Generalized Jensen functional equation on restricted domain

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Abstract. We prove the Hyers-Ulam stability on restricted domains of generalized Jensen functional equation

$$m-1 \sum_{k=0}^{m-1} f(x + b_k y) = mf(x), \ x, y \in E,$$

where $b_k = \exp(\frac{2\pi i k}{m})$ for $0 \leq k \leq m - 1$. These results are applied to study of an asymptotic behavior of these functional equation.

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1 Introduction

The question concerning the stability of functional equations has been first raised by S. M. Ulam in 1940 [38]. S. M. Ulam started the stability by the following question

*Given a group $G$, a metric group $(G', d)$, a number $\delta > 0$ and a mapping $f: G \to G'$ which satisfies the inequality $d(f(xy), f(x)f(y)) < \delta$ for all*
For all $x, y \in G$, does there exist an homomorphism $h : G \to G'$ and a constant $\gamma > 0$, depending only on $G$ and $G'$ such that $d(f(x), h(x)) \leq \gamma$ for all $x$ in $G$?


Recently, the stability problem of the K-quadratic functional equation has been investigated by a number of mathematicians, the interested reader should refer to Ait Sibaha , Bouikhalene and Elqorachi [1], B.Bouikhalene, et al. [5], A.Charifi, et al. [10], Ab.Chahbi, et al. [9] and R. Lukasik [27], see also [6],[20],[22]-[25] and [32].

In 1998 Jung [22] investigated the Hyers-Ulam stability for additive and quadratic mappings on restricted domains. In 2002, J. M. Rassias [36] improved the bounds and thus the stability results obtained by Jung. Besides, he establish the Ulam stability for more general equations of two types on a restricted domain. Finally, he apply our recent results to the asymptotic behavior of functional equations of different types.

The stability problems of several functional equations on a restricted domain have been extensively investigated by a number of authors, for example ([8], [15], [30] and [31]).

Throughout this paper, let $E, F$ be vector spaces over $K = \mathbb{Q}$ or $\mathbb{R}$ and $2 \leq m \in \mathbb{N}$. Our aim is to prove the Hyers-Ulam stability on restricted domains of generalized Jensen functional equation

\[
\sum_{k=0}^{m-1} f(x + b_k y) = mf(x), \quad x, y \in E, \tag{1.1}
\]

where $b_k = \exp\left(\frac{2\pi ik}{m}\right)$ for $0 \leq k \leq m - 1$. These results are applied to study of an asymptotic behavior of these functional equation.
2 Notations and Preliminary results

In this section, we need to introduce some notions and notations. A function $A : E \to F$ between two vector spaces $E$ and $F$ is said to be additive provided if $A(x + y) = A(x) + A(y)$ for all $x, y \in E$. In this case, it is easily seen that $A(rx) = rA(x)$ for all $x \in E$ and all $r \in \mathbb{Q}$.

Let $k \in \mathbb{N}$ and $A : E^k \to F$ be a function, then we say that $A$ is $k$-additive provided if it is additive in each variable. In addition, we say that $A$ is symmetric provided if $A(x_{\sigma(1)}, x_{\sigma(2)}, \ldots, x_{\sigma(k)}) = A(x_1, x_2, \ldots, x_k)$ whenever $x_1, x_2, \ldots, x_k \in E$ and $\sigma$ is a permutation of $(1, 2, \ldots, k)$.

Let $k \in \mathbb{N}$ and $A : E^k \to F$ be symmetric and $k$-additive and let $A_k(x) = A(x, x, \ldots, x)$ for $x \in E$ and note that $A_k(rx) = r^k A_k(x)$ whenever $x \in E$ and $r \in \mathbb{Q}$.

In this way a function $A_k : E \to F$ which satisfies for all $\lambda \in \mathbb{Q}$ and $x \in E$, $A_k(\lambda x) = \lambda^k A_k$ will be called a rational-homogeneous form of degree $k$, (assuming $A_k \neq 0$).

A function $p : E \to F$ is called a generalized polynomial (GP) function of degree $m \in \mathbb{N}$ if there exist $a_0 \in E$ and a rational-homogeneous form $A_k : E \to F$ (for $1 \leq k \leq m$) of degree $k$, such that

$$p(x) = a_0 + \sum_{k=1}^{m} A_k(x)$$

for all $x \in E$.

