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On Uniform Exponential Trichotomy in Banach Spaces

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Abstract. In this paper we consider three concepts of uniform exponential trichotomy on the half-line in the general framework of evolution operators in Banach spaces. We obtain a systematic classification of uniform exponential trichotomy concepts and the connections between them.

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1 Introduction

In the qualitative theory of dynamical systems the exponential trichotomy is one of the most important asymptotic properties playing a crucial role in the study of center manifolds (see [2], [3], [4], [5], [7]).
A dynamical system on a Banach space has an exponential trichotomy if the space can be decomposed in every moment into a direct sum of three subspaces such that the solutions are exponentially stable on stable subspaces, exponentially instable on instable subspaces and respectively exponentially bounded on the neutral subspaces. In the particular case when all neutral subspaces contain only the null vector one obtains the exponential dichotomy property.
The notion of exponential trichotomy of differential equations was introduced by S. Elaydi and O. Hajek in [4].

In the last decades, a substantial part of the asymptotic theory of dynamical systems was devoted to the study of the exponential dichotomies, the extension of the methods to the trichotomy case being difficult due to the presence of nontrivial neutral subspaces.
Characterizations for the uniform exponential trichotomy of evolution operators in Banach spaces was obtained by M. Megan and C. Stoica in [9] and [10]. The case of nonuniform exponential trichotomy was considered by L. Barreira and C. Valls in [2] and [3] who consider the problems of robustness and Lyapunov functions for nonuniform exponential trichotomy. A study of the exponential trichotomy by means of input-output techniques were obtained by B. Sasu and A. L. Sasu in [11].
In this paper we study three concepts of uniform exponential trichotomy (strong exponential trichotomy, exponential trichotomy, weak exponential trichotomy) in the general framework of evolution operators in Banach spaces, characterization and connections between these concepts are given. Generalizations of the uniform exponential trichotomy have been studied by I. Lopez-Fenner and M. Pinto in [6] and N. Lupa and M. Megan in [8].
Thus we obtain generalization for the case of uniform exponential trichotomy of the results obtained by M. G. Babuţia, T. Ceauşu and N. M. Seimeanu in [1] for the particular case of uniform exponential dichotomy.

2 Definitions, notation and preliminary results

Let X be a real or complex Banach space and $\mathcal{B}(X)$ the Banach algebra of all bounded linear operators on X. The norms on X and $\mathcal{B}(X)$ will be denoted by $\|\cdot\|$. Denote by I the identity operator on X. We also denote by Δ the set of all pairs of real nonnegative numbers (t,s) with $t \geq s$ i.e.

$$\Delta = \{(t,s) \in \mathbb{R}_+^2 \ : \ t \geq s \geq 0\}$$

and by T the set defined by

$$T = \{(t,s,t_0) \in \mathbb{R}_+^3 \ : \ t \geq s \geq t_0 \geq 0\}.$$

Definition 2.1. A map $U : \Delta \to \mathcal{B}(X)$ is called an **evolution operator** on X if

(e_1) $U(t,t) = I$ for every $t \geq 0$;

(e_2) $U(t,s)U(s,t_0) = U(t,t_0)$ for all $(t,s,t_0) \in T$.

In addition,

(e_3) if for all $(t, s) \in \Delta$ the linear operator $U(t, s)$ is bijective then we say that the evolution operator U is **reversible**;

Definition 2.2. An operator valued function $P : \mathbb{R}_+ \to \mathcal{B}(X)$ is said to be a **family of projections** on X if

$$P(t)^2 = P(t), \quad \text{for every } t \geq 0.$$

Definition 2.3. Three families of projections $P_1, P_2, P_3 : \mathbb{R}_+ \to \mathcal{B}(X)$ are called **supplementary** if for all $t \in \mathbb{R}_+$ we have

(s_1) $P_1(t) + P_2(t) + P_3(t) = I$;

(s_2) $P_i(t)P_j(t) = 0$ for all $i, j \in \{1, 2, 3\}$ with $i \neq j$.

