, considered part of the same family with TS. Many experiments with different goals was performed. The results was compared with those obtained by other highly performing algorithms. The main conclusion is that the proposed methods provides a good balance between solution quality and speed. The authors suggest that their algorithm to be included in a category that takes into account the memory usage, ie a memory-based or a memory-less methods classification.

In paper [97] the authors propose a $Variable\ Neighborhood\ Search$ (VNS) method for reducing the bandwidth of a matrix. VNS metaheuristic is a systematic change of neighborhood within the local search algorithm. In local search phase it was used an $Improved\ Hill\ Climbing$ (IHC) algorithm. The experimental results reported by authors show that their proposed methods outperforms other methods (quality and execution time) and others previous best known solutions.

Lim et. all propose in paper [78] a metaheuristic based on $Particle\ Swarm$ $Optimization$ [66] combined with $Hill-Climbing$ (PSO-HC) for bandwidth reduction. Their proposed method use in first phase PSO technique in order to generate high-quality renumbering and then a HC local search technique to refine the solution. Experimental results reported by authors show that their methods outperforms TS and GRASP methods, especially in terms

of quality solution. To note that the hybridization of these two techniques (PSO-HC) results in increasing the speed of convergence.
Also, in paper [?Pintea2011] are proposed techniques to reduce the bandwidth of matrices using ACO optimization hybridized with local search techniques. A wide description of the ACO technique and other applications of this technique in matrix optimization processes can be found in [?Pintea2014].
In paper [79] is proposed an algorithm that use a *Genetic Algorithm* (GA) combined with a simplified $Hill - Climbing$ (HC) technique for bandwidth minimization problem. The GA technique is using in global search (finding good solution patterns) while HC technique is using for local search (tune solutions to reach local optimum).
Another method that use genetic algorithms (GA) is proposed in paper [110]. The method based on a cost function is developed for a large class of matrix optimization problems: bandwidth (MBMP), antibandwidth (ABMP) and linear ordering (LOP).
In paper [53] another genetic algorithm for bandwidth reduction is proposed. An approach based in *Genetic Programming* (GP) is proposed in paper [71]. The experimental results presented by the authors show quality results but a lower execution time compared to other methods for bandwidth reduction.
In paper [72] a hyper-heuristic approach based on genetic programming is presented for evolving graph-theoretic bandwidth reduction algorithms. In the proposed GP algorithm, $Tarpeian\ Method$ (TM) [107] was used in order to control excessive code growth.
An interesting approach to solve MBMP based on genetic programing is proposed in paper [108]. The experimental results reported by authors show that proposed method outperforms in term of solution quality five of the best metaheuristics like VNS, SA, GRASP-PR, GA and TS.
In paper [61] is proposed an evolutionary approach to minimize the bandwidth. The novelty introduced by the authors consists in adding learning concepts (penalties and rewards) to the genetic algorithm.
In paper [52] the authors proposed a methodology for bandwidth reduction based on evolutionary algorithms, using several fitness function. More, the authors propose the *Seeded Approach*, a technique that consists in including in initial population of the evolutionary algorithm the results obtained with RCM algorithm.
An approach to minimize the bandwidth and profile of sparse matrices that use $Charged\ System\ Search$ (CSS) is proposed in paper [65]. CSS is a metaheuristic based on laws of Coulomb and Gauss from Electrostatics and Newtonian mechanics.
In paper [109] the authors propose an improved genetic algorithm (GA) for solving the matrix bandwidth minimization problem.

3.1.4 Algorithms for Matrix Bandwidth Optimization on Unsymmetrical Matrices

Most proposed algorithm treats bandwidth reduction problem only in the case of symmetric matrices respectively of undirected graphs. Only a few authors take into account the asymmetrical case, where algorithms must reduce a lower bandwidth and an upper bandwidth:
In paper [39] the authors propose an improved extension of their WBRA and TS proposed algorithms in [38], for bandwidth reduction in case of unsymmetric matrices;
The authors in paper [10] propose an algorithm for reducing the bandwidth of an unsymmetrical matrix based in a binary pattern of each row and column in part. The nest step consist in an alternate between ordering the rows and columns, in decreasing order, the result stand at the base of interchange processes. The problem is that isn't proposed a strategy for this double ordering.
Reid and Scott propose in [116] a few methods to reduce the bandwidth of unsymmetrical matrices. The first proposed method its an unsymmetrical variant of Reverse Cuthill-McKee algorithm [44]. In the second, the authors have adjusted the node-centroid and hill-climbing ideas of Lim, Rodrigues and Xiao [80] combined with ideas from [10] to the unsymmetrical case.