Let $F^E$ denote the vector space, over a field $K$, consisting of all maps from $E$ into $F$. For each $h \in E$, define the linear difference operator $\Delta_h$ on $F^E$ by

$$\Delta_h f(x) = f(x + h) - f(x)$$

for all $f \in F^E$ and all $x \in E$. Notice that these difference operators commute ($\Delta_{h_1}\Delta_{h_2} = \Delta_{h_2}\Delta_{h_1}$ for all $h_1, h_2 \in E$) and if $h \in E$ and $n \in \mathbb{N}$, then $\Delta_h^n$ the $n$-th iterate of $\Delta_h$ satisfies

$$\Delta_h^n f(x) = \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} f(x + kh)$$

for all $f \in F^E$ and all $x, h \in E$.

The following theorem is proved by Mazur and Orlicz [28],[29] then in greater generality by Djokovic [14].
Theorem 2.1. Let $n \in \mathbb{N}$ and $f : E \to F$ be a function between two vector spaces $E$ and $F$, then the following assertions are equivalent.

1. $\Delta^n_h f(x) = 0$ for all $x, h \in E$.

2. $\Delta_{h_n} \ldots \Delta_{h_1} f(x) = 0$ for all $x, h_1, \ldots, h_n \in E$.

3. $f$ is a generalized polynomial (GP) of degree at most $n - 1$.

3 Main results

Lemma 3.1. Suppose that $2 \leq m \in \mathbb{N}$, $b_k \in K$ for $0 \leq k \leq m$, $F$ is a real (or complex) Banach space, $\delta \geq 0$ and $f_k : E \to F$ for $0 \leq k \leq m$, be mappings fulfilling

\[ \left\| \sum_{k=0}^{m} f_k(x + b_k y) - f_k(x) \right\| \leq \delta \]  

(3.1)

for all $x, y \in E$ and assume that $b_k - b_j \neq 0$ and $b_0 \neq 0$ whenever $0 \leq j < k \leq m$. Then,

\[ \left\| \Delta_{h_{m+1}} \ldots \Delta_{h_1} f_0(x) \right\| \leq 3^m \delta \]  

(3.2)

for all $x, h_1, \ldots, h_{m+1} \in E$.

Proof. For $0 \leq k \leq m$ let $d_{jk} = b_k - b_j$ so that $d_{jk} = 0$ if $j < k$ and $d_{kk} = 0$. For $0 \leq k \leq m$ and $x, y, h_1 \in E$,

\[ (x - b_{m} h_1) + b_k (y + h_1) = x + b_k y + d_{km} h_1. \]

By using these quality and (3.1), we find that

\[ \left\| \sum_{k=0}^{m} (f_k(x + b_k y + d_{km} h_1) - f(x + b_k y)) - \sum_{k=0}^{m} (f_k(x + d_{km} h_1) - f_k(x)) \right\| \]

\[ \leq \left\| \sum_{k=0}^{m} f_k(x + b_k y + d_{km} h_1) - f_k(x - b_{m} h_1) \right\| + \left\| \sum_{k=0}^{m} f_k(x + d_{km} h_1) - f_k(x - b_{m} h_1) \right\| \]

\[ + \left\| \sum_{k=0}^{m} f_k(x + b_k y) - f_k(x) \right\| \leq 3\delta \]
and thus, since $d_{mm} = 0$,

$$\left\| \sum_{k=0}^{m-1} \Delta_{d_{km}h_1} f_k(x + b_k y) - f_k(x) \right\| \leq 3\delta \quad (3.3)$$

for all $x, y, h_1 \in E$. Repeating the argument that led from (3.1) to (3.3), we find that

$$\left\| \sum_{k=0}^{m-2} \Delta_{d_{km-1}h_2} \Delta_{d_{km}h_1} f_k(x + b_k y) - f_k(x) \right\| \leq 3\delta \quad (3.4)$$

for all $x, y, h_1, h_2 \in E$. Applying this reasoning $m - 2$ more times we are inclined to admit that

$$\left\| \Delta_{d_{01}h_m} \ldots \Delta_{d_{0m}h_1} f_0(x + b_0 y) - f_k(x) \right\| \leq 3^m \delta \quad (3.5)$$

for all $x, y, h_1, \ldots, h_m \in E$. Consequently,

$$\left\| \Delta_{x} \Delta_{d_{01}h_m} \ldots \Delta_{d_{0m}h_1} f_0(b_0 y) \right\| \leq 3^m \delta. \quad (3.6)$$

Since $d_{0k} \neq 0$ for $1 \leq k \leq m$, and $b_0 \neq 0$ the last inequality simplify asserts that

$$\left\| \Delta_{h_{m+1}} \ldots \Delta_{h_1} f_0(z) \right\| \leq 3^m \delta \quad (3.7)$$

for all $z, h_1, \ldots, h_{m+1} \in E$.

