Some Simplicial Complexes of Universal Osborn Loops

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Abstract. A loop is shown to be a universal Osborn loop if and only if it has a particular simplicial complex. A loop is shown to be a universal Osborn loop and obeys two new identities if and only if it has another particular simplicial complex. A universal Osborn loop and four of its isotopes are shown to form a rectangular pyramid in a 3-dimensional space.

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1 Introduction and Preliminaries

A loop is called an Osborn loop if it obeys any of the two identities below.

$$\text{OS}_3 : (x \cdot yz)x = xy \cdot [(x^\lambda \cdot xz) \cdot x] \quad (1.1)$$

$$\text{OS}_5 : (x \cdot yz)x = xy \cdot [(x \cdot x^p z) \cdot x] \quad (1.2)$$

For a comprehensive introduction to Osborn loops and its universality, and a detailed literature review on it, readers should check Jaiyéolá, Adéníran and Sòlárìn [3] and Jaiyéolá [4]. In this present paper, we shall follow the style and notations used in Jaiyéolá, Adéníran and Sòlárìn [3] and Jaiyéolá [4]. The only concepts and notions which will be introduced here are those that were not defined in Jaiyéolá, Adéníran and Sòlárìn [3] and Jaiyéolá [4].

Definition 1.1. Let $(L, \cdot)$ be a loop and $U, V, W \in \text{SYM}(L, \cdot)$.

1. If $(U, V, W) \in \text{AUT}(L, \cdot)$ for some V, W , then U is called *autotopic*.
2. If $(U, V, W) \in \text{AUT}(L, \cdot)$ such that $W = U, V = I$, then U is called *λ -regular*.
3. If $(U, V, W) \in \text{AUT}(L, \cdot)$ such that $U = I, W = V$, then V is called *ρ -regular*.

Stein [5] and Drisko [2] while considering the action of isotopisms and autotopisms of loops, found it convenient to think of a loop $\mathcal{Q} = (Q, \cdot, \backslash, /)$ in terms of the set $T_{\mathcal{Q}}$ of all ordered triples (x, y, z) of elements of Q such that $x \cdot y = z$. An isotopism (α, β, γ) from G to H takes $(x, y, z) \in T_G$ to $(x\alpha, y\beta, z\gamma) \in T_H$. We shall adopt his conventions at some points in time. We shall denote by $[\alpha, \beta]$, the commutator of any $\alpha, \beta \in \text{SYM}(G, \cdot)$. Let $(Q, \cdot, \backslash, /)$ be a loop, then we shall be making use of the following notations for principal isotopes of $(Q, \cdot)$.

- $(Q, *_0)$ represents $Q_{x,v}$;
- $(Q, \circ_0)$ represents $Q_{u, \phi_0(x, u, v)}$, $\phi_0(x, u, v) = (u \backslash [(uv)/(u \backslash (xv))]v)$;
- $(Q, \circ_1)$ represents $Q_{u, [u \backslash (xv)]}$;
- $(Q, *_1)$ represents $Q_{\phi_1(x, u, v), v}$, $\phi_1(x, u, v) = (u \backslash [(uv)/(u \backslash (xv))]v)$ for all $x, u, v \in Q$;
- $(Q, \circ_2)$ represents $Q_{x, \phi_2(x, u, v)}$, $\phi_2(x, u, v) = (u \backslash [(u/v)(u \backslash (xv))])$;
- $(Q, \circ_3)$ represents $Q_{[x \cdot u \backslash v]/v, [u \backslash (xv)]}$;
- $(Q, *_2)$ represents $Q_{u, e}$;
- $(Q, *_3)$ represents $Q_{e, v}$.

Let $(G, \cdot)$ be a loop and let

$$BS_2(G, \cdot) = \{\theta \in \text{SYM}(G) : G(a, b) \stackrel{\theta}{\cong} G(c, d) \text{ for some } a, b, c, d \in G\}.$$

As shown in Bryant and Schneider [1], $BS_2(G, \cdot)$ forms a group for a loop $(G, \cdot)$ and it shall be called the second Bryant-Schneider group (2nd BSG) of the loop.

Consider the following two notions in algebraic topology.

Definition 1.2. Let V_Q be a set of isotopes of a loop $(Q, \cdot)$ and let $S_Q \subseteq \mathcal{I}^{V_Q}$ such that $\phi \in S_Q$. If S_Q is a topology on V_Q , then it is called the topology of isotopes of the loop Q and the pair (V_Q, S_Q) is called a topological space of isotopes of Q if (V_Q, S_Q) is a topological space.

Based on the above notion of topological space of isotopes of a loop, the following facts are direct consequences.

Lemma 1.1. Let $(Q, \cdot)$ be a loop and let V_Q be the set of isotopes of Q . Then, $(V_Q, \mathcal{I}^{V_Q})$ is a topological space of isotopes of Q .

Lemma 1.2. Let $(Q, \cdot)$ be a G -loop and let V_Q be the set of isotopes of Q . Let $S_Q = \{X_i\}_{i \in \Omega} \subseteq \mathcal{I}^{V_Q}$ such that $\phi \in S_Q$ and $x_{i_j} \cong x_{i_k}$ for all $x_{i_j}, x_{i_k} \in X_i$. Then, (V_Q, S_Q) is a topological space of isotopes of Q .

Corollary 1.3. Let $(Q, \cdot)$ be a CC-loop or VD-loop or K -loop or Buchsteiner loop or extra loop or group. Let $S_Q = \{X_i\}_{i \in \Omega} \subseteq \mathcal{I}^{V_Q}$ such that $\phi \in S_Q$ and $x_{i_j} \cong x_{i_k}$ for all $x_{i_j}, x_{i_k} \in X_i$. Then, (V_Q, S_Q) is a topological space of isotopes of Q .

Definition 1.3. A simplicial complex is a pair (V, S) where V is a set of points called vertices and S is a given family of finite subsets, called simplexes, so that the following conditions are satisfied:

1. all points of V are simplexes;
2. any non-empty subset of a simplex is a simplex.

A simplex consisting of $(n + 1)$ points is called n -dimensional simplex.

