On the Bounds for the Derivatives of the Solutions of the Linear Volterra Integral Equations

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Abstract. Some bounds about the solution of the linear Volterra integral equations of the second type of the form

$$f(t) = 1 - \int_0^t K(t - \tau)f(\tau)d\tau = 1 - K * f$$

with unit source term and positive monotonically increasing convolution kernel were obtained in [5, 6, 8–10]. The boundedness of functions $f', f'', \dots, f^{(n)}$, ($n \in \mathbb{N}$) defined on the infinite interval $[0, \infty)$ were shown in [10, 12]. In this paper, it is shown that the results in [10, 12] for an equation in the same form can also be derived under different conditions than those of [10, 12].

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1 Introduction

Volterra integral equations arise in great many branches of science. For example they are models of epidemic diffusion, population dynamics, reaction-

diffusion in small cells and in general of evolutionary phenomena incorporating memory, [1].

In the study of linear viscoelastic materials, from measurements of the relaxation modulus $G(t)$, approximations to the corresponding creep compliance (retardation) modulus $J(t)$ are determined by solving the convolution inter-conversion equation

$$\int_0^t G(t-\tau)J(\tau)d\tau = t, \quad t \geq 0, \quad (1.1)$$

where $G \in C^1[0, \infty)$, $G(t) > 0$ and $G'(t) < 0$ for all t such that $0 \leq t < \infty$, [2]. Differentiation of (1.1) leads to

$$J(t) = \frac{1}{G(0)} - \frac{1}{G(0)} \int_0^t G'(t-\tau)J(\tau)d\tau. \quad (1.2)$$

(1.2) is the linear second type convolution Volterra integral equation. An integral equation of the form

$$f(t) = \phi(t) - \int_0^t K(t-\tau)f(\tau)d\tau = \phi(t) - K * f \quad (1.3)$$

is known as the second type linear Volterra integral equation with convolution kernel, where ϕ is the source term and K is kernel which are the known functions, f is an unknown function, [13, p.23]. Such equations are important in many applications, including rheology, risk theory and various applications of renewal theory including branching processes, queuing theory and inventory analysis, [4]. Some properties of the solution of the integral equation (1.3) are investigated on the infinite interval, [7, 11].

The way of obtaining a new equation which is equivalent to the equation of the form (1.3) is given by Theorem A named as Equivalence Theorem as follows:

Theorem 1.1.(Equivalence Theorem). [5, Theorem 1.1.1] *If*

- (1) $K \in C^1[0, \infty)$,
- (2) ϕ is locally integrable,

then the equation (1.3) is equivalent to

$$f(t) = \psi(t) - \int_0^t L(t-\tau)f(\tau)d\tau,$$

where

$$\psi(t) = \phi(t) + \int_0^t g'(t - \tau) \phi(\tau) d\tau,$$

$$L(t) = g'(t) + ag(t) + \int_0^t g(t - \tau) K'(\tau) d\tau,$$

where $a = K(0)$, g is any function such that $g \in C^1[0, \infty)$ and $g(0) = 1$.

The sufficient conditions which enable to derive the solution of (1.3) in terms of the solution of the equation

$$h(t) = 1 - \int_0^t K(t - \tau)h(\tau)d\tau = 1 - K * h \quad (1.4)$$

are given by the Convolution Theorem.

Theorem 1.2.(Convolution Theorem). [3, pp. 229-230] *Let the conditions*

$$(1) \ \phi'(t) \text{ exists for } 0 \leq t \leq T, \int_0^T |\phi'(t)| dt < \infty, \ (T > 0),$$

$$(2) \ \int_0^T |K(t)| dt < \infty$$

hold, then the solution of equation (1.3) is given of the form

$$\begin{aligned} f(t) &= h(t) \phi(0) + \int_0^t h(t - \tau) \phi'(\tau) d\tau \\ &= h(t) \phi(0) + h * \phi', \ (0 \leq t \leq T), \end{aligned} \quad (1.5)$$

where $h(t)$ is the solution of (1.4).

Therefore, if h is known, so is f . Or if the properties of h are known, we may be able to obtain certain properties of f by (1.5).