Definition 2.4. Given an evolution operator $U : \Delta \to \mathcal{B}(X)$ we say that a family of projections $P : \mathbb{R}_+ \to \mathcal{B}(X)$ is **invariant** for U if

$$P(t)U(t, s) = U(t, s)P(s)$$

for all $(t, s) \in \Delta$.
If P is invariant for U, then the family $\{RangeP(t),\ t \in \mathbb{R}_+\}$, of ranges of the projections $P(t)$ is invariant for U in the sense that if $x \in RangeP(s)$ for some $s \in \mathbb{R}_+$ then $U(t, s)x \in RangeP(t)$ for all $t \geq s$.
A characterization of invariant projections is given by

Proposition 2.1. *The family of projections $P : \mathbb{R}_+ \to \mathcal{B}(X)$ is invariant for the evolution operator $U : \Delta \to \mathcal{B}(X)$ if and only if the following two properties hold:*

(i_1) *$U(t, s)(KerP(s)) \subset KerP(t)$;*

(i_2) *$U(t, s)(RangeP(s)) \subset RangeP(t)$ for all $(t, s) \in \Delta$.*

Proof. **Necessity.** If P is invariant for U, $x \in KerP(s)$ and $y \in RangeP(s)$ then

$$P(t)U(t, s)x = U(t, s)P(s)x = 0$$

and

$$U(t, s)y = U(t, s)P(s)y = P(t)U(t, s)y \in RangeP(t)$$

for all $(t, s) \in \Delta$.
Sufficiency. We observe that if $x \in X$ then $x - P(s)x \in KerP(s)$ and $U(t, s)P(s)x \in RangeP(t)$ and hence by (i_1) it results that

$$U(t, s)(x - P(s)x) \in KerP(t)$$

which implies that

$$P(t)U(t,s)x = P(t)U(t,s)P(s)x.$$

Because, from (i_2), we have that $U(t,s)P(s) \in RangeP(t)$, it results that

$$P(t)U(t,s)P(s)x = U(t,s)P(s)x.$$

Finally, we obtain that $P(t)U(t,s)x = U(t,s)P(s)x$ and hence P is invariant for U. $\square$

Definition 2.5. A family $\mathcal{P}=\{P_1, P_2, P_3\}$ of the three supplementary projections is called **invariant for the evolution operator** U if P_1, P_2 and P_3 are invariant for U.

Definition 2.6. Let $P : \mathbb{R}_+ \to \mathcal{B}(X)$ be a family of projections on X which is invariant for the evolution operator $U : \Delta \to \mathcal{B}(X)$. We say that P is **strongly invariant** with U if for all $(t,s) \in \Delta$ the restriction of $U(t,s)$ on $RangeP(s)$ is an isomorphism from $RangeP(s)$ to $RangeP(t)$.

Remark 2.1. If the family of projections $P : \mathbb{R}_+ \to \mathcal{B}(X)$ is invariant for the reversible evolution operator $U : \Delta \to \mathcal{B}(X)$ then it also strongly invariant for U.
Indeed, if U is reversible then for all $y \in RangeP(t)$, then exists $x_0 \in X$ with $U(t,s)x_0 = y$. Then $x = P(s)x_0 \in RangeP(s)$ with

$$U(t,s)x = U(t,s)P(s)x_0 = P(t)U(t,s)x_0 = P(t)y = y.$$

Thus $U(t,s)$ is surjective from $RangeP(s)$ to $RangeP(t)$. From here and reversibility of U, we finally obtain that P is strongly invariant for U.

Remark 2.2. If a family of projections $P : \mathbb{R}_+ \to \mathcal{B}(X)$ is strongly invariant for the evolution operator $U : \Delta \to \mathcal{B}(X)$ then there exists $V : \Delta \to \mathcal{B}(X)$ such that $V(t,s)$ is an isomorphism from $RangeP(t)$ to $RangeP(s)$ and

(v_1) $U(t,s)V(t,s)P(t) = P(t)$;

(v_2) $V(t,s)U(t,s)P(s) = P(s)$,

for all $(t,s) \in \Delta$.
The map V is called the **skew-evolution operator** associated to the pair (U,P).