3.2 Algorithms for envelope reduction

In paper [9] a *spectral* method was proposed for envelope reduction of sparse matrices. Spectral methods are a class of techniques used in applied mathematics and scientific computing that have a very good convergence, which leads in some cases to a lower execution times. Even if the main purpose of the proposed method is to reduce the envelope of a matrix, the experimental results show the effectiveness of the proposed method in terms of reducing the bandwidth in some cases. The numerical results presented by authors show that their algorithm find smaller envelope sizes than those obtained by Gibbs-Poole-Stockmeyer (GPS) algorithm or Reverse Cuthill-McKee (RCM) algorithm, in some cases the reducing the envelope been more than a factor. Therefore, the proposed algorithm also improves execution times compared with GPS or RCM algorithms. An analysis of matrix envelope reduction by spectral technique is presented in paper [45].
In paper[17] is proposed a multilevel algorithm, a heuristic inspired from graph partitioning, algorithm that in general produces better envelopes in less time than other algorithms, like RCM.
Another spectral algorithm for envelope reduction in sparse matrices was

proposed in paper [75]. Mention that the spectral technique was combined with a combinatorial algorithm, which had effect in reducing the time for obtaining the solution.
In paper [73] the authors propose an hyper-heuristic approach based on genetic programming for envelope reduction problem. Based on the experimental results provided by the authors, it can be seen a remarkable quality of the solutions, especially for large instances.

3.3 Algorithms for profile reduction

One of the first algorithm for sparse matrices profile reduction was proposed by Cuthill-McKee in 1969 [28]. In the same period King [68] also propose a method to reduce the matrix profile.
In paper [99] is proposed an algorithm which can reduce matrix profile by factor 2-3. The method can be easily modified to obtain a matrix transformation to arbitrary desirable form.
An improvement of the GPS algorithm for matrix profile reduction is proposed by Wang in paper [134]. The matrix profile is reduced by 2% on average, compared with the GPS algorithm, while the execution times are close.
In paper [64] a simple ACO algorithm is proposed for nodal ordering problem to reduce the profile of sparse matrices. To improve the process, a local search procedure is included. Same authors propose a profile minimization via *charged system search* (CSS) algorithm in [65].
In paper [54] are proposed two profile reduction algorithms: the first one is an algebraic approach while the second is a systematic search method. First use the spectral information of the matrix structure while the second use the first algorithm to find a good starting point.
In paper [125] is proposed a method for profile reduction that is based in graph theory, namely the concept of the convexity. The results of the proposed algorithm are very good in the cases of graphs with complex configuration (non-planar graphs with rings).
An extension of tabu search for profile reduction is proposed in paper [39]. The main modification is made on the cost function.
A heuristic method to reduce the profile of a symmetric and sparse matrix is proposed in paper [56]. Comparations with other known and performant profile reduction methods are also presented in this paper. To mention that the tests were performed for large size matrices (greater than 25000).
In paper [120] Sloan propose an algorithm for reducing the profile of symmetric matrices. The experimental result from this paper indicates that the proposed algorithm is more faster, requires less memory for data storage and

it is more simple than RCM or GPS algorithms.
In paper [118] is proposed a GRASP algorithm combined with Path Relinking to solve the *SumCut* minimization problem in graph. Note that the SumCut problem is equivalent to the Profile problem [115], the differences consist in the numbering of the nodes.
In paper [111] the authors propose an automatic node resequencing algorithm for matrix profile reduction for the constraint problems from the fluid-solid interaction analysis.
An algorithm for profile reduction, designed to be time efficient is proposed in paper [123].
In paper [77] a compound algorithm that reduce the matrix profile is proposed.
Boutora and all. propose in paper [18] a method for profile reduction with application in Finite Elements Method (FEM) analysis.

3.4 Algorithms for average bandwidth reduction

The term *average bandwidth* and an algorithm for reducing the average bandwidth were proposed for first time in paper [86]. The proposed algorithm, based on the direct matrix formulation, is a heuristic approach, more exactly a Greedy approach that uses the arrangement of nonzero elements in relation to zero elements from each line of matrix, separately on each side of the main diagonal.
The approach proposed in the paper [87, 88] is a metaheuristic that relies on applying successive transformations directly to the matrix. The method combines a greedy-like selection of rows/columns to be swapped with a genetic inspired crossover. Another feature of the proposed method consists in using an elitist selection of the active nodes in the search tree, leading to an effective pruning technique. An approach corresponds to a depth first search, similar to preorder traversal of trees, it is also analyzed. Other heuristics are also presented in this paper, such as those that using the symmetry of matrices in combination with techniques from genetic algorithms.
In order to improve the solving in parallel of equations systems by iterative methods in paper [91] is proposed a bandwidth reduction algorithm that for a given/desired partitioning it is aimed achieving the best possible preconditioning in the case of parallel solving of equations systems.