Definition 1.4. Let V_Q be a set of isotopes of a loop $(Q, \cdot)$ and let $S_Q \subseteq \mathcal{I}^{V_Q}$. If $K_Q = (V_Q, S_Q)$ is a simplicial complex, then K_Q is called a trivial simplicial complex of isotopes of the loop Q .

Definition 1.5. Let V_Q be a set of isotopes of a loop $(Q, \cdot)$ and let $S_Q = \{X_i\}_{i \in \Omega} \subseteq \mathcal{I}^{V_Q}$ such that $x_{i_j} \cong x_{i_k}$ for all $x_{i_j}, x_{i_k} \in X_i$. If $K_Q = (V_Q, S_Q)$ is a simplicial complex, then K_Q is called a non-trivial simplicial complex of isotopes or simplicial complex of isotopes of the loop Q .

The facts below follow suite.

Lemma 1.4. Let $(Q, \cdot)$ be a loop and let V_Q be the set of isotopes of Q . Then, $(V_Q, \mathcal{I}^{V_Q})$ is a trivial simplicial complex of isotopes of Q .

Lemma 1.5. Let $(Q, \cdot)$ be a G -loop and let V_Q be the set of isotopes of Q . Let $S_Q = \{X_i\}_{i \in \Omega} \subseteq \mathcal{Z}^{V_Q}$ such that $x_{i_j} \cong x_{i_k}$ for all $x_{i_j}, x_{i_k} \in X_i$. Then, (V_Q, S_Q) is a simplicial complex of isotopes of Q .

Corollary 1.6. Let $(Q, \cdot)$ be a CC-loop or VD-loop or K -loop or Buchsteiner loop or extra loop or group. Let $S_Q = \{X_i\}_{i \in \Omega} \subseteq \mathcal{Z}^{V_Q}$ such that $x_{i_j} \cong x_{i_k}$ for all $x_{i_j}, x_{i_k} \in X_i$. Then, (V_Q, S_Q) is a simplicial complex of isotopes of Q .

Definition 1.6. Let $K = (V, S)$ and $K' = (V', S')$ be two simplicial complexes. A simplicial map $f : K \rightarrow K'$ is a set map $f : V \rightarrow V'$ satisfying the property: for every simplex $x \in S$, the image $f(x) \in S'$.

In this work, the notion of simplicial complex is used to characterize universal Osborn loops. The following results are important for the set objective.

Theorem 1.7. (Jaíyéolá, Adéníran and Sòlárín [3])

Let $\mathcal{Q} = (Q, \cdot, \backslash, /)$ be a loop and $\gamma_0(x, u, v) = \mathbb{R}_v R_{[u \backslash (xv)]} \mathbb{L}_u L_x$ for all $x, u, v \in Q$, then $\mathcal{Q}$ is a universal Osborn loop if and only if the commutative diagram

$$\begin{array}{ccc}
 & & (Q, \circ_0) \\
 & \nearrow (R_{\phi_0(x, u, v)}, L_u, I) & \downarrow (\gamma_0, \gamma_0, \gamma_0) \text{ isomorphism} \\
 (Q, \cdot) & \xrightarrow[\text{principal isotopism}]{(R_v, L_x, I)} & (Q, *_0)
 \end{array} \quad (1.3)$$

holds.

Theorem 1.8. (Jaíyéolá [4])

Let $\mathcal{Q} = (Q, \cdot, \backslash, /)$ be a loop and $\gamma_1(x, u, v) = \mathbb{R}_v R_{[u \backslash (xv)]} \mathbb{L}_u L_x$ for all $x, u, v \in Q$, then $\mathcal{Q}$ is a universal Osborn loop if and only if the commutative diagram

$$\begin{array}{ccc}
 & & (Q, *_1) \\
 & \nearrow (R_v, L_{\phi_1(x, u, v)}, I) & \downarrow (\gamma_1, \gamma_1, \gamma_1) \text{ isomorphism} \\
 (Q, \cdot) & \xrightarrow[\text{principal isotopism}]{(R_{[u \backslash (xv)]}, L_u, I)} & (Q, \circ_1)
 \end{array} \quad (1.4)$$

holds.

Theorem 1.9. (Jaíyéolá, Adéníran and Sòlárín [3])

Let $\mathcal{Q} = (Q, \cdot, \backslash, /)$ be a loop and $\gamma_0(x, u, v) = \mathbb{R}_v R_{[u \backslash (xv)]} \mathbb{L}_u L_x$ for all $x, u, v \in Q$, then $\mathcal{Q}$ is a universal Osborn loop implies the commutative diagram

$$\begin{array}{ccc}
 & & (Q, \circ_2) \\
 & \nearrow (R_{\phi_2(x, u, v)}, L_x, I) & \uparrow (\gamma_0, \gamma_0, \gamma_0) \text{ isomorphism} \\
 (Q, \cdot) & \xrightarrow[\text{principal isotopism}]{(I, L_u, I)} & (Q, *_2)
 \end{array} \quad (1.5)$$

holds.

Theorem 1.10. (*Jaíyéolá [4]*)

Let $\mathcal{Q} = (Q, \cdot, \backslash, /)$ be a loop and $\gamma_1(x, u, v) = \mathbb{R}_v R_{[u \backslash (xv)]} \mathbb{L}_u L_x$ for all $x, u, v \in Q$, then $\mathcal{Q}$ is a universal Osborn loop implies the commutative diagram

$$\begin{array}{ccc}
 & & (Q, \circ_3) \\
 & \nearrow^{(R_{[u \backslash (xv)]}, L_{[x \cdot u \backslash v]}, I)} & \uparrow \text{isomorphism} \\
 (Q, \cdot) & \xrightarrow[(\text{principal isotopism})]{(R_v, I, I)} & (Q, *_3)
 \end{array} \tag{1.6}$$

holds.

Lemma 1.11. (*Drisko [2]*)

Let $\mathcal{Q} = (Q, \cdot, \backslash, /)$ be a loop. Then $Q_{f,g} \cong Q_{c,d}$ if and only if there exists $(\alpha, \beta, \gamma) \in \text{AUT}(\mathcal{Q})$ such that $(f, g, fg)(\alpha, \beta, \gamma) = (c, d, cd)$.

