

# On a Certain Extension of the Hurwitz-Lerch Zeta Function

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**Abstract.** Our purpose in this paper is to consider a generalized form of the extended Hurwitz-Lerch Zeta function. For this extended Hurwitz-Lerch Zeta function, we obtain some classical properties which includes various integral representations, a differential formula, Mellin transforms and certain generating relations. We further consider an application to probability distributions and also point out some important special cases of the main results.

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## 1 Introduction

Let  $\mathbb{N}$ ,  $\mathbb{Z}^-$ ,  $\mathbb{R}$ ,  $\mathbb{R}^+$  and  $\mathbb{C}$  denote the sets of positive integers, negative integers, real numbers, positive real numbers and complex numbers, respectively, and also, let  $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$  and  $\mathbb{Z}_0^- := \mathbb{Z}^- \cup \{0\}$ .

The Hurwitz-Lerch Zeta function  $\Phi(z, s, a)$  is defined by (see, *e.g.*, [5, p.

27, Eq. 1.11(1)]; see also [11, p. 121] and [12, p. 194]):

$$\Phi(z, s, a) := \sum_{n=0}^{\infty} \frac{z^n}{(n+a)^s} \quad (1.1)$$

$$(a \in \mathbb{C} \setminus \mathbb{Z}_0^-; s \in \mathbb{C} \text{ when } |z| < 1; \Re(s) > 1 \text{ when } |z| = 1).$$

Also, the integral representation of  $\Phi(z, s, a)$  (see, *e.g.*, [5, p. 27, Eq. 1.11(3)]; see also [12, p. 194, Eq. 2.5(4)]) is given by

$$\Phi(z, s, a) = \frac{1}{\Gamma(s)} \int_0^{\infty} \frac{t^{s-1} e^{-at}}{1 - ze^{-t}} dt = \frac{1}{\Gamma(s)} \int_0^{\infty} \frac{t^{s-1} e^{-(a-1)t}}{e^t - z} dt \quad (1.2)$$

$$(\Re(a) > 0; \Re(s) > 0 \text{ when } |z| \leq 1 (z \neq 1); \Re(s) > 1 \text{ when } z = 1).$$

Certain forms of generalizations of the Hurwitz-Lerch Zeta function  $\Phi(z, s, a)$  have been considered by various authors, see, *e.g.*, [3–7]). Goyal and Laddha [8, p. 100, Eq. (1.5)] and Garg *et al.* [7, p. 313, Eq. (1.7)] studied certain functions which are, respectively, defined by

$$\Phi_{\mu}^*(z, s, a) := \sum_{n=0}^{\infty} \frac{(\mu)_n}{n!} \frac{z^n}{(n+a)^s} \quad (1.3)$$

$$(\mu \in \mathbb{C}; a \in \mathbb{C} \setminus \mathbb{Z}_0^-; s \in \mathbb{C} \text{ when } |z| < 1; \Re(s - \mu) > 1 \text{ when } |z| = 1)$$

and

$$\Phi_{\lambda, \mu; \nu}(z, s, a) := \sum_{n=0}^{\infty} \frac{(\lambda)_n (\mu)_n}{(\nu)_n n!} \frac{z^n}{(n+a)^s} \quad (1.4)$$

$(\lambda, \mu \in \mathbb{C}; \nu, a \in \mathbb{C} \setminus \mathbb{Z}_0^-; s \in \mathbb{C} \text{ when } |z| < 1; \Re(s + \nu - \lambda - \mu) > 1 \text{ when } |z| = 1)$ , where  $(\lambda)_\nu$  denotes the usual Pochhammer symbol (for  $\lambda \in \mathbb{C}$ ) defined by (see [12, p. 2 and p. 5]):

$$\begin{aligned} (\lambda)_\nu &:= \begin{cases} 1 & (n = 0) \\ \lambda(\lambda + 1) \dots (\lambda + \nu - 1) & (\nu \in \mathbb{N}) \end{cases} \\ &= \frac{\Gamma(\lambda + \nu)}{\Gamma(\lambda)} \quad (\lambda \in \mathbb{C} \setminus \mathbb{Z}_0^-), \end{aligned} \quad (1.5)$$

where  $\Gamma(\lambda)$  is the familiar (Euler's) Gamma function (see, *e.g.*, [12, Chapter 1]).