We assume throughout that $t \in [0, \infty)$ and $n \in \mathbb{N}_2 = \{2, 3, \dots\}$.

2 The Main Results

Theorem C. *Let us consider the equation*

$$f(t) = 1 - \int_0^t K(t - \tau) f(\tau) d\tau = 1 - K * f. \quad (2.1)$$

If the conditions

(1) $K \in C^2[0, \infty)$, $K(t) > 0$, $K'(t) > 0$ and $K''(t) \leq 0$ for all t ,

(2) $4b \leq a^2$

hold, then the solution of (2.1) satisfies the inequalities $|f'(t)| \leq 3a/2$ and $|f''(t)| \leq (3a^2 + 4b)/2$ for all t , where $a = K(0)$ and $b = K'(0)$.

Proof. Firstly, we can assert that f provides the inequality $|f(t)| \leq 1$ for all t , [6, Theorem B]. By choosing $g(t) = e^{-\gamma t}$ ($\gamma = \gamma_0 = a/2$) in Theorem A (Equivalence Theorem), we get the equivalent equation of (2.1) of the form

$$f(t) = e^{-\gamma t} - \int_0^t L(t - \tau) f(\tau) d\tau = e^{-\gamma t} - L * f, \quad (2.2)$$

where

$$L(t) = (a - \gamma)e^{-\gamma t} + K' * e^{-\gamma t} = \left(\frac{a}{2}\right)e^{-\gamma t} + K' * e^{-\gamma t} > 0$$

and thus,

$$\begin{aligned} L'(t) &= (\gamma^2 - a\gamma + b)e^{-\gamma t} + K'' * e^{-\gamma t} \\ &= \left(-\frac{a^2}{4} + b\right)e^{-\gamma t} + K'' * e^{-\gamma t} \leq 0 \end{aligned}$$

for all t . On the other hand, differentiation of the equivalent equation (2.2) leads to

$$f'(t) = -\gamma e^{-\gamma t} - L(0)f(t) - L' * f.$$

Taking $\gamma = \gamma_0 = a/2$, the inequality

$$\begin{aligned} |f'(t)| &\leq \gamma + \frac{a}{2}|f(t)| + \int_0^t |L'(t - \tau)||f(\tau)|d\tau \\ &\leq \gamma + \frac{a}{2} - \int_0^t L'(t - \tau)d\tau = \gamma + \frac{a}{2} + L(0) - L(t) \\ &= \frac{a}{2} + \frac{a}{2} + \frac{a}{2} - L(t) = \frac{3a}{2} - L(t) \leq \frac{3a}{2} \end{aligned} \quad (2.3)$$

is obtained for all t . From (2.1),

$$f'(t) = -af(t) - K' * f$$

and so

$$f''(t) = -af'(t) - bf(t) - K'' * f.$$

By the inequalities $|f(t)| \leq 1$ and (2.3), we have

$$\begin{aligned}
 |f''(t)| &\leq a|f'(t)| + b|f(t)| + \int_0^t |K''(t-\tau)||f(\tau)|d\tau \\
 &\leq a\left(\frac{3a}{2}\right) + b - \int_0^t K''(t-\tau)d\tau \\
 &= \frac{3a^2}{2} + b + K'(0) - K'(t) \\
 &= \frac{3a^2}{2} + 2b - K'(t) \leq \frac{1}{2}(3a^2 + 4b)
 \end{aligned}$$

for all t . □

Theorem 1. *Let us consider the equation*

$$f_1(t) = 1 - \int_0^t K_1(t-\tau) f_1(\tau) d\tau = 1 - K_1 * f_1. \quad (2.4)$$

If

- (1) $K_1 \in C^3[0, \infty)$, $K_1(t) > 0$, $K_1'(t) > 0$, $K_1''(t) > 0$ and $K_1'''(t) \leq 0$ for all t ,
- (2) $a_{10}^2 < 4a_{11}$,
- (3) $3a_{11} \leq a_{10}^2$,
- (4) $2a_{10}^3 - 9a_{10}a_{11} + 27a_{12} \leq 0$,

then the solution of (2.4) satisfies the inequalities

$$|f_1'(t)| \leq \frac{7a_{10}}{3}, \quad |f_1''(t)| \leq \frac{1}{9}(7a_{10}^2 + 36a_{11})$$

and

$$|f_1'''(t)| \leq \frac{1}{9}(7a_{10}^3 + 57a_{10}a_{11} + 36a_{12})$$

for all t , where $a_{10} = K_1(0)$, $a_{11} = K_1'(0)$ and $a_{12} = K_1''(0)$.