Theorem 1.12. (*Bryant and Schneider [1]*)

Let $(Q, \cdot, \backslash, /)$ be a quasigroup. If $Q_{a,b} \stackrel{I}{\cong} Q_{c,d}$ if and only if $c \cdot b, a \cdot d \in N_\mu(Q_{a,b})$ and $a \cdot b = c \cdot d$.

2 Main Results

Theorem 2.1. Let $\mathcal{Q} = (Q, \cdot, \backslash, /)$ be a universal Osborn loop. Then, the following are necessary and sufficient for each other.

1. $(Q, \circ_0) \stackrel{I}{\cong} (Q, \circ_1)$.
2. $(Q, *_0) \stackrel{I}{\cong} (Q, *_1)$.
3. $\mathcal{Q}$ is a boolean group.

Proof. By combining the commutative diagrams in Equation 1.3 and Equa-

tion 1.4, we have the commutative diagram below.

$$\begin{array}{ccccc}
 (Q, \circ_1) & & (Q, \circ_1) & & (Q, \circ_1) \\
 \uparrow & & \uparrow \gamma_{01}^\circ & & \uparrow \\
 & & (Q, \circ_0) & \xrightarrow{\gamma_0} & \\
 \nearrow (R_{\phi_0}, L_u, I) & & & & \\
 (Q, \cdot) & \xrightarrow{(R_v, L_x, I)} & (Q, *_0) & \xrightarrow{\gamma_{01}^*} & (Q, *_1) \\
 \downarrow \omega & & & & \downarrow \gamma_1 \\
 (Q, *_1) & & & & (Q, *_1)
 \end{array} \tag{2.1}$$

where $\chi = (R_{[u \setminus (xv)]}, L_u, I)$ and $\omega = (R_v, L_{\phi_1}, I)$. Let

$$(Q, \circ_0) \xrightarrow[\text{isotopism}]{(\delta_{01}^\circ, \varepsilon_{01}^\circ, \pi_{01}^\circ)} (Q, \circ_1).$$

So, from Equation 2.1,

$$\begin{aligned}
 (R_{\phi_0(x,u,v)}, L_u, I)(\delta_{01}^\circ, \varepsilon_{01}^\circ, \pi_{01}^\circ) &= (R_{[u \setminus (xv)]}, L_u, I) \Rightarrow \\
 (R_{\phi_0(x,u,v)} \delta_{01}^\circ, L_u \varepsilon_{01}^\circ, \pi_{01}^\circ) &= (R_{[u \setminus (xv)]}, L_u, I) \Leftrightarrow \\
 R_{\phi_0(x,u,v)} \delta_{01}^\circ &= R_{[u \setminus (xv)]}, \quad L_u \varepsilon_{01}^\circ = L_u \quad \text{and} \quad \pi_{01}^\circ = I \Leftrightarrow \\
 \delta_{01}^\circ &= R_{\phi_0(x,u,v)}^{-1} R_{[u \setminus (xv)]}, \quad \varepsilon_{01}^\circ = L_u^{-1} L_u = I \quad \text{and} \quad \pi_{01}^\circ = I.
 \end{aligned}$$

Thus, $(Q, \circ_0) \cong (Q, \circ_1)$ iff $\delta_{01}^\circ = \varepsilon_{01}^\circ = I$ iff

$$\begin{aligned}
 R_{\phi_0(x,u,v)}^{-1} R_{[u \setminus (xv)]} &= I \Leftrightarrow \phi_0(x, u, v) = [u \setminus (xv)] \\
 (u \setminus ([uv]/(u \setminus (xv)))v) &= [u \setminus (xv)] \Leftrightarrow x \setminus (uv) = u \setminus (xv).
 \end{aligned}$$

Similarly, by using the procedure above, it can be shown that $(Q, *_0) \cong (Q, *_1)$ iff $x \setminus (uv) = u \setminus (xv)$.

Keeping in mind that every Osborn loop of exponent 2 is an abelian group, hence, a Boolean group. This completes the proof. $\square$

Remark 2.1. It can be observed that in a universal Osborn loop $\mathcal{Q} = (Q, \cdot, \setminus, /)$ and for $\gamma_0(x, u, v)$ and $\gamma_1(x, u, v)$ of Theorem 1.7 and Theorem 1.8,

$\gamma_0(x, u, v) = \gamma_1(x, u, v)$ if and only if $[\mathbb{L}_u L_x, \mathbb{R}_v R_{[u \setminus (xv)]}] = I$ for all $x, u, v \in Q$.

The proof of Theorem 2.1 can also be achieved by making use of Theorem 1.12. Take $a = u$, $b = \phi_0(x, u, v)$, $c = u$ and $d = u \setminus (xv)$. Then, $(Q, \circ_0) \stackrel{I}{\cong} (Q, \circ_1)$ iff

1. $u\phi_0(x, u, v) \in N_\mu((Q, \circ_0))$,
2. $u[u \setminus (xv)] \in N_\mu((Q, \circ_0))$,
3. $u\phi_0(x, u, v) = u[u \setminus (xv)] \Leftrightarrow Q$ is a Boolean group.

Theorem 2.2. *Let $Q = (Q, \cdot, \setminus, /)$ be a universal Osborn loop. Then $(Q, \circ_0) \cong (Q, \circ_1)$ if and only if there exists $(I, \beta, \gamma) \in AUT(Q)$ such that*

$$uv = xR_v \mathbb{L}_u \beta^{-1} L_u \mathbb{R}_v \cdot xR_v \mathbb{L}_u = xR_v \gamma^{-1} \mathbb{R}_v \cdot xR_v \mathbb{L}_u \quad (2.2)$$

for all $x, u, v \in Q$.