The corresponding integral representations of (1.3) and (1.4) are given, respectively, by

$$\Phi_{\mu}^{*}(z, s, a) = \frac{1}{\Gamma(s)} \int_0^{\infty} \frac{t^{s-1} e^{-at}}{(1 - ze^{-t})^{\mu}} dt = \frac{1}{\Gamma(s)} \int_0^{\infty} \frac{t^{s-1} e^{-(a-1)t}}{(e^t - z)^{\mu}} dt \quad (1.6)$$

$$(\Re(a) > 0; \Re(s) > 0 \quad \text{when } |z| \leq 1 (z \neq 1); \Re(s) > 1 \quad \text{when } z = 1)$$

and

$$\Phi_{\lambda, \mu; \nu}(z, s, a) := \frac{1}{\Gamma(s)} \int_0^{\infty} t^{s-1} e^{-at} {}_2F_1(\lambda, \mu; \nu; ze^{-t}) dt \quad (1.7)$$

$$(\Re(a) > 0; \Re(s) > 0 \quad \text{when } |z| \leq 1 (z \neq 1); \Re(s) > 1 \quad \text{when } z = 1).$$

In their paper, Chaudhry *et al.* [1] presented the following extension of the Beta function:

$$B(x, y; p) = \int_0^1 t^{x-1} (1-t)^{y-1} \exp\left(-\frac{p}{t(1-t)}\right) dt \quad (\Re(p) > 0), \quad (1.8)$$

and showed that this extension has certain connections with the Macdonald, Error, and Whittaker functions.

Subsequently, Chaudhry *et al.* [2] used the above function  $B(x, y; p)$  to provide an extension of the hypergeometric function which is given by

$$F_p(a, b; c; z) = \sum_{n=0}^{\infty} (a)_n \frac{B(b+n, c-b; p)}{B(b, c-b)} \frac{z^n}{n!} \quad (1.9)$$

$$(p \geq 0, |z| < 1; \Re(c) > \Re(b) > 0).$$

Evidently, the special cases of (1.8) and (1.9) when  $p = 0$  reduce immediately to the classical Eulerian Beta and Gauss's hypergeometric function, respectively.

Motivated by these aforementioned considerations, we first give an extension of the generalized Hurwitz-Lerch Zeta function (1.4). We then investigate certain properties of the extended generalized Hurwitz-Lerch Zeta function and derive for this class of functions various integral representations, a derivative formula, Mellin transform and certain generating relations. An application to probability distributions is also considered and we also point out some special cases of the main results.

## 2 Extended Generalized Hurwitz-Lerch Zeta Function

In terms of the extended Beta function  $B(x, y; p)$  defined by (1.8), we propose a new extension of generalized Hurwitz-Lerch Zeta function by changing

$$\frac{(\mu)_n}{(\nu)_n} = \frac{B(\mu + n, \nu - \mu)}{B(\mu, \nu - \mu)} \rightarrow \frac{B(\mu + n, \nu - \mu; p)}{B(\mu, \nu - \mu)}$$

in (1.4) and thus, define this new extended form as follows:

$$\Phi_{\lambda, \mu; \nu}(z, s, a; p) := \sum_{n=0}^{\infty} \frac{(\lambda)_n}{n!} \frac{B(\mu + n, \nu - \mu; p)}{B(\mu, \nu - \mu)} \frac{z^n}{(n + a)^s} \quad (2.1)$$

( $p \geq 0$ ;  $\lambda, \mu \in \mathbb{C}$ ;  $\nu, a \in \mathbb{C} \setminus \mathbb{Z}_0^-$ ;  $s \in \mathbb{C}$  when  $|z| < 1$ ;  $\Re(s + \nu - \lambda - \mu) > 1$  when  $|z| = 1$ ).

**Remark 2.1.** The following special or limit cases of the generalized Hurwitz-Lerch Zeta function  $\Phi_{\lambda, \mu; \nu}(z, s, a; p)$  defined by (2.1) are worth noting.