Proof. One can immediately see from Theorem 3.1 of [6] that f_1 provides the inequality $|f_1(t)| \leq 2$ for all t . By choosing $g(t) = e^{-\gamma t}$ and $\gamma = \gamma_1 = a_{10}/3$ in the Equivalence Theorem, we get the equivalent equation to (2.4) of the form

$$f_1(t) = e^{-\gamma t} - \int_0^t L_1(t-\tau) f_1(\tau) d\tau = e^{-\gamma t} - L_1 * f_1, \quad (2.5)$$

where

$$L_1(t) = (a_{10} - \gamma)e^{-\gamma t} + K_1' * e^{-\gamma t}.$$

Thus,

$$L_1'(t) = (\gamma^2 - a_{10}\gamma + a_{11})e^{-\gamma t} + K_1'' * e^{-\gamma t}$$

and

$$L_1''(t) = (-\gamma^3 + a_{10}\gamma^2 - a_{11}\gamma + a_{12})e^{-\gamma t} + K_1''' * e^{-\gamma t}.$$

It is known that L_1 satisfies all of the conditions of Theorem *C* from the proof of Theorem 3.1 of [6]. Hence, by Theorem *C*, the first two derivatives of the solution of the equation $h_1(t) = 1 - L_1 * h_1$ verify the inequalities

$$|h_1'(t)| \leq \frac{3L_1(0)}{2} = \frac{3(a_{10} - \gamma)}{2} = a_{10} = y_{11}$$

and

$$\begin{aligned} |h_1''(t)| &\leq \frac{1}{2} [3(L_1(0))^2 + 4L_1'(0)] \\ &= \frac{1}{2} [3(a_{10} - \gamma)^2 + 4(\gamma^2 - a_{10}\gamma + a_{11})] \\ &= \frac{1}{9}(2a_{10}^2 + 18a_{11}) = y_{12} \end{aligned}$$

for all t . By the Convolution Theorem, the solution f_1 of the equivalent equation (2.5) can be written by means of h_1 of the form

$$f_1(t) = h_1(t) - \gamma h_1 * e^{-\gamma t}. \quad (2.6)$$

Thus, we have by (2.6) that

$$f_1'(t) = h_1'(t) - \gamma (h_1(0)e^{-\gamma t} + h_1' * e^{-\gamma t}) \quad (2.7)$$

which yields

$$f_1''(t) = h_1''(t) + \gamma^2 e^{-\gamma t} - \gamma h_1'(0)e^{-\gamma t} - \gamma h_1'' * e^{-\gamma t}. \quad (2.8)$$

By (2.7) and (2.8),

$$\begin{aligned} |f_1'(t)| &\leq |h_1'(t)| + \gamma + \gamma \int_0^t |h_1'(t - \tau)| e^{-\gamma \tau} d\tau \\ &\leq y_{11} + \gamma + \gamma y_{11} \int_0^t e^{-\gamma \tau} d\tau = y_{11} + \gamma + y_{11} - y_{11}e^{-\gamma t} \\ &= 2y_{11} + \gamma - y_{11}e^{-\gamma t} \leq 2y_{11} + \gamma = \frac{7a_{10}}{3} = x_{11} \end{aligned} \quad (2.9)$$

and

$$\begin{aligned}
 |f_1''(t)| &\leq |h_1''(t)| + \gamma^2 + \gamma|h_1'(0)| + \gamma \int_0^t |h_1''(t-\tau)|e^{-\gamma\tau} d\tau \\
 &\leq y_{12} + \gamma^2 + \gamma y_{11} + \gamma y_{12} \int_0^t e^{-\gamma\tau} d\tau \\
 &= 2y_{12} + \gamma^2 + \gamma y_{11} - y_{12}e^{-\gamma t} \\
 &\leq 2y_{12} + \gamma^2 + \gamma y_{11} = \frac{1}{9}(8a_{10}^2 + 36a_{11}) = b_{12}
 \end{aligned} \tag{2.10}$$

for all t and $\gamma = \gamma_1 = a_{10}/3$.