Proof. Following Lemma 1.11, $(Q, \circ_0) \cong (Q, \circ_1)$ if and only if there exists $(\alpha, \beta, \gamma) \in AUT(Q)$ such that

$$\begin{aligned} (u, \phi_0(x, u, v), u\phi_0(x, u, v))(\alpha, \beta, \gamma) &= (u, [u \setminus (xv)], xv) \Leftrightarrow \\ (u\alpha, \phi_0(x, u, v)\beta, (u\phi_0(x, u, v))\gamma) &= (u, [u \setminus (xv)], xv) \Leftrightarrow \\ u\alpha = u, \phi_0(x, u, v)\beta &= [u \setminus (xv)] \text{ and } (u\phi_0(x, u, v))\gamma = xv \Leftrightarrow \\ \alpha = I, \{u \setminus ([(uv)/(u \setminus (xv))] v)\} \beta &= u \setminus (xv) \text{ and } \{[(uv)/(u \setminus (xv))] v\} \gamma = xv \Leftrightarrow \\ \alpha = I, [(uv)/(u \setminus (xv))] R_v \mathbb{L}_u \beta &= xR_v \mathbb{L}_u \text{ and } [(uv)/(u \setminus (xv))] R_v \gamma = xR_v \Leftrightarrow \\ \alpha = I, (uv)/(u \setminus (xv)) &= xR_v \mathbb{L}_u \beta^{-1} L_u \mathbb{R}_v \text{ and } [(uv)/(u \setminus (xv))] = xR_v \gamma^{-1} \mathbb{R}_v \\ \Leftrightarrow \alpha = I, uv &= xR_v \mathbb{L}_u \beta^{-1} L_u \mathbb{R}_v \cdot xR_v \mathbb{L}_u \text{ and } uv = xR_v \gamma^{-1} \mathbb{R}_v \cdot xR_v \mathbb{L}_u \Leftrightarrow \end{aligned}$$

there exists $(I, \beta, \gamma) \in AUT(Q)$ such that

$$uv = xR_v \mathbb{L}_u \beta^{-1} L_u \mathbb{R}_v \cdot xR_v \mathbb{L}_u = xR_v \gamma^{-1} \mathbb{R}_v \cdot xR_v \mathbb{L}_u.$$

□

Remark 2.2. If the autotopism (α, β, γ) in Theorem 2.2 is the identity autotopism, then we shall have the equivalence of 1. and 3. of Theorem 2.1.

Corollary 2.3. *Let $Q = (Q, \cdot, \setminus, /)$ be a universal Osborn loop. Then $(Q, \circ_0) \cong (Q, \circ_1)$ implies that there exists $(I, \beta, \gamma) \in AUT(Q)$ such that $\gamma = \mathbb{L}_u \beta L_u$ for all $u \in Q$. Hence,*

1. $\gamma = \beta$ iff $[\beta, L_u] = I$ or $[\gamma, L_u] = I$. Thence, β is a ρ -regular permutation.

2. $\gamma = L_u$ iff $\beta = L_u$. Thence, $\mathcal{Q}$ is an abelian group.

Proof. The proof of these follows from the fact in Theorem 2.2 that

$$xR_v\mathbb{L}_u\beta^{-1}L_u\mathbb{R}_v \cdot xR_v\mathbb{L}_u = xR_v\gamma^{-1}\mathbb{R}_v \cdot xR_v\mathbb{L}_u \Rightarrow$$

$$\mathbb{L}_u\beta L_u = \gamma \text{ for all } u \in Q. \quad \square$$

Theorem 2.4. Let $\mathcal{Q} = (Q, \cdot, \backslash, /)$ be a universal Osborn loop. Then $(Q, *_0) \cong (Q, *_1)$ if and only if there exists $(\delta, I, \pi) \in AUT(\mathcal{Q})$ such that

$$uv = x \cdot x\delta R_v\mathbb{L}_u = x \cdot xR_v\pi\mathbb{L}_u \quad (2.3)$$

for all $x, u, v \in Q$.

Proof. Following Lemma 1.11, $(Q, *_0) \cong (Q, *_1)$ if and only if there exists $(\delta, \varepsilon, \pi) \in AUT(\mathcal{Q})$ such that $(x, v, xv)(\delta, \varepsilon, \pi) = (\phi_1(x, u, v), v, \phi_1(x, u, v)v)$. The procedure of the proof of the remaining part is similar to that of Theorem 2.2. $\square$

Remark 2.3. If the autotopism $(\delta, \varepsilon, \pi)$ in Theorem 2.4 is the identity autotopism, then we shall have the equivalence of 2. and 3. of Theorem 2.1.

Corollary 2.5. Let $\mathcal{Q} = (Q, \cdot, \backslash, /)$ be a universal Osborn loop. Then $(Q, *_0) \cong (Q, *_1)$ implies that there exists $(\delta, I, \pi) \in AUT(\mathcal{Q})$ such that $\pi = \mathbb{R}_v\delta R_v$ for all $v \in Q$. Hence,

1. $\pi = \delta$ iff $[\delta, R_v] = I$ or $[\pi, R_v] = I$. Thence, δ is a λ -regular permutation.
2. $\delta = R_v$ iff $\pi = R_v$. Thence, $\mathcal{Q}$ is an abelian group.

Proof. The proof of these follows from the fact in Theorem 2.4 that

$$x \cdot x\delta R_v\mathbb{L}_u = x \cdot xR_v\pi\mathbb{L}_u \Rightarrow$$

$$\pi = \mathbb{R}_v\delta R_v \text{ for all } v \in Q. \quad \square$$

Theorem 2.6. Let $\mathcal{Q} = (Q, \cdot, \backslash, /)$ be a universal Osborn loop. Then $(Q, \circ_0) \cong (Q, \circ_1)$ and $(Q, *_0) \cong (Q, *_1)$ if and only if there exists $(I, \beta, \gamma), (\delta, I, \pi) \in AUT(\mathcal{Q})$ such that

$$\begin{aligned} uv &= xR_v\mathbb{L}_u\beta^{-1}L_u\mathbb{R}_v \cdot xR_v\mathbb{L}_u = \\ xR_v\gamma^{-1}\mathbb{R}_v \cdot xR_v\mathbb{L}_u &= x \cdot x\delta R_v\mathbb{L}_u = x \cdot xR_v\pi\mathbb{L}_u \end{aligned} \quad (2.4)$$

for all $x, u, v \in Q$

Proof. This is achieved by simply combining Theorem 2.2 and Theorem 2.4. $\square$

Theorem 2.7. Let $\mathcal{Q} = (Q, \cdot, \backslash, /)$ be a universal Osborn loop. If $(Q, \circ_0) \stackrel{\gamma_{01}^\circ}{\cong} (Q, \circ_1)$ and $(Q, *_0) \stackrel{\gamma_{01}^*}{\cong} (Q, *_1)$, then $\gamma_0 \gamma_{01}^* \gamma_1 = \gamma_{01}^\circ$.