**Case 1.** If we set  $\lambda = 1$ , then (2.1) yields the following special case of another extended form of the generalized Hurwitz-Lerch Zeta function  $\Phi_{\mu; \nu}^{1,1}(z, s, a; p)$  studied earlier by Lin and Srivastava [9, p. 727, Eq. (8)] (with  $\rho = \sigma = 1$ ):

$$\Phi_{\mu; \nu}^{1,1}(z, s, a; p) := \Phi_{1, \mu; \nu}(z, s, a; p) := \sum_{n=0}^{\infty} \frac{B(\mu + n, \nu - \mu; p)}{B(\mu, \nu - \mu)} \frac{z^n}{(n + a)^s} \quad (2.2)$$

( $p \geq 0$ ;  $\mu \in \mathbb{C}$ ;  $\nu, a \in \mathbb{C} \setminus \mathbb{Z}_0^-$ ;  $s \in \mathbb{C}$  when  $|z| < 1$ ;  $\Re(s + \nu - \mu) > 1$  when  $|z| = 1$ ).

**Case 2.** Upon setting  $\lambda = \nu = 1$  in (2.1), we have the extended generalized Hurwitz-Lerch Zeta function  $\Phi_{\mu}^*(z, s, a; p)$  of Goyal and Laddha [8] in the following form:

$$\Phi_{\mu}^*(z, s, a; p) := \Phi_{1, \mu; 1}(z, s, a; p) := \sum_{n=0}^{\infty} \frac{B(\mu + n, 1 - \mu; p)}{B(\mu, 1 - \mu)} \frac{z^n}{(n + a)^s} \quad (2.3)$$

( $p \geq 0$ ;  $\mu \in \mathbb{C}$ ;  $a \in \mathbb{C} \setminus \mathbb{Z}_0^-$ ;  $s \in \mathbb{C}$  when  $|z| < 1$ ;  $\Re(s + 1 - \mu) > 1$  when  $|z| = 1$ ).

**Case 3.** A limit case of the generalized Hurwitz-Lerch function  $\Phi_{\lambda,\mu;\nu}(z, s, a; p)$  defined by (2.1) (which is particularly of interest in our present investigation) is given by

$$\Phi_{\mu;\nu}^*(z, s, a; p) := \lim_{|\lambda| \rightarrow \infty} \left\{ \Phi_{\lambda,\mu;\nu} \left( \frac{z}{\lambda}, s, a; p \right) \right\} := \sum_{n=0}^{\infty} \frac{B(\mu + n, \nu - \mu; p)}{B(\mu, \nu - \mu)} \frac{z^n}{n!(n + a)^s} \tag{2.4}$$

( $p \geq 0; \mu \in \mathbb{C}; \nu, a \in \mathbb{C} \setminus \mathbb{Z}_0^-; s \in \mathbb{C}$  when  $|z| < 1; \Re(s + \nu - \mu) > 1$  when  $|z| = 1$ ).

**Remark 2.2.** The special cases of (2.1) and (2.3) when  $p = 0$  are easily seen to reduce to (1.4) and (1.3), respectively, in view of the following relationships of the functions (defined above):

$$\Phi_{\lambda,\mu;\nu}(z, s, a; 0) = \Phi_{\lambda,\mu;\nu}(z, s, a) \quad \text{and} \quad \Phi_{1,\mu;1}(z, s, a; 0) = \Phi_{\mu}^*(z, s, a),$$

Also, for  $p = 0$  in (2.2), we have the following special case of the generalized Hurwitz-Lerch function of Lin and Srivastava [9, p. 727, Eq. (8)] (with  $\rho = \sigma = 1$ ):

$$\Phi_{1,\mu;\mu}(z, s, a; 0) = \Phi_{\mu;\nu}^{1,1}(z, s, a).$$

### 3 Integral Representations and Derivative Formula

In this section, we present certain integral representations and a derivative formula for the extended Hurwitz-Lerch Zeta function defined by (2.1).

**Theorem 3.1.** *The following integral representation holds true:*

$$\Phi_{\lambda,\mu;\nu}(z, s, a; p) := \frac{1}{\Gamma(s)} \int_0^{\infty} t^{s-1} e^{-at} F_p(\lambda, \mu; \nu; ze^{-t}) dt \tag{3.1}$$

( $p \geq 0; \Re(a) > 0; \Re(s) > 0$  when  $|z| \leq 1$  ( $z \neq 1$ );  $\Re(s) > 1$  when  $z = 1$ ).