On the other hand, differentiation of the equivalent equation (2.5) leads to

$$f_1''(t) = \gamma^2 e^{-\gamma t} - L_1(0)f_1'(t) - L_1'(0)f_1(t) - L_1'' * f_1.$$

Taking $\gamma = \gamma_1 = a_{10}/3$, the inequality

$$\begin{aligned}
 |f_1''(t)| &\leq \gamma^2 + L_1(0)|f_1'(t)| + L_1'(0)|f_1(t)| + \\
 &\quad + \int_0^t |L_1''(t-\tau)||f_1(\tau)| d\tau \\
 &\leq \gamma^2 + (a_{10} - \gamma)x_{11} + (\gamma^2 - a_{10}\gamma + a_{11})2 - 2 \int_0^t L_1''(\tau) d\tau \\
 &= \gamma^2 + (a_{10} - \gamma)x_{11} + 4(\gamma^2 - a_{10}\gamma + a_{11}) - 2L_1'(t) \\
 &\leq \gamma^2 + (a_{10} - \gamma)x_{11} + 4(\gamma^2 - a_{10}\gamma + a_{11}) \\
 &= \frac{1}{9}(7a_{10}^2 + 36a_{11}) = c_{12}
 \end{aligned} \tag{2.11}$$

is obtained for all t . From (2.10) and (2.11), we get the inequality

$$|f_1''(t)| \leq \min\{b_{12}, c_{12}\} = c_{12} = \frac{1}{9}(7a_{10}^2 + 36a_{11}) = x_{12} \tag{2.12}$$

for all t . From (2.4),

$$f_1'''(t) = -a_{10}f_1''(t) - a_{11}f_1'(t) - a_{12}f_1(t) - K_1''' * f_1.$$

By the inequalities $|f_1(t)| \leq 2$, (2.9) and (2.12), we have the inequality

$$\begin{aligned}
 |f_1'''(t)| &\leq a_{10}|f_1''(t)| + a_{11}|f_1'(t)| + a_{12}|f_1(t)| + \int_0^t |K_1'''(t-\tau)||f_1(\tau)| d\tau \\
 &\leq a_{10}x_{12} + a_{11}x_{11} + 2a_{12} - 2 \int_0^t K_1'''(\tau) d\tau \\
 &= a_{10}x_{12} + a_{11}x_{11} + 4a_{12} - 2K_1''(t) \\
 &\leq a_{10}x_{12} + a_{11}x_{11} + 4a_{12} \\
 &= \frac{1}{9}(7a_{10}^3 + 57a_{10}a_{11} + 36a_{12})
 \end{aligned}$$

for all t . □

Theorem 2. *Let us consider the equation*

$$f_2(t) = 1 - \int_0^t K_2(t - \tau) f_2(\tau) d\tau = 1 - K_2 * f_2. \quad (2.13)$$

If

- (1) $K_2 \in C^4[0, \infty)$, $K_2(t) > 0$, $K_2'(t) > 0$, $K_2''(t) > 0$, $K_2'''(t) > 0$ and $K_2^{(4)}(t) \leq 0$ for all t ,
- (2) $a_{20}^2 < 3a_{21}$,
- (3) $\frac{8}{3}a_{21} \leq a_{20}^2$,
- (4) $a_{20}^3 - 4a_{20}a_{21} + 8a_{22} \leq 0$,
- (5) $-3a_{20}^4 + 16a_{20}^2a_{21} - 64a_{20}a_{22} + 256a_{23} \leq 0$,

then the solution of (2.13) satisfies the inequalities

$$|f_2'(t)| \leq \frac{15a_{20}}{4}, \quad |f_2''(t)| \leq \frac{1}{8}(-a_{20}^2 + 64a_{21}),$$

$$|f_2'''(t)| \leq \frac{1}{64}(-45a_{20}^3 + 544a_{20}a_{21} + 512a_{22})$$

and

$$|f_2^{(4)}(t)| \leq \frac{1}{64}(-45a_{20}^4 + 536a_{20}^2a_{21} + 752a_{20}a_{22} + 512a_{21}^2 + 512a_{23})$$

for all t , where $a_{20} = K_2(0)$, $a_{21} = K_2'(0)$, $a_{22} = K_2''(0)$ and $a_{23} = K_2'''(0)$.