Remark 2.3. If $P : \mathbb{R}_+ \to \mathcal{B}(X)$ is invariant for the reversible evolution operator $U : \Delta \to \mathcal{B}(X)$ then the skew-evolution operator associated to the pair (U,P) is $V(t,s) = U(t,s)^{-1}$ for all $(t,s) \in \Delta$.

Example 2.1. Let $X = \mathbb{R}^3$ with the norm

$$\|(x_1, x_2, x_3)\| = |x_1| + |x_2| + |x_3|$$

and let $U : \Delta \to \mathcal{B}(X)$ be the evolution operator defined by

$$U(t,s)(x_1,x_2,x_3) = \begin{cases} (x_1 e^{s-t}, x_2 e^{t-s}, x_3 e^{t-s}), & \text{if } t \geq s > 0 \text{ or } t = s = 0 \\ (x_1 e^{-t}, 0, x_3 e^{t}), & \text{if } t > s = 0 \end{cases}$$

and let $P : \mathbb{R}_+ \to \mathcal{B}(X)$ be the family of projections defined by

$$P(t)(x_1,x_2,x_3) = \begin{cases} (0,0,x_3), & \text{if } t = 0 \\ (-x_2 e^{-2t}, x_2, x_3), & \text{if } t > 0. \end{cases}$$

One can see that $U(t,s)P(s) = P(t)U(t,s)$ but P is not strongly invariant for U because the restriction of $U(1,0)$ from $RangeP(0)$ to $RangeP(1)$ is not surjective. Indeed, for $y = (-\frac{1}{e^2}, 0, 1) = P(1)(0,1,0) \in RangeP(1)$, it does not exists $x = (0,0,x_3) \in RangeP(0)$ with $y = U(1,0)x$ because we would obtain $(-\frac{1}{e^2}, 0, 1) = (0,0,x_3)$, which is impossible.

Definition 2.7. A family $\mathcal{P} = \{P_1, P_2, P_3\}$ of three supplementary projections is called **compatible** with the evolution operator $U : \Delta \to \mathcal{B}(X)$ if

(c_1) P_1 is invariant for U;

(c_2) P_2 and P_3 are strongly invariant for U.

For an evolution operator $U : \Delta \to \mathcal{B}(X)$ and $\mathcal{P}$ compatible with U, we will denote by $V_2(t,s)$ and $V_3(t,s)$ the skew-evolution operators associated to the pairs (U, P_2) and (U, P_3) respectively.

3 Uniform exponential trichotomy

Let $\mathcal{P} = \{P_1, P_2, P_3\}$ be a family of three supplementary projections which is invariant for the evolution operator $U : \Delta \to \mathcal{B}(X)$.

Definition 3.1. We say that the pair $(U, \mathcal{P})$ is **uniformly exponentially trichotomic** (and denote u.e.t) if there are $N \geq 1$, $\alpha, \beta > 0$ such that

(uet_1) $e^{\alpha(t-s)}\|U(t,s)P_1(s)x\| \leq N\|P_1(s)x\|$;

(uet_2) $e^{\alpha(t-s)}\|P_2(s)x\| \leq N\|U(t,s)P_2(s)x\|$;

(uet_3) $\|U(t,s)P_3(s)x\| \leq Ne^{\beta(t-s)}\|P_3(s)x\|$;

(uet_4) $\|P_3(s)x\| \leq Ne^{\beta(t-s)}\|U(t,s)P_3(s)x\|$

for all $(t,s,x) \in \Delta \times X$.

A characterization of the u.e.t. property is given by

Proposition 3.1. *The pair $(U,\mathcal{P})$ is u.e.t. if and only if there exist $N \geq 1$, $\alpha > 0$ and $\beta > 0$ such that*

(uet_1') $e^{\alpha(t-s)}\|U(t,t_0)P_1(t_0)x_0\| \leq N\|U(s,t_0)P_1(t_0)x_0\|$;

(uet_2') $e^{\alpha(t-s)}\|U(s,t_0)P_2(t_0)x_0\| \leq N\|U(t,t_0)P_2(t_0)x_0\|$;

(uet_3') $\|U(t,t_0)P_3(t_0)x_0\| \leq Ne^{\beta(t-s)}\|U(s,t_0)P_3(t_0)x_0\|$;

(uet_4') $\|U(s,t_0)P_3(t_0)x_0\| \leq Ne^{\beta(t-s)}\|U(t,t_0)P_3(t_0)x_0\|$

for all $(t,s,t_0) \in T$ and $x_0 \in X$.