*Proof.* Using the following Eulerian integral:

$$\frac{1}{(n + a)^s} := \frac{1}{\Gamma(s)} \int_0^{\infty} t^{s-1} e^{-(n+a)t} dt \quad (\min\{\Re(s), \Re(a)\} > 0; n \in \mathbb{N}_0) \tag{3.2}$$

in (2.1), we have

$$\Phi_{\lambda,\mu;\nu}(z, s, a; p) = \sum_{n=0}^{\infty} \frac{(\lambda)_n}{n!} \frac{B(\mu+n, \nu-\mu; p)}{B(\mu, \nu-\mu)} \left( \frac{1}{\Gamma(s)} \int_0^{\infty} t^{s-1} e^{-(n+a)t} dt \right).$$

Interchanging the order of summation and integration under the condition stated in Theorem 3.1, we get

$$\Phi_{\lambda,\mu;\nu}(z, s, a; p) := \frac{1}{\Gamma(s)} \int_0^{\infty} t^{s-1} e^{-at} \left( \sum_{n=0}^{\infty} (\lambda)_n \frac{B(\mu+n, \nu-\mu; p)}{B(\mu, \nu-\mu)} \frac{(ze^{-t})^n}{n!} \right) dt.$$

Now using the definition (1.9) to express the involved sum as an extended hypergeometric function, we are led to the desired result.  $\square$

**Theorem 3.2.** *The following integral representations hold true:*

$$\Phi_{\lambda,\mu;\nu}(z, s, a; p) = \frac{e^{-2p} \Gamma(\nu)}{\Gamma(\mu) \Gamma(\nu-\mu)} \int_0^{\infty} \frac{x^{\mu-1}}{(1+x)^\nu} \exp\left(-p\left(x + \frac{1}{x}\right)\right) \Phi_{\lambda}^* \left( \frac{zx}{1+x}, s, a \right) dx \quad (3.3)$$

$$(p \geq 0; \Re(\nu) > \Re(\mu) > 0)$$

and

$$\begin{aligned} \Phi_{\lambda,\mu;\nu}(z, s, a; p) &= \frac{e^{-2p} \Gamma(\nu)}{\Gamma(s) \Gamma(\mu) \Gamma(\nu-\mu)} \\ &\times \int_0^{\infty} \int_0^{\infty} \frac{t^{s-1} e^{-at} x^{\mu-1}}{(1+x)^\nu} \exp\left(-p\left(x + \frac{1}{x}\right)\right) \left(1 - \frac{zx e^{-t}}{1+x}\right)^{-\lambda} dt dx \end{aligned} \quad (3.4)$$

$$(p \geq 0; \Re(\nu) > \Re(\mu) > 0, \min\{\Re(s), \Re(a)\} > 0),$$

provided that the integrals in the right-hand sides of the assertions (3.3) and (3.4) converge.

*Proof.* By setting  $\alpha = \mu + n$  and  $\beta = \nu - \mu$  in the following integral representation of the extended Beta-function (see, e.g., [1, p. 22, Equation (2.8)]):

$$B(\alpha, \beta; p) = e^{-2p} \int_0^{\infty} \frac{x^{\alpha-1}}{(1+x)^{\alpha+\beta}} \exp\left(-p\left(x + \frac{1}{x}\right)\right) dx \quad (\Re(p) > 0), \quad (3.5)$$

$$(p \geq 0; \Re(\alpha) > 0, \Re(\beta) > 0)$$

we find that

$$B(\mu + n, \nu - \mu; p) = e^{-2p} \int_0^\infty \frac{x^{\mu+n-1}}{(1+x)^{\nu+n}} \exp\left(-p\left(x + \frac{1}{x}\right)\right) dx, \quad (3.6)$$

which by appealing to the (2.1) and using (1.3) immediately yields the first assertion (3.3) of Theorem 3.2.

Also, in view of (3.1) and (3.6), we also obtain

$$\begin{aligned} \Phi_{\lambda,\mu;\nu}(z, s, a; p) &:= \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} e^{-at} \sum_{n=0}^\infty (\lambda)_n \frac{B(\mu + n, \nu - \mu; p)}{B(\mu, \nu - \mu)} \frac{(ze^{-t})^n}{n!} dt \\ &= \frac{e^{-2p} \Gamma(\nu)}{\Gamma(s)\Gamma(\mu)\Gamma(\nu - \mu)} \int_0^\infty \int_0^\infty \frac{t^{s-1} e^{-at} x^{\mu-1}}{(1+x)^\nu} \exp\left(-p\left(x + \frac{1}{x}\right)\right) \\ &\quad \sum_{n=0}^\infty \frac{(\lambda)_n}{n!} \left(\frac{zxe^{-t}}{1+x}\right)^n dt dx, \end{aligned} \quad (3.7)$$