Proof. Firstly, it is immediate by Theorem 3.3 of [6] that f_2 provides the inequality $|f_2(t)| \leq 4$ for all t . By choosing the function g and the number γ as $g(t) = e^{-\gamma t}$ and $\gamma = \gamma_2 = a_{20}/4$ in the Equivalence Theorem, we derive the equivalent equation to (2.13) of the form

$$f_2(t) = e^{-\gamma t} - \int_0^t L_2(t - \tau) f_2(\tau) d\tau = e^{-\gamma t} - L_2 * f_2, \quad (2.14)$$

where

$$L_2(t) = (a_{20} - \gamma)e^{-\gamma t} + K_2' * e^{-\gamma t}.$$

Thus,

$$\begin{aligned} L_2'(t) &= (\gamma^2 - a_{20}\gamma + a_{21})e^{-\gamma t} + K_2'' * e^{-\gamma t}, \\ L_2''(t) &= (-\gamma^3 + a_{20}\gamma^2 - a_{21}\gamma + a_{22})e^{-\gamma t} + K_2''' * e^{-\gamma t} \end{aligned}$$

and

$$L_2'''(t) = (\gamma^4 - a_{20}\gamma^3 + a_{21}\gamma^2 - a_{22}\gamma + a_{23})e^{-\gamma t} + K_2^{(4)} * e^{-\gamma t}.$$

It is known that L_2 satisfies all of the conditions of Theorem 1 from the proof of Theorem 3.3 in [6]. Hence, the first three derivatives of the solution of the equation $h_2(t) = 1 - L_2 * h_2$ must verify the inequalities

$$|h_2'(t)| \leq \frac{7L_2(0)}{3} = \frac{7(a_{20} - \gamma)}{3} = \frac{7a_{20}}{4} = y_{21},$$

$$\begin{aligned} |h_2''(t)| &\leq \frac{1}{9} [7(L_2(0))^2 + 36L_2'(0)] \\ &= \frac{1}{9} [7(a_{20} - \gamma)^2 + 36(\gamma^2 - a_{20}\gamma + a_{21})] \\ &= \frac{1}{16} (-5a_{20}^2 + 64a_{21}) = y_{22} \end{aligned}$$

and

$$\begin{aligned} |h_2'''(t)| &\leq \frac{1}{9} [7(L_2(0))^3 + 57L_2(0)L_2'(0) + 36L_2''(0)] \\ &= \frac{1}{9} [7(a_{20} - \gamma)^3 + 57(a_{20} - \gamma)(\gamma^2 - a_{20}\gamma + a_{21}) + \\ &\quad + 36(-\gamma^3 + a_{20}\gamma^2 - a_{21}\gamma + a_{22})] \\ &= \frac{1}{8} (-3a_{20}^3 + 30a_{20}a_{21} + 32a_{22}) = y_{23} \end{aligned}$$

for all t from Theorem 1. By the Convolution Theorem, the solution f_2 of the equivalent equation (2.14) can be written by means of h_2 of the form $f_2(t) = h_2(t) - \gamma h_2 * e^{-\gamma t}$. Thus

$$\begin{aligned} f_2'(t) &= h_2'(t) - \gamma e^{-\gamma t} - \gamma h_2' * e^{-\gamma t}, \\ f_2''(t) &= h_2''(t) + \gamma^2 e^{-\gamma t} - \gamma h_2'(0)e^{-\gamma t} - \gamma h_2'' * e^{-\gamma t} \end{aligned}$$

and

$$f_2'''(t) = h_2'''(t) - \gamma^3 e^{-\gamma t} + \gamma^2 h_2'(0)e^{-\gamma t} - \gamma h_2''(0)e^{-\gamma t} - \gamma h_2''' * e^{-\gamma t}.$$