\begin{proof}

\end{proof}

**Theorem 3.2.** Suppose that $E$ is a vector space and $F$ is real (or complex) Banach space, $2 \leq m \in \mathbb{N}$, $b_k = \exp(\frac{2ik\pi}{m})$ for $0 \leq k \leq m - 1$ and $\delta \geq 0$. If $f : E \to F$ satisfies

$$\left\| \sum_{k=0}^{m-1} f(x + b_k y) - mf(x) \right\| \leq \delta \quad (3.8)$$

for all $x, y \in E$. Then there exists a unique generalized polynomial (GP) $q : E \to F$ at most $m - 1$ such that $q(0) = 0$ and

$$\left\| f(x) - f(0) - q(x) \right\| \leq 3^{m-1} \delta \text{ for all } x \in E.$$  

Moreover

$$\sum_{k=1}^{m} q(x + b_k y) - mq(x) = 0 \text{ for all } x, y \in E.$$
Proof. If we put $f_k = f$ for each $k$, in (3.8), we get

$$\left\| \sum_{k=0}^{m-1} f_k(x + b_k y) - f_k(x) \right\| \leq \delta$$

(3.9)

Then, by Lemma 3.1

$$\| \Delta_{h_m} \cdots \Delta_{h_1} f(z) \| \leq 3^{m-1} \delta$$

for all $z, h_1, \ldots, h_m \in E$.

Hence, according to (Theorem II [20]), there exists a generalized polynomial (GP) $q : E \to F$, of degree at most $m - 1$ such that

$$\| f(x) - q(x) \| \leq 3^{m-1} \delta$$

(3.10)

and

$$q(x) = f(0) + \sum_{j=1}^{m-1} A_j(x)$$

(3.11)

where $A_j : E^j \to F$ are symmetric, $j$-additive mappings.

By (3.10) and (3.11), we obtain

$$\left\| \sum_{k=0}^{m-1} (q(x + b_k y) - mq(x)) \right\| \leq \left\| \sum_{k=0}^{m-1} (q(x + b_k y) - f(x + b_k y)) \right\| + \| mf(x) - mq(x) \|$$

so,

$$\left\| \sum_{k=0}^{m-1} (q(x + b_k y) - mq(x)) \right\| \leq 2m.3^{m-1} \delta$$

(3.12)

for all $x, y \in E$.

Now (3.12) says, in light of (3.11), that

$$\left\| \sum_{j=1}^{m-1} \left( \sum_{k=0}^{m-1} A_j(x + b_k y) - mA_j(x) \right) \right\| \leq 2m.3^{m-1} \delta$$

(3.13)

for all $x, y \in E$. In (3.13), replace $x$ by $rx$ and $y$ by $ry$ ($r \in \mathbb{Q}$) to conclude that, for all $x, y \in E$ and all $r \in \mathbb{Q}$,

$$\left\| \sum_{j=1}^{m-1} r^j \left( \sum_{k=0}^{m-1} A_j(x + b_k y) - mA_j(x) \right) \right\| \leq 2m.3^{m-1} \delta$$

(3.14)
By continuity, (3.14) holds for all real $r$, and all $x, y \in E$. Now suppose that $\phi : F \rightarrow \mathbb{R}$ is a continuous linear functional. Then

$$\left\| \sum_{j=1}^{m-1} r^j \phi \left( \sum_{k=0}^{m-1} (A_j(x + b_ky) - mA_j(x)) \right) \right\| \leq ||\phi||2m.3^{m-1}\delta$$

(3.15)

for all $x, y \in E$ and all $r \in \mathbb{R}$. Since a real polynomial function is bounded if and only if it is constant, from the last inequality we surmise that, for $1 \leq j \leq m - 1$,

$$\phi \left( \sum_{k=0}^{m-1} A_j(x + b_ky) - mA_j(x) \right) = 0$$

(3.16)

for all $x, y \in E$. Since this is so for every continuous linear functional $\phi : F \rightarrow \mathbb{R}$, by the Hahn-Banach theorem,

$$\sum_{k=0}^{m-1} A_j(x + b_ky) - mA_j(x) = 0 \text{ for } x, y \in E \text{ and } 1 \leq j \leq m - 1.$$  

(3.17)