which in view of the binomial expansion

$$(1 - zt)^{-a} = \sum_{n=0}^\infty (a)_n \frac{(zt)^n}{n!}$$

leads us to the second assertion (3.4) of Theorem 3.2. □

**Theorem 3.3.** *The following integral representation holds true:*

$$\Phi_{\lambda,\mu;\nu}(z, s, a; p) := \frac{1}{\Gamma(\lambda)} \int_0^\infty t^{\lambda-1} e^{-t} \Phi_{\mu;\nu}^*(zt, s, a; p) dt \quad (3.8)$$

( $p \geq 0; \Re(\lambda) > 0, \Re(a) > 0; \Re(s) > 0$  when  $|z| \leq 1$  ( $z \neq 1$ );  $\Re(s) > 1$  when  $z = 1$ ), where  $\Phi_{\mu;\nu}^*(z, s, a; p)$  is the limiting case defined by (2.4).

*Proof.* Using the integral representation of the Pochhammer symbol  $(\lambda)_n$ :

$$(\lambda)_n = \frac{1}{\Gamma(\lambda)} \int_0^\infty t^{\lambda+n-1} e^{-t} dt$$

in (2.1) and inverting the order of summation and integration (which is permissible under the conditions stated with Theorem 3.3), we get

$$\Phi_{\lambda,\mu;\nu}(z, s, a; p) = \frac{1}{\Gamma(\lambda)} \int_0^\infty t^{\lambda-1} e^{-t} \sum_{n=0}^\infty \frac{B(\mu + n, \nu - \mu; p)}{B(\mu, \nu - \mu)} \frac{(zt)^n}{n!(n+a)^s} dt. \quad (3.9)$$

Applying now (2.4), we get the desired integral representation. □

We next derive the following derivative formula for the function  $\Phi_{\lambda,\mu;\nu}(z, s, a; p)$  defined by (2.1).

**Theorem 3.4.** *The following derivative formula for  $\Phi_{\lambda,\mu;\nu}(z, s, a; p)$  holds true:*

$$\frac{d^n}{dz^n} \{\Phi_{\lambda,\mu;\nu}(z, s, a; p)\} = \frac{(\lambda)_n (\mu)_n}{(\nu)_n} \Phi_{\lambda+n, \mu+n; \nu+n}(z, s, a+n; p) \quad (n \in \mathbb{N}_0). \quad (3.10)$$

*Proof.* Differentiating (2.1) with respect to  $z$ , we obtain

$$\frac{d}{dz} \{\Phi_{\lambda,\mu;\nu}(z, s, a; p)\} = \sum_{n=1}^{\infty} \frac{(\lambda)_n}{(n-1)!} \frac{B(\mu+n, \nu-\mu; p)}{B(\mu, \nu-\mu)} \frac{z^{n-1}}{(n+a)^s}. \quad (3.11)$$

which upon replacing  $n$  by  $n+1$  in the right-hand side of (3.11), and using the identities that

$$B(\mu, \nu-\mu) = \frac{\nu}{\mu} B(\mu+1, \nu-\mu) \quad \text{and} \quad (a)_{n+1} = a(a+1)_n$$

gives the derivative formula that

$$\frac{d}{dz} \{\Phi_{\lambda,\mu;\nu}(z, s, a; p)\} = \frac{\lambda\mu}{\nu} \Phi_{\lambda+1, \mu+1; \nu+1}(z, s, a+1; p).$$

A repeated application of the this above procedure  $n$ -times gives us the general formula (3.10).  $\square$

## 4 Mellin Transform and Generating Relations

In this section, we obtain Mellin transform and certain generating relations for the extended Hurwitz-Lerch Zeta function  $\Phi_{\lambda,\mu;\nu}(z, s, a; p)$  defined by (2.1).

The Mellin transform of a suitable integrable function  $f(t)$  with index  $\alpha$  is defined, as usual, by

$$\mathcal{M}\{f(\tau) : \tau \rightarrow \alpha\} := \int_0^{\infty} \tau^{\alpha-1} f(\tau) d\tau, \quad (4.1)$$

provided that the improper integral in (4.1) exists.