Hence, the inequalities

$$\begin{aligned}
 |f_2'(t)| &\leq |h_2'(t)| + \gamma + \gamma \int_0^t |h_2'(t - \tau)| e^{-\gamma\tau} d\tau \\
 &\leq y_{21} + \gamma + \gamma y_{21} \int_0^t e^{-\gamma\tau} d\tau = y_{21} + \gamma + y_{21} - y_{21}e^{-\gamma t} \\
 &= 2y_{21} + \gamma - y_{21}e^{-\gamma t} \leq 2y_{21} + \gamma = \frac{15a_{20}}{4} = x_{21},
 \end{aligned}$$

$$\begin{aligned}
 |f_2''(t)| &\leq |h_2''(t)| + \gamma^2 + \gamma|h_2'(0)| + \gamma \int_0^t |h_2''(t - \tau)| e^{-\gamma\tau} d\tau \\
 &\leq y_{22} + \gamma^2 + \gamma y_{21} + \gamma y_{22} \int_0^t e^{-\gamma\tau} d\tau = 2y_{22} + \gamma^2 + \gamma y_{21} - y_{22}e^{-\gamma t} \\
 &\leq 2y_{22} + \gamma^2 + \gamma y_{21} = \frac{1}{8}(-a_{20}^2 + 64a_{21}) = x_{22}
 \end{aligned}$$

and

$$\begin{aligned}
 |f_2'''(t)| &\leq |h_2'''(t)| + \gamma^3 + \gamma^2|h_2'(0)| + \gamma|h_2''(0)| + \\
 &\quad + \gamma \int_0^t |h_2'''(t - \tau)| e^{-\gamma\tau} d\tau \\
 &\leq y_{23} + \gamma^3 + \gamma^2 y_{21} + \gamma y_{22} + \gamma y_{23} \int_0^t e^{-\gamma\tau} d\tau \\
 &= 2y_{23} + \gamma^3 + \gamma^2 y_{21} + \gamma y_{22} - y_{23}e^{-\gamma t} \\
 &\leq 2y_{23} + \gamma^3 + \gamma^2 y_{21} + \gamma y_{22} \\
 &= \frac{1}{64}(-45a_{20}^3 + 544a_{20}a_{21} + 512a_{22}) = b_{23} \quad (2.15)
 \end{aligned}$$

are obtained for all t and $\gamma = \gamma_2 = a_{20}/4$.

On the other hand, differentiation of the equivalent equation (2.14) leads to

$$f_2'''(t) = -\gamma^3 e^{-\gamma t} - L_2(0)f_2''(t) - L_2'(0)f_2'(t) - L_2''(0)f_2(t) - L_2''' * f_2.$$

Taking $\gamma = \gamma_2 = a_{20}/4$, the inequality

$$\begin{aligned}
 |f_2'''(t)| &\leq \gamma^3 + L_2(0)|f_2''(t)| + L_2'(0)|f_2'(t)| + \\
 &\quad + L_2''(0)|f_2(t)| + \int_0^t |L_2'''(t-\tau)||f_2(\tau)|d\tau \\
 &\leq \gamma^3 + (a_{20} - \gamma)x_{22} + (\gamma^2 - a_{20}\gamma + a_{21})x_{21} + \\
 &\quad + 4(-\gamma^3 + a_{20}\gamma^2 - a_{21}\gamma + a_{22}) - 4 \int_0^t L_2'''(\tau)d\tau \\
 &= \gamma^3 + (a_{20} - \gamma)x_{22} + (\gamma^2 - a_{20}\gamma + a_{21})x_{21} + \\
 &\quad + 8(-\gamma^3 + a_{20}\gamma^2 - a_{21}\gamma + a_{22}) - 4L_2''(t) \\
 &\leq \gamma^3 + (a_{20} - \gamma)x_{22} + (\gamma^2 - a_{20}\gamma + a_{21})x_{21} + \\
 &\quad + 8(-\gamma^3 + a_{20}\gamma^2 - a_{21}\gamma + a_{22}) \\
 &= \frac{1}{64}(-26a_{20}^3 + 496a_{20}a_{21} + 512a_{22}) = c_{23} \tag{2.16}
 \end{aligned}$$

is obtained for all t .