Proof. **Necessity.** If $(U,\mathcal{P})$ is u.e.t. then by Definition 3.1 there exist $N \geq 1$, $\alpha > 0$ and $\beta > 0$ such that

$$\begin{aligned}
e^{\alpha(t-s)}\|U(t,t_0)P_1(t_0)x_0\| &= e^{\alpha(t-s)}\|U(t,s)P_1(s)U(s,t_0)P_1(t_0)x_0\| \leq \\
&\leq N\|P_1(s)U(s,t_0)P_1(t_0)x_0\| = \\
&= N\|U(s,t_0)P_1(t_0)x_0\| \\
e^{\alpha(t-s)}\|U(s,t_0)P_2(t_0)x_0\| &= e^{\alpha(t-s)}\|P_2(s)U(s,t_0)P_2(t_0)x_0\| \leq \\
&\leq N\|U(t,s)U(s,t_0)P_2(t_0)x_0\| = \\
&= N\|U(t,t_0)P_2(t_0)x_0\| \\
\|U(t,t_0)P_3(t_0)x_0\| &= \|U(t,s)P_3(s)U(s,t_0)P_3(t_0)x_0\| \leq \\
&\leq Ne^{\beta(t-s)}\|P_3(s)U(s,t_0)P_3(t_0)x_0\| = \\
&= Ne^{\beta(t-s)}\|U(s,t_0)P_3(t_0)x_0\| \\
\|U(s,t_0)P_3(t_0)x_0\| &= \|P_3(s)U(s,t_0)P_3(t_0)x_0\| \leq \\
&\leq Ne^{\beta(t-s)}\|U(t,s)P_3(s)U(s,t_0)P_3(t_0)x_0\| = \\
&= Ne^{\beta(t-s)}\|U(t,t_0)P_3(t_0)x_0\|
\end{aligned}$$

for all $(t,s,t_0) \in T$ and $x_0 \in X$.
Sufficiency. It results by putting $s = t_0$. □

For the case of reversible evolution operators we have

Proposition 3.2. *Let $\mathcal{P} = \{P_1, P_2, P_3\}$ be a family of three supplementary projections which is invariant for the reversible evolution operator $U : \Delta \to \mathcal{B}(X)$. The pair $(U, \mathcal{P})$ is u.e.t. if and only if there are $N \geq 1$, $\alpha > 0$ and $\beta > 0$ such that*

(uet_1'') $e^{\alpha(t-s)}\|U(t,s)P_1(s)x\| \leq N\|P_1(s)x\|$;

(uet_2'') $e^{\alpha(t-s)}\|U(t,s)^{-1}P_2(t)x\| \leq N\|P_2(t)x\|$;

(uet_3'') $\|U(t,s)P_3(s)x\| \leq Ne^{\beta(t-s)}\|P_3(s)x\|$;

(uet_4'') $\|U(t,s)^{-1}P_3(t)x\| \leq Ne^{\beta(t-s)}\|P_3(t)x\|$

for all $(t, s, x) \in \Delta \times X$.

Proof. It is sufficient to prove the equivalences $(uet_2) \Longleftrightarrow (uet_2'')$ and $(uet_4) \Longleftrightarrow (uet_4'')$. For the equivalence $(uet_2) \Longrightarrow (uet_2'')$ we observe that

$$e^{\alpha(t-s)}\|U(t,s)^{-1}P_2(t)x\| = e^{\alpha(t-s)}\|P_2(s)U(t,s)^{-1}P_2(t)x\| \leq \\ \leq N\|U(t,s)P_2(s)U(t,s)^{-1}P_2(t)x\| = N\|P_2(t)x\|$$

for all $(t, s, x) \in \Delta \times X$.
Conversely, if (uet_2'') holds then

$$e^{\alpha(t-s)}\|P_2(s)x\| = e^{\alpha(t-s)}\|U(t,s)^{-1}P_2(t)U(t,s)P_2(s)x\| \leq \\ \leq N\|P_2(t)U(t,s)P_2(s)x\| = N\|U(t,s)P_2(s)x\|$$

for all $(t, s, x) \in \Delta \times X$.
The proof of $(uet_4) \Longleftrightarrow (uet_4'')$ is similar. □