**Theorem 4.1.** *The Mellin transform of the function  $\Phi_{\lambda,\mu;\nu}(z, s, a; p)$  defined by (2.1) is given by*

$$\begin{aligned} & \mathcal{M} \{ \Phi_{\lambda,\mu;\nu}(z, s, a; p)(z, s, a) : p \rightarrow \alpha \} \\ &= \frac{\Gamma(\alpha)B(\mu + \alpha, \nu - \mu + \alpha)}{B(\mu, \nu - \mu)} \Phi_{\lambda,\mu+\alpha,\nu+2\alpha}(z, s, a) \end{aligned} \tag{4.2}$$

$$(\Re(\alpha) > 0 \quad \text{and} \quad \Re(\lambda + \alpha) > 0).$$

*Proof.* Using the definition (4.1) of the Mellin transform, we find from (2.1) that

$$\begin{aligned} \mathcal{M} \{ \Phi_{\lambda,\mu;\nu}(z, s, a; p) : p \rightarrow \alpha \} &:= \int_0^\infty p^{\alpha-1} \Phi_{\lambda,\mu;\nu}(z, s, a; p) dp \\ &= \int_0^\infty p^{\alpha-1} \left( \sum_{n=0}^\infty \frac{(\lambda)_n}{n!} \frac{B(\mu + n, \nu - \mu; p)}{B(\mu, \nu - \mu)} \frac{z^n}{(n + a)^s} \right) dp \\ &= \sum_{n=0}^\infty \frac{(\lambda)_n}{n!} \frac{z^n}{(n + a)^s} \frac{1}{B(\mu, \nu - \mu)} \int_0^\infty p^{\alpha-1} B(\mu + n, \nu - \mu; p) dp. \end{aligned}$$

Applying the following result given by ([1, p. 21, Eq. (2.1)])

$$\int_0^\infty p^{\alpha-1} B(x, y; p) dp = \Gamma(\alpha)B(x + \alpha, y + \alpha) \tag{4.3}$$

$$(\Re(\alpha) > 0, \quad \Re(x + \alpha) > 0, \quad \Re(y + \alpha) > 0,$$

we obtain

$$\begin{aligned} \mathcal{M} \{ \Phi_{\lambda,\mu;\nu}(z, s, a; p) : p \rightarrow \alpha \} &= \\ &= \Gamma(\alpha) \sum_{n=0}^\infty \frac{(\lambda)_n}{n!} \frac{z^n}{(n + a)^s} \frac{B(\mu + \alpha + n, \nu - \mu + \alpha)}{B(\mu, \nu - \mu)} \\ &= \frac{\Gamma(\alpha) B(\mu + \alpha, \nu - \mu + \alpha)}{B(\mu, \nu - \mu)} \sum_{n=0}^\infty \frac{(\lambda)_n(\mu + \alpha)_n}{(\nu + 2\alpha)_n} \frac{z^n}{n!(n + a)^s}, \end{aligned} \tag{4.4}$$

which upon using (2.1) gives the desired Mellin transform representation.  $\square$

**Remark 4.1.** The special case of (4.2) when  $\alpha = 1$  yields the following integral of the function  $\Phi_{\lambda,\mu;\nu}(z, s, a; p)$  with respect to the variable  $p$  in terms of the generalized Hurwitz-Lerch Zeta function defined by (1.4):

$$\int_0^{\infty} \Phi_{\lambda,\mu;\nu}(z, s, a; p) dp = \frac{\Gamma(\alpha)B(\mu + 1, \nu - \mu + 1)}{B(\mu, \nu - \mu)} \Phi_{\lambda,\mu+1,\nu+2}(z, s, a). \quad (4.5)$$

We next derive the following generating relations for the function  $\Phi_{\lambda,\mu;\nu}(z, s, a; p)$  defined by (2.1).