Proof. The commutative diagram in Equation 2.1 proves this. $\square$

Corollary 2.8. Let $\mathcal{Q} = (Q, \cdot, \backslash, /)$ be a universal Osborn loop. If $(Q, \circ_0) \cong (Q, \circ_1)$ and $(Q, *_0) \cong (Q, *_1)$, then the following are necessary and sufficient for each other.

1. $\beta = I$.	4. $\pi = I$.	6. $(Q, *_0) \stackrel{I}{\cong} (Q, *_1)$.
2. $\gamma = I$.		7. $\mathcal{Q}$ is a boolean group.
3. $\delta = I$.	5. $(Q, \circ_0) \stackrel{I}{\cong} (Q, \circ_1)$.	

Proof. To prove the equivalence of 1. to 4. and 7., use Equation 2.4 of Theorem 2.6. The proof of the equivalence of 5. to 7. follows from Theorem 2.1. $\square$

Remark 2.4. Corollary 2.8 is a very important result in this study. It gives us the main distinctions between Theorem 2.1 and Theorem 2.6. That is, the necessary and sufficient condition(s) under which the isomorphisms $(Q, \circ_0) \cong (Q, \circ_1)$ and $(Q, *_0) \cong (Q, *_1)$ will be trivial. And the condition(s) is when any of the autotopic permutations of β , γ , δ and π of Theorem 2.2 and Theorem 2.4 is equal to the identity mapping.

Next, it is important to deduce the actual definitions of the autotopic mappings β , γ , δ , π and the isomorphisms γ_{01}^* and γ_{01}° . Recall that by the necessary part of Lemma 1.11, if $\mathcal{Q} = (Q, \cdot, \backslash, /)$ is a loop and $Q_{f,g} \stackrel{\theta}{\cong} Q_{c,d}$, then there exists $(A, B, C) \in AUT(\mathcal{Q})$ such that $(f, g, fg)(A, B, C) = (c, d, cd)$. According to the proof of this,

$$\begin{aligned} (A, B, C) &= (R_g \theta R_d^{-1}, L_f \theta L_c^{-1}, \theta) \Leftrightarrow \\ A &= R_g \theta R_d^{-1}, B = L_f \theta L_c^{-1} \text{ and } C = \theta. \end{aligned} \quad (2.5)$$

Thus,

$$\begin{aligned} I &= \alpha = R_{\phi_o(x,u,v)} \gamma_{01}^\circ R_{[u \setminus (xv)]}^{-1}, \beta = L_u \gamma_{01}^\circ L_u^{-1} \text{ and } \gamma = \gamma_{01}^\circ \\ \gamma_{01}^\circ &= \mathbb{R}_{\phi_o(x,u,v)} R_{[u \setminus (xv)]}, \beta = L_u \mathbb{R}_{\phi_o(x,u,v)} R_{[u \setminus (xv)]} \mathbb{L}_u^{-1} \text{ and} \\ \gamma &= \mathbb{R}_{\phi_o(x,u,v)} R_{[u \setminus (xv)]} \end{aligned}$$

and

$$\begin{aligned}\delta &= R_v \gamma_{01}^* R_v^{-1}, \quad I = \varepsilon = L_x \gamma_{01}^* L_{\phi_1(x,u,v)}^{-1} \text{ and } \pi = \gamma_{01}^* \\ \delta &= R_v \gamma_{01}^* \mathbb{R}_v^{-1}, \quad \gamma_{01}^* = \mathbb{L}_x L_{\phi_1(x,u,v)} \text{ and } \pi = \gamma_{01}^* \\ \delta &= R_v \mathbb{L}_x L_{\phi_1(x,u,v)} \mathbb{R}_v^{-1}, \quad \gamma_{01}^* = \mathbb{L}_x L_{\phi_1(x,u,v)} \text{ and } \pi = \mathbb{L}_x L_{\phi_1(x,u,v)}.\end{aligned}$$

Therefore, Theorem 2.2 and Theorem 2.4 can now be restated as follows.

Theorem 2.9. *Let $\mathcal{Q} = (Q, \cdot, \backslash, /)$ be a universal Osborn loop. Then $(Q, \circ_0) \stackrel{\gamma_{01}^\circ}{\cong} (Q, \circ_1)$ if and only if*

$$y \cdot u \backslash [(uz)\psi_0] = (yz)\psi_0 \text{ and } uv = xR_v(R_v\psi_0)^{-1} \cdot xR_v\mathbb{L}_u \quad (2.6)$$

where $\psi_0 = \mathbb{R}_{\phi_o(x,u,v)} R_{[u \backslash (xv)]}$ for all $x, y, z, u, v \in Q$

Proof. Simply substitute

$$\beta = L_u \mathbb{R}_{\phi_o(x,u,v)} R_{[u \backslash (xv)]} \mathbb{L}_u^{-1} \text{ and } \gamma = \mathbb{R}_{\phi_o(x,u,v)} R_{[u \backslash (xv)]}$$

into Equation 2.2 of Theorem 2.2. $\square$

Theorem 2.10. *Let $\mathcal{Q} = (Q, \cdot, \backslash, /)$ be a universal Osborn loop. Then $(Q, *_0) \stackrel{\gamma_{01}^*}{\cong} (Q, *_1)$ if and only if*

$$[(yv)\psi_1]/v \cdot z = (yz)\psi_1 \text{ and } uv = x \cdot u \backslash [(xv)\psi_1] \quad (2.7)$$

where $\psi_1 = \mathbb{L}_x L_{\phi_1(x,u,v)}$ for all $x, y, z, u, v \in Q$

Proof. Simply substitute

$$\delta = R_v \mathbb{L}_x L_{\phi_1(x,u,v)} \mathbb{R}_v^{-1} \text{ and } \pi = \mathbb{L}_x L_{\phi_1(x,u,v)}$$

into Equation 2.3 of Theorem 2.4. $\square$

Lemma 2.11. *Let $\mathcal{Q} = (Q, \cdot, \backslash, /)$ be a loop.*

1. *$\mathcal{Q}$ is a universal Osborn loop and obeys Equation 2.6 if and only if $\gamma_0, \gamma_{01}^\circ \in BS_2(\mathcal{Q})$.*
2. *$\mathcal{Q}$ is a universal Osborn loop and obeys Equation 2.7 if and only if $\gamma_1, \gamma_{01}^* \in BS_2(\mathcal{Q})$.*