Proposition 3.3. *Let $\mathcal{P} = \{P_1, P_2, P_3\}$ be a family of three supplementary projections which is compatible with the evolution operator $U : \Delta \to \mathcal{B}(X)$. The pair $(U, \mathcal{P})$ is u.e.t. if and only if there are $N \geq 1$, $\alpha > 0$ and $\beta > 0$ such that*

(uet_1''') $e^{\alpha(t-s)}\|U(t,s)P_1(s)x\| \leq N\|P_1(s)x\|$;

(uet_2''') $e^{\alpha(t-s)}\|V_2(t,s)P_2(t)x\| \leq N\|P_2(t)x\|$;

(uet_3''') $\|U(t,s)P_3(s)x\| \leq Ne^{\beta(t-s)}\|P_3(s)x\|$;

(uet_4''') $\|V_3(t,s)P_3(t)x\| \leq Ne^{\beta(t-s)}\|P_3(t)x\|$

for all $(t, s, x) \in \Delta \times X$.

Proof. It is sufficient to prove that $(uet_2) \Longleftrightarrow (uet_2^{'''})$ and $(uet_4) \Longleftrightarrow (uet_4^{'''})$. For $(uet_2) \Longrightarrow (uet_2^{'''})$ we observe that if (uet_2) holds then

$$\begin{aligned} e^{\alpha(t-s)}\|V_2(t,s)P_2(t)x\| &= e^{\alpha(t-s)}\|P_2(s)V_2(t,s)P_2(t)x\| \le \\ &\le N\|U(t,s)P_2(s)V_2(t,s)P_2(t)x\| = \\ &= N\|P_2(t)U(t,s)V_2(t,s)P_2(t)x\| = N\|P_2(t)x\| \end{aligned}$$

for all $(t,s,x) \in \Delta \times X$.
Similarly, by $(uet_2^{'''})$ it results that

$$e^{\alpha(t-s)}\|P_2(s)x\| = e^{\alpha(t-s)}\|V_2(t,s)U(t,s)P_2(s)x\| \le N\|U(t,s)P_2(s)x\|$$

for all $(t,s,x) \in \Delta \times X$.
The proof of the equivalence $(uet_4) \Longleftrightarrow (uet_4^{'''})$ is similar. □

4 Uniform strong exponential trichotomy

Let $\mathcal{P} = \{P_1, P_2, P_3\}$ be a family of three supplementary projections which is compatible with the evolution operator $U : \Delta \to \mathcal{B}(X)$. We denote by $V_2, V_3 : \Delta \to \mathcal{B}(X)$ the skew-evolution operators associated to the pairs (U, P_2) and (U, P_3) respectively.

Definition 4.1. We say that the pair $(U, \mathcal{P})$ is **uniformly strongly exponentially trichotomic** (and denote u.s.e.t) if there exist $N \ge 1$, $\alpha > 0$ and $\beta > 0$ such that

$(uset_1)$ $e^{\alpha(t-s)}\|U(t,s)P_1(s)x\| \le N\|x\|$;

$(uset_2)$ $e^{\alpha(t-s)}\|V_2(t,s)P_2(t)x\| \le N\|x\|$;

$(uset_3)$ $\|U(t,s)P_3(s)x\| \le Ne^{\beta(t-s)}\|x\|$;

$(uset_4)$ $\|V_3(t,s)P_3(t)x\| \le Ne^{\beta(t-s)}\|x\|$
for all $(t,s,x) \in \Delta \times X$.