**Theorem 4.2.** *The following generating function for  $\Phi_{\lambda,\mu;\nu}(z, s, a; p)$  in (2.1) holds true:*

$$\sum_{n=0}^{\infty} (\lambda)_n \Phi_{\lambda+n,\mu;\nu}(z, s, a; p) \frac{t^n}{n!} = (1-t)^{-\lambda} \Phi_{\lambda+n,\mu;\nu}\left(\frac{z}{1-t}, s, a; p\right) \quad (4.6)$$

$$(p \geq 0, \lambda \in \mathbb{C} \text{ and } |t| < 1).$$

*Proof.* Let the left-hand side of the assertion (4.6) be denoted by  $S$ . Then, from the definition (2.1), we have

$$\begin{aligned} S &= \sum_{n=0}^{\infty} (\lambda)_n \left\{ \sum_{k=0}^{\infty} (\lambda+n)_k \frac{B(\mu+k, \nu-\mu; p)}{B(\mu, \nu-\mu)} \frac{z^k}{k!(k+a)^s} \right\} \frac{t^n}{n!} \\ &= \sum_{k=0}^{\infty} (\lambda)_k \frac{B(\mu+k, \nu-\mu; p)}{B(\mu, \nu-\mu)} \left\{ \sum_{n=0}^{\infty} (\lambda+k)_n \frac{t^n}{n!} \right\} \frac{z^k}{k!(k+a)^s} \end{aligned}$$

upon reversal of the order of summation and use of the identity  $(\lambda)_n(\lambda+n)_k = (\lambda)_k(\lambda+k)_n$ .

Using the binomial expansion

$$(1-t)^{-\lambda-k} = \sum_{n=0}^{\infty} (\lambda+k)_n \frac{t^n}{n!} \quad (|t| < 1),$$

and interpreting in terms of (2.1) as a function of the form  $\Phi_{\lambda+n,\mu;\nu}\left(\frac{z}{1-t}, s, a; p\right)$ , we are thus lead to the assertion (4.6) of Theorem 4.2.  $\square$

Our next result on the generating function is contained in the following:

**Theorem 4.3.** *The following generating function for  $\Phi_{\lambda,\mu;\nu}(z, s, a; p)$  in (2.1) holds true:*

$$\sum_{n=0}^{\infty} \frac{(s)_n}{n!} \Phi_{\lambda,\mu;\nu}(z, s+n, a; p) t^n = \Phi_{\lambda,\mu;\nu}(z, s, a-t; p) \quad (4.7)$$

$(p \geq 0, \lambda \in \mathbb{C} \text{ and } |t| < |a|; s \neq 1).$

*Proof.* Using (2.1) in the right-hand side of the assertion (4.7), we have

$$\begin{aligned} \Phi_{\lambda,\mu;\nu}(z, s, a-t; p) &= \sum_{k=0}^{\infty} (\lambda)_k \frac{B(\mu+k, \nu-\mu; p)}{B(\mu, \nu-\mu)} \frac{z^k}{k!(k+a-t)^s} \\ &= \sum_{k=0}^{\infty} (\lambda)_k \frac{B(\mu+k, \nu-\mu; p)}{B(\mu, \nu-\mu)} \frac{z^k}{k!(k+a)^s} \left(1 - \frac{t}{k+a}\right)^{-s} \\ &= \sum_{k=0}^{\infty} (\lambda)_k \frac{B(\mu+k, \nu-\mu; p)}{B(\mu, \nu-\mu)} \frac{z^k}{k!(k+a)^s} \left\{ \sum_{n=0}^{\infty} \frac{(s)_n}{n!} \frac{t^n}{(k+a)^n} \right\} \\ &= \sum_{n=0}^{\infty} \frac{(s)_n}{n!} \left\{ \sum_{k=0}^{\infty} (\lambda)_k \frac{B(\mu+k, \nu-\mu; p)}{B(\mu, \nu-\mu)} \frac{z^k}{k!(k+a)^{s+n}} \right\} t^n. \end{aligned}$$

The desired assertion (4.7) of Theorem 4.3 follows now on making use of (2.1). □

## 5 Application to the Probability Distributions

In this concluding section, we consider a general probability distribution involving the extended generalized Hurwitz-Lerch Zeta function (2.1) defined as follow:

**Definition 5.1.** *A random variable  $\xi$  is said to be the generalized Hurwitz distributed if its probability density function is given by*

$$f_{\xi}(a) =: \begin{cases} \frac{s\Phi_{\lambda,\mu;\nu}(z, s+1, a; p)}{\Phi_{\lambda,\mu;\nu}(z, s, 1; p)}, & a \geq 1 \\ 0 & \text{otherwise,} \end{cases} \quad (5.1)$$

since one can easily verify that

$$\int_1^{\infty} f_{\xi}(a) da = 1,$$

where it is tacitly assumed that the arguments  $z$ ,  $s$ ,  $p$  and the parameters  $\lambda$ ,  $\mu$  and  $\nu$  are fixed and suitably constrained so that the probability density function  $f_\xi(a)$  remains nonnegative.