Proof. This follows by combining Theorem 1.7, Theorem 1.8, Theorem 2.2 and Theorem 2.4 $\square$

Remark 2.5. It is a self exercise to confirm if $(Q, \circ_0) \stackrel{\gamma_{01}^\circ}{\cong} (Q, \circ_1)$ and $(Q, *_0) \stackrel{\gamma_{01}^*}{\cong} (Q, *_1)$ in some universal Osborn loops like Moufang loops and extra loops by simply verifying Equation 2.6 and Equation 2.7. Furthermore, the relation $\gamma_0 \gamma_{01}^* \gamma_1 = \gamma_{01}^\circ$ of Theorem 2.7 is justifiable as well. It must be noted also, that in any universal Osborn loop $\mathcal{Q}$, Equation 2.6 and Equation 2.7 are necessary and sufficient conditions for $\gamma_{01}^*, \gamma_{01}^\circ \in BS_2(\mathcal{Q})$.

By combining the commutative diagrams in Equation 1.5 and Equation 1.6, we have the commutative diagram below.

$$\begin{array}{ccccc}
 (Q, \circ_3) & & (Q, \circ_3) & & (Q, \circ_3) \\
 \uparrow & & \uparrow \gamma_{23}^\circ & & \uparrow \\
 & & (Q, \circ_2) & & \\
 (R_a, L_b, I) \nearrow (R_{\phi_2}, L_x, I) & & \searrow \gamma_0 & & \\
 (Q, \cdot) & \xrightarrow{(I, L_u, I)} & (Q, *_2) & & \\
 \downarrow (R_v, I, I) & & \searrow \gamma_{23}^* & & \downarrow \gamma_1 \\
 (Q, *_3) & & & & (Q, *_3)
 \end{array} \tag{2.8}$$

where $a = [u \setminus (xv)]$ and $b = \{[x \cdot u \setminus v]/v\}$.

Theorem 2.12. *Let $\mathcal{Q} = (Q, \cdot, \setminus, /)$ be a universal Osborn loop. Then $(Q, \circ_2) \cong (Q, \circ_3)$ if and only if there exists $(\lambda, \mu, \nu) \in \text{AUT}(\mathcal{Q})$ such that*

$$\lambda = R_{u \setminus v} \mathbb{R}_v, \quad \mu = L_u \mathbb{L}_{u \setminus v} \text{ and } [x \cdot x R_v \mathbb{L}_u \mu^{-1}] \nu = x \lambda \cdot x R_v \mathbb{L}_u \tag{2.9}$$

for all $x, u, v \in Q$.

Proof. Following Lemma 1.11, $(Q, \circ_2) \cong (Q, \circ_3)$ if and only if there exists $(\lambda, \mu, \nu) \in \text{AUT}(\mathcal{Q})$ such that $(x, \phi_2(x, u, v), x \phi_2(x, u, v))(\lambda, \mu, \nu) = ([x \cdot u \setminus v]/v, [u \setminus (xv)], \{[x \cdot u \setminus v]/v\}[u \setminus (xv)])$. The procedure of the proof of the remaining part is similar to that of Theorem 2.2. $\square$

Lemma 2.13. *Let $\mathcal{Q} = (Q, \cdot, \setminus, /)$ be a universal Osborn loop. Then $(Q, \circ_2) \stackrel{\gamma_{23}^\circ}{\cong} (Q, \circ_3)$ if and only if there exists $(\lambda, \mu, \gamma_{23}^\circ) \in \text{AUT}(\mathcal{Q})$ such that*

$$\begin{aligned}
 \gamma_{23}^\circ &= \mathbb{R}_{\phi_2(x, u, v)} R_{u \setminus v} \mathbb{R}_v R_{[u \setminus (xv)]} = \mathbb{L}_x L_u \mathbb{L}_{u \setminus v} L_{\{[x \cdot u \setminus v]/v\}} \text{ and} \\
 [x \cdot x R_v \mathbb{L}_u \mu^{-1}] \gamma_{23}^\circ &= x \lambda \cdot x R_v \mathbb{L}_u
 \end{aligned} \tag{2.10}$$

for all $x, u, v \in Q$.

Proof. Considering the commutative diagram in Equation 2.8 and using Equation 2.5,

$$\lambda = R_{\phi_2(x, u, v)} \gamma_{23}^\circ R_{[u \setminus (xv)]}^{-1}, \quad \mu = L_x \gamma_{23}^\circ L_{\{[x \cdot u \setminus v]/v\}}^{-1} \text{ and } \nu = \gamma_{23}^\circ.$$

The final conclusion follows from Theorem 2.12. $\square$

Corollary 2.14. *Let $\mathcal{Q} = (Q, \cdot, \backslash, /)$ be a universal Osborn loop. $\gamma_{23}^\circ \in BS_2(\mathcal{Q})$ if and only if there exists $(\lambda, \mu, \gamma_{23}^\circ) \in AUT(\mathcal{Q})$ such that*

$$\begin{aligned} \gamma_{23}^\circ &= \mathbb{R}_{\phi_2(x,u,v)} R_{u \backslash v} \mathbb{R}_v R_{[u \backslash (xv)]} = \mathbb{L}_x L_u \mathbb{L}_{u \backslash v} L_{\{[x \cdot u \backslash v]/v\}} \text{ and} \\ [x \cdot x R_v \mathbb{L}_u \mu^{-1}] \gamma_{23}^\circ &= x \lambda \cdot x R_v \mathbb{L}_u \end{aligned} \quad (2.11)$$

for all $x, u, v \in Q$.