Remark 4.1. The pair $(U, \mathcal{P})$ is u.s.e.t. if and only if there are $N \ge 1$, $\alpha > 0$ and $\beta > 0$ such that

$(uset_1^{'})$ $e^{\alpha(t-s)}\|U(t,s)P_1(s)\| \le N$;

$(uset_2^{'})$ $e^{\alpha(t-s)}\|V_2(t,s)P_2(t)\| \le N$;

$(uset_3^{'})$ $\|U(t,s)P_3(s)\| \le Ne^{\beta(t-s)}$;

$(uset'_4)$ $\|V_3(t,s)P_3(t)\| \leq Ne^{\beta(t-s)}$ for all $(t,s) \in \Delta$.

From here it results a necessary condition for u.s.e.t. which is given by

Remark 4.2. If $(U, \mathcal{P})$ is u.s.e.t. then the family $\mathcal{P}$ is bounded i.e. there is $N \geq 1$ such that

$$\|P_1(t)\| \leq N, \quad \|P_2(t)\| \leq N, \quad \|P_3(t)\| \leq N$$

for all $t \geq 0$.

From Definition 4.1 and Proposition 3.3 it results the connection between the concepts of u.e.t. and u.s.e.t. It is given by

Remark 4.3. If the pair $(U, \mathcal{P})$ is u.s.e.t., then it is u.e.t. The converse of the previous implication is not valid, phenomenon illustrated by

Example 4.1. Let $X = l^\infty(\mathbb{N}, \mathbb{R})$ be the Banach space of all bounded real-valued sequences endowed with the norm

$$\|x\| = \sup_{n\in\mathbb{N}} |x_n|, \qquad \text{where } x = (x_0, x_1, ...x_n, ...) \in X.$$

For every $t \in \mathbb{R}_+$ we define $P_1(t), P_2(t), P_3(t) : X \longrightarrow X$ by

$$P_1(t)x = y_1(t), \quad P_2(t)x = y_2(t), \quad P_3(t)x = y_3(t)$$

where

$$y_1(t)(n) = \begin{cases} x_n + tx_{n+1}, & n = 3k \\ 0, & n \neq 3k \end{cases}, \quad y_2(t)(n) = \begin{cases} -tx_{n+1}, & n = 3k \\ x_n, & n = 3k+1 \\ 0, & n = 3k+2 \end{cases}$$

and

$$y_3(t)(n) = \begin{cases} x_n, & n = 3k+2 \\ 0, & n \neq 3k+2 \end{cases} \qquad \text{where } x_n = x(n),\ n \in \mathbb{N}.$$

It is easy to verify that for all $t, s \geq 0$ and $x \in X$ we have

(i) $P_1(t) + P_2(t) + P_3(t) = I$;

(ii) $P_i(t)P_j(t) = \delta_{ij}P_i(t)$;

(iii) $P_1(t)P_j(s) = \begin{cases} P_1(s), & j = 1 \\ 0, & j = 2,3 \end{cases}$

(iv) $P_2(t)P_j(s) = \begin{cases} P_2(t), & j = 2 \\ 0, & j \neq 2 \end{cases}$

(v) $P_3(t)P_j(s) = P_j(s)P_3(t) = \begin{cases} P_3(t), & j = 3 \\ 0, & j \neq 3 \end{cases}$

(vi) $\|P_1(t)x\| = \sup\limits_{n \geq 0} |x_{3n} + tx_{3n+1}|$;

(vii) $\|P_2(s)x\| = \max\{1, s\} \sup\limits_{n \geq 0} |x_{3n+1}|$;

(viii) $\|P_3(t)x\| = \sup\limits_{n \geq 0} |x_{3n+2}|$;

(ix) $\|P_1(t)\| = t + 1$;

(x) $\|P_2(t)\| = \max\{1, t\}$;

(xi) $\|P_3(t)\| = 1$.