**Theorem 5.1.** *Let  $\xi$  be a continuous random variable with its probability density function defined by (5.1). Then, the moment generating function  $M(t)$  of the random variable  $\xi$  is given by*

$$M(t) = E_s[e^{\xi t}] = \sum_{n=0}^{\infty} E_s[\xi^n] \frac{t^n}{n!}, \quad (5.2)$$

where the moments  $E_s[\xi^n]$  of order  $n$  are given by

$$E_s[\xi^n] = \sum_{k=0}^n \frac{n!}{(n-k)!} \frac{\Gamma(s-k)}{\Gamma(s)} \frac{\Phi_{\lambda,\mu;\nu}(z, s-k, 1; p)}{\Phi_{\lambda,\mu;\nu}(z, s, 1; p)}. \quad (5.3)$$

*Proof.* The assertion in (5.2) can be derived easily by using the series expansion of  $e^{\xi t}$ . To establish (5.3), we observe that

$$\frac{d}{da} \{\Phi_{\lambda,\mu;\nu}(z, s, a; p)\} = -s\Phi_{\lambda,\mu;\nu}(z, s+1, a; p), \quad (5.4)$$

which follows readily from (2.1), and thus from the definition of  $E_s[\xi^n]$ , we have

$$\begin{aligned} E_s[\xi^n] &= \int_1^\infty a^n f_\xi(a) da \\ &= \frac{s}{\Phi_{\lambda,\mu;\nu}(z, s, 1; p)} \int_1^\infty a^n \Phi_{\lambda,\mu;\nu}(z, s+1, a; p) da \\ &= -\frac{1}{\Phi_{\lambda,\mu;\nu}(z, s, 1; p)} \int_1^\infty a^n \frac{d}{da} \{\Phi_{\lambda,\mu;\nu}(z, s, a; p)\} da \\ &= \left[ -\frac{a^n \Phi_{\lambda,\mu;\nu}(z, s, a; p)}{\Phi_{\lambda,\mu;\nu}(z, s, 1; p)} \right]_{a=1}^\infty + \\ &\quad + \frac{n}{\Phi_{\lambda,\mu;\nu}(z, s, 1; p)} \int_1^\infty a^{n-1} \Phi_{\lambda,\mu;\nu}(z, s, a; p) da \\ &= 1 - \lim_{a \rightarrow \infty} \left\{ \frac{a^n \Phi_{\lambda,\mu;\nu}(z, s, a; p)}{\Phi_{\lambda,\mu;\nu}(z, s, 1; p)} \right\} + \\ &\quad + \frac{n}{\Phi_{\lambda,\mu;\nu}(z, s, 1; p)} \int_1^\infty a^{n-1} \Phi_{\lambda,\mu;\nu}(z, s, a; p) da \\ &= 1 + \frac{n}{\Phi_{\lambda,\mu;\nu}(z, s, 1; p)} \int_1^\infty a^{n-1} \Phi_{\lambda,\mu;\nu}(z, s, a; p) da, \end{aligned} \quad (5.5)$$

where, in addition to the derivative property (5.4), we have used the following limit formula:

$$\begin{aligned} \lim_{a \rightarrow \infty} \{a^n \Phi_{\lambda, \mu; \nu}(z, s, a; p)\} &= \lim_{a \rightarrow \infty} \left\{ \frac{a^n}{\Gamma(s)} \int_0^\infty t^{s-1} e^{-at} F_p(\lambda, \mu; \nu; ze^{-t}) dt \right\} \\ &= \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} \lim_{a \rightarrow \infty} \{a^n e^{-at}\} F_p(\lambda, \mu; \nu; ze^{-t}) dt \\ &= 0. \end{aligned} \quad (5.6)$$

Consequently, we have the following reduction formula for  $E_s[\xi^n]$ :

$$E_s[\xi^n] = 1 + \frac{\Phi_{\lambda, \mu; \nu}(z, s-1, 1; p)}{\Phi_{\lambda, \mu; \nu}(z, s, 1; p)} \frac{n}{s-1} E_{s-1}[\xi^{n-1}], \quad (5.7)$$

and by iterating the recurrence (5.5), we arrive at the desired result (5.3).  $\square$

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