Proof. This follows from Lemma 2.13. $\square$

Corollary 2.15. *Let $\mathcal{Q} = (Q, \cdot, \backslash, /)$ be a loop. $\mathcal{Q}$ is a universal Osborn loop and $\gamma_{23}^\circ \in BS_2(\mathcal{Q})$ implies $\gamma_0 \in BS_2(\mathcal{Q})$ and there exists $(\lambda, \mu, \gamma_{23}^\circ) \in AUT(\mathcal{Q})$ such that*

$$\begin{aligned} \gamma_{23}^\circ &= \mathbb{R}_{\phi_2(x,u,v)} R_{u \backslash v} \mathbb{R}_v R_{[u \backslash (xv)]} = \mathbb{L}_x L_u \mathbb{L}_{u \backslash v} L_{\{[x \cdot u \backslash v]/v\}} \text{ and} \\ [x \cdot x R_v \mathbb{L}_u \mu^{-1}] \gamma_{23}^\circ &= x \lambda \cdot x R_v \mathbb{L}_u \end{aligned} \quad (2.12)$$

for all $x, u, v \in Q$.

Proof. This follows from Theorem 1.9 and Lemma 2.13. $\square$

Simplicial Complex of Isotopes of a Universal Osborn Loop

Theorem 2.16. *Let $(Q, \cdot)$ be a loop. Let $V_0(Q) = \{(Q, \cdot), (Q, \circ_0), (Q, *_0)\}$ and $S_0(Q) = \{\{(Q, \cdot)\}, \{(Q, \circ_0)\}, \{(Q, *_0)\}, \{(Q, \circ_0), (Q, *_0)\}\}$. Then, $(Q, \cdot)$ is a universal Osborn loop if and only if $K_0(Q) = (V_0(Q), S_0(Q))$ is a simplicial complex of isotopes of $(Q, \cdot)$.*

Proof. This is proved with the help of Theorem 1.7. $\square$

Theorem 2.17. *Let $(Q, \cdot)$ be a loop. Let $V_1(Q) = \{(Q, \cdot), (Q, \circ_1), (Q, *_1)\}$ and $S_1(Q) = \{\{(Q, \cdot)\}, \{(Q, \circ_1)\}, \{(Q, *_1)\}, \{(Q, \circ_1), (Q, *_1)\}\}$. Then, $(Q, \cdot)$ is a universal Osborn loop if and only if $K_1(Q) = (V_1(Q), S_1(Q))$ is a simplicial complex of isotopes of $(Q, \cdot)$.*

Proof. This is proved with the help of Theorem 1.8. $\square$

Theorem 2.18. *Let $(Q, \cdot)$ be a loop. Let $V_2(Q) = \{(Q, \cdot), (Q, \circ_2), (Q, *_2)\}$ and $S_2(Q) = \{\{(Q, \cdot)\}, \{(Q, \circ_2)\}, \{(Q, *_2)\}, \{(Q, \circ_2), (Q, *_2)\}\}$. If $(Q, \cdot)$ is a universal Osborn loop, then $K_2(Q) = (V_2(Q), S_2(Q))$ is a simplicial complex of isotopes of $(Q, \cdot)$.*

Proof. This is proved with Theorem 1.9. $\square$

Theorem 2.19. *Let $(Q, \cdot)$ be a loop. Let $V_3(Q) = \{(Q, \cdot), (Q, \circ_3), (Q, *_3)\}$ and $S_3(Q) = \{\{(Q, \cdot)\}, \{(Q, \circ_3)\}, \{(Q, *_3)\}, \{(Q, \circ_3), (Q, *_3)\}\}$. If $(Q, \cdot)$ is a universal Osborn loop, then $K_3(Q) = (V_3(Q), S_3(Q))$ is a simplicial complex of isotopes of $(Q, \cdot)$.*

Proof. This is proved with the aid of Theorem 1.10. $\square$

Corollary 2.20. *Let $(Q, \cdot)$ be a loop. Let $V_i(Q) = \{(Q, \cdot), (Q, \circ_i), (Q, *_i)\}$ and $S_i(Q) = \{\{(Q, \cdot)\}, \{(Q, \circ_i)\}, \{(Q, *_i)\}, \{(Q, \circ_i), (Q, *_i)\}\}$ for $i = 0, 1$. Then, $(Q, \cdot)$ is a universal Osborn loop if and only if $K_{01}(Q) = K_0(Q) \cup K_1(Q) = (V_0(Q) \cup V_1(Q), S_0(Q) \cup S_1(Q))$ is a simplicial complex of isotopes of $(Q, \cdot)$.*

Proof. This follows from Theorem 2.16 and Theorem 2.17. $\square$

Corollary 2.21. *Let $(Q, \cdot)$ be a loop. Let $V_i(Q) = \{(Q, \cdot), (Q, \circ_i), (Q, *_i)\}$ and $S_i(Q) = \{\{(Q, \cdot)\}, \{(Q, \circ_i)\}, \{(Q, *_i)\}, \{(Q, \circ_i), (Q, *_i)\}\}$ for $i = 2, 3$. If $(Q, \cdot)$ is a universal Osborn loop, then $K_{23}(Q) = K_2(Q) \cup K_3(Q) = (V_2(Q) \cup V_3(Q), S_2(Q) \cup S_3(Q))$ is a simplicial complex of isotopes of $(Q, \cdot)$.*

Proof. This follows from Theorem 2.18 and Theorem 2.19. $\square$

Corollary 2.22. *Let $(Q, \cdot)$ be a loop. Let $V_i(Q) = \{(Q, \cdot), (Q, \circ_i), (Q, *_i)\}$ and $S_i(Q) = \{\{(Q, \cdot)\}, \{(Q, \circ_i)\}, \{(Q, *_i)\}, \{(Q, \circ_i), (Q, *_i)\}\}$ for $i = 0, 1, 2, 3$. If $(Q, \cdot)$ is a universal Osborn loop, then*

$K_{0123}(Q) = \bigcup_{i=0}^3 K_i(Q) = \left(\bigcup_{i=0}^3 V_i(Q), \bigcup_{i=0}^3 S_i(Q) \right)$ is a simplicial complex of isotopes of $(Q, \cdot)$.