3.5 Algorithms for matrix antibandwidth optimization

The Matrix Antibandwidth Maximization Problem (MABMP) and algorithms for MABMP were proposed by Leung in 1984 [76]. The proposed

method consists in finding a graph labeling with a value of antibandwidth greater than a predetermined value k. The proposed polynomial time algorithms are good for several classes of graphs.
An improved genetic algorithm for antibandwidth reduction is proposed in [110]. The method based on a cost function is developed for a large class matrix optimization problems: bandwidth (MBMP), antibandwidth (MABMP) and linear ordering (LOP).
Upper bounds for antibandwidth, related to the independent and chromatic numbers of graph, are proposed in [137].
In paper [114] the authors proposed algorithms for the antibandwidth problem designed for several classes of special graphs: two dimensional meshes, hypercubes and tori. The case of three dimensional meshes is treated in [128].
Heuristics based on GRASP for antibandwidth problem, with some results in the case of general graphs, were proposed in [32].
In paper [83] the authors proposed a heuristic algorithm based on the variable neighborhood search (VNS) methodology. The experimental results shows that the proposed method outperforms old methods from solution quality point of view. Same authors proposed a hybrid metaheuristic for the cyclic antibandwidth problem in paper [84]. The proposed method hybridizes the *Artificial Bee Colony* with *Tabu Search*.
A *Memetic Algorithm* (MA) for antibandwidth problem is proposed in paper [126]. The main features of the proposed algorithm are a heuristic to generate a good quality initial population and a local search operator based on Hill-Climbing technique.
Another memetic algorithm for antibandwidth optimization is proposed by Bansal in paper [8]. The proposed algorithm is designed for the case of general graphs.

3.6 Evolution and trends in bandwidths optimization algorithms

In the last subsections, a large number of algorithms for bandwidth, envelope, profile, average bandwidth and antibandwidth optimization were briefly reviewed.
In Figure 7 can be seen a diagram of the evolution in time of these algorithms in terms of their number (published papers), separately for each category in part: exact, heuristics and metaheuristics.
After analyzing these data, several conclusions can be drawn:
- because these problems are NP-complete [101], the number of heuristic and metaheuristic algorithms is dominant, compared to the exact methods;
- the large number of algorithms proposed in the past two decades show the rising importance of the issue;

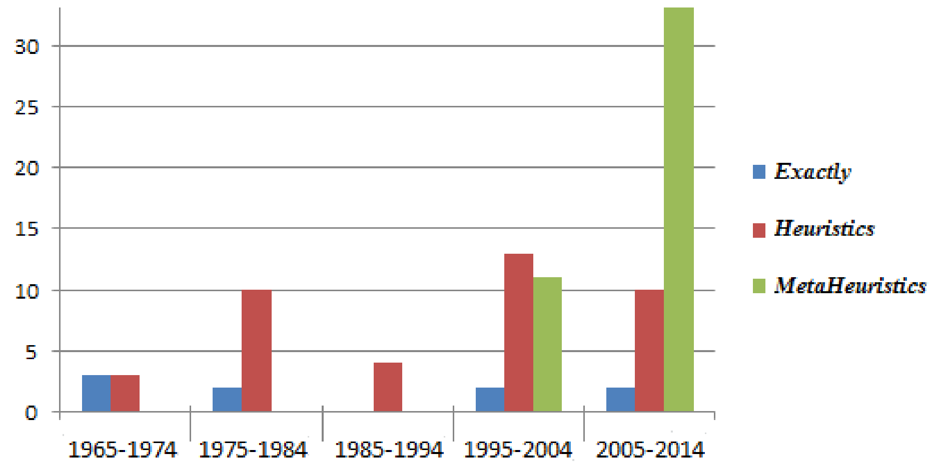

Figure 7: Time evolution of algorithms for bandwidths optimization

- the domination of metaheuristics methods in the last decade is evident.
It can be concluded that the growth of performances, both in quality and runtime, requires the use of hybrid methodology, ie the hybridization of powerful optimization techniques from mathematics (HC, TS, GRASP etc) with techniques from artificial intelligence (GA, GP, PSO etc).
At the same time, for old and new, exact and heuristic, a parallelization of bandwidths optimization algorithms is required, because this is the chance to greatly reduce the execution time.