From condition (3) of Theorem 2,

$$b_{23} - c_{23} = \frac{a_{20}(48a_{21} - 19a_{20}^2)}{64} \leq 0$$

and so by (2.15) and (2.16), we obtain the inequality

$$|f_2'''(t)| \leq \min\{b_{23}, c_{23}\} = b_{23} = \frac{-45a_{20}^3 + 544a_{20}a_{21} + 512a_{22}}{64} = x_{23}$$

for all t . From (2.13),

$$f_2^{(4)}(t) = -a_{20}f_2'''(t) - a_{21}f_2''(t) - a_{22}f_2'(t) - a_{23}f_2(t) - K_2^{(4)} * f_2.$$

By using the inequalities obtained above and $|f_2(t)| \leq 4$, the bound of $f_2^{(4)}$ is found as

$$\begin{aligned}
 |f_2^{(4)}(t)| &\leq a_{20}|f_2'''(t)| + a_{21}|f_2''(t)| + a_{22}|f_2'(t)| + a_{23}|f_2(t)| + \\
 &\quad + \int_0^t |K_2^{(4)}(t-\tau)||f_2(\tau)|d\tau \\
 &\leq a_{20}x_{23} + a_{21}x_{22} + a_{22}x_{21} + 4a_{23} - 4 \int_0^t K_2^{(4)}(\tau)d\tau \\
 &= a_{20}x_{23} + a_{21}x_{22} + a_{22}x_{21} + 8a_{23} - 4K_2'''(t) \\
 &\leq a_{20}x_{23} + a_{21}x_{22} + a_{22}x_{21} + 8a_{23} \\
 &= \frac{1}{64}(-45a_{20}^4 + 536a_{20}^2a_{21} + 752a_{20}a_{22} + 512a_{21}^2 + 512a_{23})
 \end{aligned}$$

for all t which completes the proof. $\square$

Theorem 3. *Let us consider the equation*

$$f_3(t) = 1 - \int_0^t K_3(t - \tau) f_3(\tau) d\tau = 1 - K_3 * f_3. \quad (2.17)$$

If

- (1) $K_3 \in C^5[0, \infty)$, $K_3(t) > 0$, $K_3'(t) > 0, \dots, K_3^{(4)}(t) > 0$ and $K_3^{(5)}(t) \leq 0$ for all t ,
- (2) $a_{30}^2 < \frac{8}{3}a_{31}$,
- (3) $\frac{5}{2}a_{31} \leq a_{30}^2$,
- (4) $4a_{30}^3 - 15a_{30}a_{31} + 25a_{32} \leq 0$,
- (5) $-3a_{30}^4 + 15a_{30}^2a_{31} - 50a_{30}a_{32} + 125a_{33} \leq 0$,
- (6) $4a_{30}^3 - 25a_{30}a_{31} + 125a_{32} > 0$,
- (7) $4a_{30}^5 - 25a_{30}^3a_{31} + 125a_{30}^2a_{32} - 625a_{30}a_{33} + 3125a_{34} \leq 0$,

then

$$|f_3'(t)| \leq \frac{31a_{30}}{5}, \quad (2.18)$$

$$|f_3''(t)| \leq \frac{1}{25}(-52a_{30}^2 + 400a_{31}), \quad (2.19)$$

$$|f_3'''(t)| \leq \frac{1}{125}(-316a_{30}^3 + 1500a_{30}a_{31} + 2000a_{32}), \quad (2.20)$$

$$|f_3^{(4)}(t)| \leq \frac{1}{625}(-1031a_{30}^4 + 2400a_{30}^2a_{31} + 10750a_{30}a_{32} + 10000a_{31}^2 + 10000a_{33}) \quad (2.21)$$

and

$$|f_3^{(5)}(t)| \leq \frac{1}{625}(-1031a_{30}^5 + 820a_{30}^3a_{31} + 17500a_{30}a_{31}^2 + 9450a_{30}^2a_{32} + 13875a_{30}a_{33} + 20000a_{31}a_{32} + 10000a_{34}) \quad (2.22)$$

for all t , where $a_{30} = K_3(0)$, $a_{31} = K_3'(0)$, $a_{32} = K_3''(0)$, $a_{33} = K_3'''(0)$ and $a_{34} = K_3^{(4)}(0)$.