Proof. This is proved by combining Corollary 2.20 and Corollary 2.21. $\square$

Theorem 2.23. *Let $(Q, \cdot)$ be a loop. Let $V_{01}(Q) = \{(Q, \cdot), (Q, \circ_0), (Q, *_0), (Q, \circ_1), (Q, *_1)\}$ and $S_{10}(Q) = \{\{(Q, \cdot)\}, \{(Q, \circ_0)\}, \{(Q, *_0)\}, \{(Q, \circ_1)\}, \{(Q, *_1)\}, \{(Q, \circ_0), (Q, *_0)\}, \{(Q, \circ_1), (Q, *_1)\}, \{(Q, \circ_0), (Q, \circ_1)\}, \{(Q, *_0), (Q, *_1)\}, \{(Q, \circ_0), (Q, *_1)\}, \{(Q, \circ_1), (Q, *_0)\}, \{(Q, \circ_0), (Q, \circ_1), (Q, *_0)\}\}$,*

$\{(Q, \circ_0), (Q, \circ_1), (Q, *_1)\}, \{(Q, *_0), (Q, *_1), (Q, \circ_0)\},$
 $\{(Q, *_0), (Q, *_1), (Q, \circ_1)\}, \{(Q, \circ_0), (Q, \circ_1), (Q, *_0), (Q, *_1)\}$. Then, $(Q, \cdot)$
 is a universal Osborn loop and obey Equation 2.6 and Equation 2.7 if and
 only if $K_{10}(Q) = (V_{01}(Q), S_{10}(Q))$ is a simplicial complex of isotopes of
 $(Q, \cdot)$.

Proof. This is proved with the aid of Theorem 2.16, Theorem 2.17, Theorem 2.9 and Theorem 2.10. $\square$

Theorem 2.24. Let $(Q, \cdot)$ be a universal Osborn loop.

Let $V_i(Q) = \{(Q, \cdot), (Q, \circ_i), (Q, *_i)\},$

$S_i(Q) = \left\{ \{(Q, \cdot)\}, \{(Q, \circ_i)\}, \{(Q, *_i)\}, \{(Q, \circ_i), (Q, *_i)\} \right\}$ and

$K_i = (V_i(Q), S_i(Q))$ for $i = 0, 1, 2, 3$. Define $f_{ij} : K_i \rightarrow K_j$ as

$$f_{ij} : \begin{cases} (Q, \cdot) & \mapsto (Q, \cdot) \\ (Q, \circ_i) & \mapsto (Q, \circ_j) \\ (Q, *_i) & \mapsto (Q, *_j) \end{cases} \quad i, j = 0, 1, 2, 3 \text{ such that } i \neq j.$$

Then, f_{ij} is a simplicial map.

Proof. This is proved by Theorem 2.16, Theorem 2.17, Theorem 2.18 and Theorem 2.19. $\square$

Theorem 2.25. Let $(G, \cdot)$ and $(H, \star)$ be two loop isotopes under the triple (A, B, C) . For $D \in \{A, B, C\}$, if $D = E_1 E_2 \cdots E_i \cdots E_n$, $E_i : G \rightarrow H$, $i = 1, \dots, n$ been bijections such that there does not exist $r \geq n$ for which $D = E_1 E_2 \cdots E_i \cdots E_r$, then the length of D , $|D| = n$ units. If $D = I$, the identity mapping, then $|D| = 0$. The length of the isotopism $(G, \cdot) \xrightarrow[\text{Isotopism}]{(A, B, C)} (H, \star)$ is giving by $|(A, B, C)| = |A| + |B| + |C|$ units. For an

isotopism $(G, \cdot) \xrightarrow[\text{Isotopism}]{(A, B, C)} (H, \star)$, let the two loops $(G, \cdot)$ and $(H, \star)$ represent points in a 3-dimensional space and let an isotopism from $(G, \cdot)$ to $(H, \star)$ be a line with $(G, \cdot)$ and $(H, \star)$ as end-points. The set of loops $V_{01}(Q) = \{(Q, \cdot), (Q, \circ_0), (Q, *_0), (Q, \circ_1), (Q, *_1)\}$ where $(Q, \cdot)$ is a universal Osborn loop, form a rectangular pyramid with apex $(Q, \cdot)$.

Proof. We shall make use of the combined commutative diagram (2.1) as shown in the proof of Theorem 2.1. There are four isotopes of $(Q, \cdot)$ as shown in the combined commutative diagram (2.1), namely $(Q, \circ_i), (Q, *_i)$ for $i = 0, 1$. The length of each of the isotopisms $(R_{[u \setminus \{xv\}]}, L_u, I), (R_{\phi_0}, L_u, I), (R_v, L_{\phi_1}, I), (R_v, L_x, I)$ is 2 units. The length of each of the isomorphisms

$\gamma_0(x, u, v) = \mathbb{R}_v R_{[u \setminus (xv)]} \mathbb{L}_u L_x$ and $\gamma_1(x, u, v) = \mathbb{R}_v R_{[u \setminus (xv)]} \mathbb{L}_u L_x$ is 12 units. The length of each of the isomorphisms $\gamma_{01}^\circ = \mathbb{R}_{\phi_o(x, u, v)} R_{[u \setminus (xv)]}$ and $\gamma_{01}^* = \mathbb{L}_x L_{\phi_1(x, u, v)}$ is 6 units. Hence, the four loop isotopes $(Q, \circ_i), (Q, *_i)$ for $i = 0, 1$ of $(Q, \cdot)$ form a rectangle. Thus, taking $(Q, \cdot)$ as an apex and the four isotopism as lines drawn from the apex to the four vertices of the rectangle, we have a rectangular pyramid. $\square$

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