3.7 Algorithms performances for bandwidths optimization in sparse matrices

As was shown in the previous section, the interest in bandwidths reduction algorithms on sparse matrices greatly increased in the past two decades and that due to a huge practical applications in which such processes help. This increased interest has materialized in a large number of proposed algorithms. And here is a difficult problem of the user: he must choose the most suitable algorithm from a large available set. Unfortunately, most often, the user can not test a large number of algorithms and especially can not test on a large and varied instances of the final problem. That is, choosing the most suitable algorithm to their own needs is not an easy problem.
Classification with a good accuracy of bandwidths optimization algorithms in terms of solution quality and especially in terms of execution time are nearly impossible to be done. The main reasons are:
-in terms of solution quality there are not published experimental results for the same input in all published works, ie the test matrices are different from

one work to another in most cases and thus can not be made a comparative analysis of methods;
- in terms of execution times, the problem is even more difficult because the proposed algorithms in different works were run on different computers as hardware-software configuration;
- also, it is practically impossible an exact reimplementation of all proposed algorithms by one user and then run these implementations on same hardware conditions. The main reason is the brief description of the algorithms in some works.

We have tried to make such a classification but due to the reasons mentioned above we can not draw a final line. Our study was based only on results published in 20 papers (published until 2012) and analysis of over 200 related matrices. Our conclusions can be summarized as follows:
- in terms of quality solution, it seems that the exact algorithms proposed by Corso [25] and Caprara [20] have the best results for small and medium matrices. In the cases of matrices with dimensions higher than 1000, these algorithms become inefficient in terms of execution time. In such situations heuristic and metaheuristics algorithms such as those proposed in [72, 82, 97, 108] seem to be good choices.
- in terms of execution time, Gibbs-Poole-Stockmeyer algorithm [47] ensures in general, regardless of the size of the matrix, a minimum execution time, lower than execution times reported for other algorithms.

4 Discussions about theoretical an practical approaches of matrix bandwidths problems

4.1 Comparative approaches of matrix bandwidths

4.1.1 Average Bandwidth and Minimum Linear Arrangement

Average Bandwidth (*mbw*) and Minimum Linear Arrangement ($MinLA$) indicators appear to be very close or even similar but:
a) *mbw* does not depend on the connectivity of the matrix associated graph;
b) $MinLA$ value depends on graph labeling, i.e. vertices must be labeled 1,2,$\cdots$,n;
c) *mbw* is based on matrix geometry while $MinLA$ is based on graph theory;
d) *mbw* is a more intuitive representation for solving systems of equations (condition number, inverse, transpose, Jacobian, determinant etc);
e) *mbw* does not depend on the matrix symmetry, ie directed-undirected

graph, so *mbw* is more general than $MinLA$;
f) *mbw* reduction allows adding additional criteria to control the distribution of non-zeros according to the desired number of partitions or/and in determining the best partitioning in parallel solving of system equations [91];
g) *mbw* reduction allow an exact method for determining the interchange opportunity [89].
Note: from the relations for calculating Pr (Profile) and $MinLA$ we can conclude that $MinLA$ is much closer to Pr than other measures in matrices:

$$Pr(G,\sigma) = \sum_{u}(\sigma(u) - min_v(\sigma(v))) \tag{25}$$

$$MinLA(G) = min_{\sigma} \sum_{u,v} |\sigma(u) - \sigma(v)| \tag{26}$$

4.1.2 Bandwidth and Envelope/Profile

In general the *bandwidth* (*bw*) and $envelope/profile$ (Env/Pr) are not minimized by the same rows-columns interchange but a small bandwidth of matrix involve a small envelope/profile [26].
So it is desirable a permutation to reduce simultaneously both indicators, as it is shown and achieved by the method proposed by Gibbs-Poole-Stockmeyer (GPS) [47]. Also, in paper [85] is proposed an algorithm -derived from GPS- to reduce the bandwidth and the profile at the same time, but the proposed method work only on particular graphs.