Thus $\mathcal{P} = \{P_1, P_2, P_3\}$ is a family of supplementary projections.
It is easy to see that $U : \Delta \to \mathcal{B}(X)$ defined by

$$U(t, s) = e^{2(s-t)}P_1(s) + e^{2(t-s)}P_2(t) + P_3(s).$$

It is easy to verify that U is an evolution operator with

$$U(t, s)P_1(s) = P_1(t)U(t, s) = e^{2(s-t)}P_1(s)$$

$$U(t, s)P_2(s) = P_2(t)U(t, s) = e^{2(t-s)}P_2(t)$$

$$U(t, s)P_3(s) = P_3(t)U(t, s) = P_3(s) = P_3(t)$$

for all $(t, s) \in \Delta$.
Thus the family $\mathcal{P} = \{P_1, P_2, P_3\}$ is invariant for U.
Moreover the families of projections P_2 and P_3 are strongly invariant for U with

$$V_2(t, s)P_2(t) = e^{2(s-t)}P_2(s)$$

and

$$V_3(t, s)P_3(t) = P_3(s)$$

for all $(t, s) \in \Delta$.
Thus the family $\mathcal{P} = \{P_1, P_2, P_3\}$ is compatible with U.
We observe that

$$\|U(t, s)P_1(s)x\| = e^{2(s-t)}\|P_1(s)x\| \leq e^{-(t-s)}\|P_1(s)x\|$$

$$\|V_2(t,s)P_2(t)x\| = e^{2(s-t)}\|P_2(s)x\| \le e^{-(t-s)}\|P_2(t)x\|$$

$$\|U(t,s)P_3(s)x\| = \|P_3(s)x\| \le e^{(t-s)}\|P_3(s)x\|$$

$$\|V_3(t,s)P_3(t)x\| = \|P_3(t)x\| \le e^{(t-s)}\|P_3(t)x\|$$

for all $(t,s) \in \Delta$ and all $x \in X$.
By Proposition 3.2 it is results that $(U, \mathcal{P})$ is u.e.t.
Because

$$\sup_{t\ge 0} \|P_1(t)\| = \infty$$

by Remark 4.3 it follows that $(U, \mathcal{P})$ is not u.s.e.t.

5 Uniform weak exponential trichotomy

Let $\mathcal{P} = \{P_1, P_2, P_3\}$ be a family of three supplementary projections which is compatible with an evolution operator $U : \Delta \to \mathcal{B}(X)$.

Definition 5.1. We say that the pair $(U, \mathcal{P})$ is **uniformly weakly exponentially trichotomic** (and denote u.w.e.t) if there are $N \ge 1$, $\alpha > 0$ and $\beta > 0$ such that

$(uwet_1)$ $e^{\alpha(t-s)}\|U(t,s)P_1(s)\| \le N\|P_1(s)\|$;

$(uwet_2)$ $e^{\alpha(t-s)}\|V_2(t,s)P_2(t)\| \le N\|P_2(t)\|$;

$(uwet_3)$ $\|U(t,s)P_3(s)\| \le Ne^{\beta(t-s)}\|P_3(s)\|$;

$(uwet_4)$ $\|V_3(t,s)P_3(t)\| \le Ne^{\beta(t-s)}\|P_3(t)\|$

for all $(t,s) \in \Delta$.

Remark 5.1. From Definition 5.1 and Proposition 3.3 it results that if $(U, \mathcal{P})$ is u.e.t. then it is u.w.e.t.

From Remarks 4.3 and 5.1 we obtain

Remark 5.2. The connections between the trichotomy concepts considered in this paper are

$$u.s.e.t \implies u.e.t. \implies u.w.e.t.$$

The pair $(U, \mathcal{P})$ considered in Example 4.1 is u.w.e.t. (because it is u.e.t.) but not and u.s.e.t.
We pose as an **open problem** to give an example of a pair $(U, \mathcal{P})$ which is u.w.e.t. and not u.e.t.
A connection between the trichotomy concepts considered in this paper is presented in

Proposition 5.1. *For every pair $(U, \mathcal{P})$ where $\mathcal{P}$ is compatible with U the following statements are equivalent*

(1) $(U, \mathcal{P})$ *is u.s.e.t.;*

(2) $(U, \mathcal{P})$ *is u.e.t. and $\mathcal{P}$ is bounded;*

(3) $(U, \mathcal{P})$ *is u.w.e.t. and $\mathcal{P}$ is bounded.*

Proof. The implications (1) $\Longrightarrow$ (2) $\Longrightarrow$ (3) result from Remarks 4.2, 4.3 and 5.1.
The implication (3) $\Longrightarrow$ (1) is immediate from Definition 5.1 and Remark 4.1.

$\square$

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