Then, each $q$ is a generalized polynomial (GP) of degree at most $m - 1$ and from (3.17) we find that

$$\sum_{k=0}^{m-1} q(x + b_ky) - mq(x) = 0 \text{ for } x, y \in E \text{ and } 1 \leq j \leq m - 1.$$  

(3.18)

Finally, let $p$ be another generalized polynomial (GP) solution of (1.1) of degree at most $m - 1$ such that

$$||f(x) - f(0) - p(x)|| \leq 3^{m-1}\delta,$$

and $p(0) = 0$. Then, from (3.18) and (3.10) we get that $q - p$ is generalized polynomial (GP) of degree at most $m - 1$ such that $||q(x) - p(x)|| \leq 2.3^{m-1}, x \in E$. Thus, necessarily $q = p + q(0) - p(0)$. Since $q(0) = p(0) = 0$ we get that $q = p$ witch ends the proof.

**Theorem 3.3.** Let $d > 0$ and $\delta \geq 0$ be fixed. If a mapping $f : E \rightarrow F$ satisfies the generalized Jensen inequality (3.8) for all $x, y \in E$, with $||x|| + ||y|| \geq d$, then there exists a unique generalized polynomial (GP) $q : E \rightarrow F$ such that $q(0) = 0$ and

$$||f(x) - f(0) - q(x)|| \leq 3^m\delta$$

for all $x \in E$.

Moreover

$$\sum_{k=0}^{m-1} q(x + b_ky) - mq(x) = 0 \text{ for all } x, y \in E.$$
Proof. Assume that $||x|| + ||y|| < d$ and let $t \in E$ such that $||t|| = d$ where $x = 0$ and

$$t = \left(1 + \frac{d}{||x||}\right)x,$$

where $x \neq 0$. We note that $||t|| = ||x|| + d \geq d, ||x|| + ||t|| \geq d$ and

$$||t|| + ||x + b_k y|| \geq d.$$

We have for all $b_j, 0 \leq j \leq m - 1$ is a root of unity, if we put $b_j = e^{i\theta}$, we obtain

$$||x + b_j t||^2 = \left(2 \left(1 + \frac{d}{||x||}\right)(1 + \cos \theta) + \frac{d^2}{||x||^2}\right)||x||^2 \geq d^2.$$

Thus, from (3.8) and the new functional identity we get

$$m \left[ \sum_{k=0}^{m-1} f(x + b_k y) - mf(x) \right] = \sum_{k=0}^{m-1} \left( mf(x + b_k y) - \sum_{n=0}^{m-1} f(x + b_k y + b_n t) \right) + \sum_{n=0}^{m-1} (-mf(x + b_n t) + \sum_{k=0}^{m-1} f(x + b_n t + b_k y)) + m \left[ \sum_{n=0}^{m-1} f(x + b_n t) - mf(x) \right],$$

then

$$m \left\| \sum_{k=0}^{m-1} f(x + b_k y) - mf(x) \right\| \leq 3m \delta.$$

Therefore,

$$\left\| \sum_{k=0}^{m-1} f(x + b_k y) - mf(x) \right\| \leq 3 \delta.$$

for all $x, y \in E$. Now, Applying Theorem 3.2 we get the result.

Corollary 3.4. A mapping $f : E \to F$ satisfies the functional equation (1.1) if and only if the asymptotic condition

$$\left\| \sum_{k=0}^{m-1} f(x + b_k y) - mf(x) \right\| \to 0, \text{ as } ||x|| + ||y|| \to \infty. \quad (3.19)$$

holds true.
By the asymptotic condition (3.19), there exists a sequence $\delta_n$ monotonically decreasing to 0 such that

$$\left\| \sum_{k=0}^{m-1} f(x + b_k y) - mf(x) \right\| \leq \delta_n$$

(3.20)

for all $x, y \in E$ with $||x|| + ||y|| \geq n$. Hence, it follows from Theorem 3.3 that there exists a unique generalized polynomial $q_n : E \rightarrow F$ such that $q_n(0) = 0$ and

$$||f(x) - f(0) - q_n(x)|| \leq 3^m \delta_n \text{ for all } x \in E.$$  \hspace{1cm} (3.21)

for all $x \in E$. Since $\delta_n$ is a monotonically decreasing sequence, the generalized polynomial $q_n$ satisfies (3.21) for all $N \geq n$. The uniqueness of $q_n$ implies for all $N \geq n$, $q_N = q_n$. Hence, by letting $n \rightarrow \infty$, in (3.21) we conclude that $f = f(0) + q_N$ is generalized polynomial of degree at most $m - 1$, and it satisfied the functional equation (1.1).

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**References**


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