4.1.3 Bandwidth and Average Bandwidth

The relevance of *average bandwidth* in relation with *bandwidth* is given by the following considerations:
- bandwidth (*bw*) reduction is improper for preconditioning systems of equations in parallel solving of them due to the large fraction of nonzeros outside the main block diagonals, while average bandwidth (*mbw*) reduction ensure a greater compactness of the nonzero values along and around the main diagonal of the associated matrix. As a practical application, *mbw* reducing process help in solving the equations systems in parallel, specifically to a better load balancing of processors and to a decrease in communication processes between processors [90];
- the increased relevance of *mbw* compared to *bw* is given by the fact that the *bw* value can be high due to a small number of elements (even one) which are far away from the main diagonal while most non-zero elements are very close to the main diagonal. Locating the nonzero around the main diagonal in a

high percentage (small value for *mbw*) even if a small number of non-zero elements on the diagonal are close to the main diagonal (large value for *bw*) represents a major advantage in parallel solving systems of equations;
- at the same time, note that considering only *mbw* value in a preconditioning process is risky, because a small value of *mbw* does not necessarily mean a band of nonzero elements near the main diagonal. Therefore, considering only together the values of *bw* and *mbw* can provide a complete picture of the nonzero elements location relative to the main diagonal [87].

4.2 Some applications of matrix bandwidths optimizations

There are a huge practical applications in which matrix bandwidths optimizations help. At the same time, it is impossible to determine the number and all types of applications where a bandwidth optimization is used, because not all end users publish a paper on this. And this makes it more difficult to choose the best tested method for a particular problem. But several real examples can be a good aid. That is, in the following we enumerate some real applications where bandwidth optimization was used, based on published papers. In the following, some example embodiments are presented, which can be a user guide. Further examples will be presented grouped into two categories. The first includes cases in which the work relates directly to a math problem, such as "sparse matrix-vector multiplication optimizations based on matrix bandwidth reduction". Here, in general are proposed new algorithms, often adapted to final problem to be solved, where the final problem is generally a math problem. The second category includes works in which the bandwidth reduction is just an auxiliary/preconditioning process, such as "the bandwidth reduction algorithm used is that proposed by Cuthill and McKee". Here, in general, consecrated algorithms are used to solve the final problem.

4.2.1 Math Applications of bandwidths optimization algorithms

In paper [92] it is shown that bandwidth reduction on associated matrix of equations system in the case of parallel conjugate gradient reduces the execution time, especially by reducing communication time. A modified version of conjugate gradient (CG) that decreases the execution time with approx. 30% and a detailed analysis of operations performed on a parallel machine are also presented.
Paper [90] show the average bandwidth reduction relevance in the case of linear equations systems solving by parallel implementations of gaussian elimination and conjugate gradient. It has also been shown in [90] that in some

cases such preconditioning by reducing the average bandwidth leads to a drastic decrease of the efficiency of parallel computing process in terms of convergence and global runtime. To note that this unwanted effect was also observed in the case of preconditioning by bandwidth reducing. Several factors that cause this undesired effects and corrections in the case of parallel iterative methods that use the Jacobian have been proposed in paper [91].

In paper [36] the authors show the influence of the bandwidth reduction in preconditioning of solving equations systems by conjugate gradient method.

The paper [95] presents experimental results on parallel variants of some iterative methods for solving systems of equations (Newton, Chebyshev, Conjugate Gradient). One of the conclusions is that the preconditioning of the sparse systems (linear and nonlinear) by bandwidth reduction has a positive impact for all methods analyzed, such as for example, reducing the asymptotical error in the cases of well conditioned systems.

Another work that treats the influence of preconditioning by reducing bandwidth in solving systems of equations through Kyrlov methods (GMRES and CG) is [46]. The main conclusion is that the bandwidth optimization decrease the number of iterations in the iterative methods.

In technical report presented in paper [51] that describes studies, developing and evaluating the solving of sparse linear systems using Krylov methods on scalable architectures, the bandwidth reduction was the choice for preconditioner.

In paper [113] it is show that bandwidth reduction is useful in the case of solving linear systems by Go-Away algorithm [12]. Relations between the number of operations necessary for a matrix factorization or the number of operations needed to solve a linear system and the bandwidth are also provided and proved in this paper.

It is shown in paper [69] that in the case of very large matrices with small bandwidth, a strong reduction of runtime can be obtained in the process of computing eigenvectors of block tridiagonal matrices. Also a significant improvement in numerical accuracy was observed when a preconditioning process by reducing bandwidth has previously been applied.

A bandwidth reducing process can improve the overall speedup of Matrix-Vector Multiplication in the case of sparse matrices (SpMV) with 16% in single-precision and with 12.6% in double-precision operations when Graphics Processing Units (GPU) are used [136]. The same advantage for the same problem is highlighted in [103].

In papers [42, 100] it is also show that in case of solving sparse linear systems, the bandwidth reduction for their associated matrices has a significant impact on the total computation required, as a result of ´fill in´ that occurs during factorization.

Profile reduction can be used in implementing the parallel algorithms as it simplifies the computation in each of the processors [54]. Further, if the processors support vector instructions, exploiting fine grain parallelism would be extremely simple with such reordering.
In paper [3] is proposed an algorithm for partitioning a sparse matrix into a matrix that has blocks on the diagonal, useful in parallel computing. In first part an ordering algorithm that reduces the profile of the matrix is applied, in order to obtain a block-diagonal partition.

4.2.2 Real Life Applications of bandwidths optimization algorithms

In paper [23] is proposed an algorithm with polynomial complexity useful in quantum chemistry. A bandwidth reduction process - as a kind of preconditioning, which aims to obtain a matrix closer to one band diagonal- that use Reverse CutHill-McKee algorithm is used in order to increase the accuracy of calculations in the proposed algorithm.
In paper [50] a method for geodetical network adjustment problems is proposed. In these real problems, very large overdetermined systems of linear equations must be solved. Most of these systems can be sparse and consequently a preconditioning process, such as bandwidth reduction, should be applied to improve the algorithm performance.
A parallel algorithm that utilizes the non-zero structure of sparse matrices that occurs in seismic tomographic problems is proposed in [60]. After a prior preparation of matrices, Reverse Cuthill-Mckee algorithm is applied to different components/submatrix to reduce their bandwidth. Thus, the preconditioning by bandwidth reduction leads mainly to the reduction in communications between processors.
The bandwidth reduction method proposed by authors in [21] produce a substantial speed-up in the numerical simulation of micro-strip circuits, between 5 times and 18 times.
The bandwidth reduction is equivalent in VLSI optimization with minimizing the delay communication between electronic components, when solving the routing problem of VLSI layout [13, 132].
In paper [1] bandwidth reduction is a part of computer-aided design techniques for placement and routing in 3D integrated circuits, more precisely in the wire-length minimization process.
In saving large hypertext media [11] a RCM bandwidth reduction is used.
The advantage of matrices with small bandwidth is highlighted in the paper [41] which reviews several finite element formulations used to solve structural acoustics and fluid-structure interaction problems.
In paper [15] is proposed an approach for computing graph entropy, a mea-

sure for understanding a network structure and for compressing large complex networks. A matrix bandwidth reduction process is used to generate a sequence which possesses an entropy rate approaching the lower entropy rate bound.
The work [40] show the advantages of a matrix bandwidth reduction in using it with the NASA structural analysis computer program, NASTRAN. The bandwidth reduction algorithm used is that proposed by Cuthill and McKee. Better running times were obtained if a bandwidth reduction algorithm was used in a geometric problem derived from public transportation network [96]. The bandwidth reduction of sparse matrices can be uses in cluster identification [98]. Reverse Cuthill-McKee and Kings algorithm are used in this study but the influence of these algorithms is relatively small.
In paper [138] some implications of graph bandwidth in artificial intelligence are showed. First consist in the fact that the bandwidth of the constraint graph of a constraint satisfaction problem (CPS) serves as a measure of its decomposability. The second application is that the bandwidth of a search ordering provides a measure of its quality, ie backtracking with a small bandwidth ordering generally results in a smaller search tree.
Paper [67] describe an application in archeology, where it is necessary to serialize different artifacts. In this process a preconditioning that consist in a profile reduction of an associated matrix it is necessary.
The profile reduction has been used in information retrieval to browse hypertext [16].
The profile reduction process arises in the Human Genome Project [63] that consist in elucidating the genetic information of humans as well as other species.
In paper [111] the authors propose an automatic node resequencing algorithm for matrix bandwidth and profile reduction, for the constraint problems from the fluid-solid interaction analysis.

5 Conclusion

The double perspective of some matrix indicators from two points of view, matrix geometry and graph theory- gives a clear and large perspective of them.
The evolution in time presentation of the methods for reducing bandwidths in sparse matrices show the importance, actuality and trends of these concerns.

The presentation of some numerical and real life applications of the bandwidths reduction in matrices may help to make a choice and especially to produce new ideas for applying these optimizations.
An important conclusion is that the metaheuristics and the parallelization are the chances to improve the performance (quality and execution time) of the algorithms for bandwidths optimization in sparse matrices.

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