Proof. It is immediate by Theorem 3 of [8] that the inequality $|f_3(t)| \leq 8$ is held for all t . If the function g is chosen as $g(t) = e^{-\gamma t}$ ($\gamma = \gamma_3 = a_{30}/5$) in the Equivalence Theorem, we get the equivalent equation to (2.17) as

$$f_3(t) = e^{-\gamma t} - \int_0^t L_3(t-\tau) f_3(\tau) d\tau = e^{-\gamma t} - L_3 * f_3, \quad (2.23)$$

where

$$L_3(t) = (a_{30} - \gamma)e^{-\gamma t} + K_3' * e^{-\gamma t}.$$

Thus,

$$\begin{aligned} L_3'(t) &= (\gamma^2 - a_{30}\gamma + a_{31})e^{-\gamma t} + K_3'' * e^{-\gamma t}, \\ L_3''(t) &= (-\gamma^3 + a_{30}\gamma^2 - a_{31}\gamma + a_{32})e^{-\gamma t} + K_3''' * e^{-\gamma t}, \\ L_3'''(t) &= (\gamma^4 - a_{30}\gamma^3 + a_{31}\gamma^2 - a_{32}\gamma + a_{33})e^{-\gamma t} + K_3^{(4)} * e^{-\gamma t} \end{aligned}$$

and

$$L_3^{(4)}(t) = (-\gamma^5 + a_{30}\gamma^4 - a_{31}\gamma^3 + a_{32}\gamma^2 - a_{33}\gamma + a_{34})e^{-\gamma t} + K_3^{(5)} * e^{-\gamma t}.$$

It is known that L_3 satisfies all of the conditions of Theorem 2 by the proof of Theorem 3 in [8]. Hence, the first four derivatives of the solution of the equation $h_3(t) = 1 - L_3 * h_3$ verify the inequalities

$$|h_3'(t)| \leq \frac{15L_3(0)}{4} = 3a_{30},$$

$$|h_3''(t)| \leq \frac{1}{8} [-(L_3(0))^2 + 64L_3'(0)] = \frac{1}{25}(-34a_{30}^2 + 200a_{31}),$$

$$\begin{aligned} |h_3'''(t)| &\leq \frac{1}{64} [-45(L_3(0))^3 + 544L_3(0)L_3'(0) + 512L_3''(0)] \\ &= \frac{1}{125}(-149a_{30}^3 + 650a_{30}a_{31} + 1000a_{32}) \end{aligned}$$

and

$$\begin{aligned} |h_3^{(4)}(t)| &\leq \frac{1}{64} [-45(L_3(0))^4 + 536(L_3(0))^2L_3'(0) + 752L_3(0)L_3''(0) + \\ &\quad + 512(L_3'(0))^2 + 512L_3'''(0)] \\ &= \frac{1}{625}(-432a_{30}^4 + 775a_{30}^2a_{31} + 4875a_{30}a_{32} + 5000a_{31}^2 + 5000a_{33}) \end{aligned}$$

for all t and $\gamma = \gamma_3$ from Theorem 2. By the Convolution Theorem, the solution f_3 of the equivalent equation (2.23) can be written by means of h_3 of the form $f_3(t) = h_3(t) - \gamma h_3 * e^{-\gamma t}$. Thus,

$$\begin{aligned} f_3'(t) &= h_3'(t) - \gamma e^{-\gamma t} - \gamma h_3' * e^{-\gamma t}, \\ f_3''(t) &= h_3''(t) + \gamma^2 e^{-\gamma t} - \gamma h_3'(0) e^{-\gamma t} - \gamma h_3'' * e^{-\gamma t}, \\ f_3'''(t) &= h_3'''(t) - \gamma^3 e^{-\gamma t} + \gamma^2 h_3'(0) e^{-\gamma t} - \gamma h_3''(0) e^{-\gamma t} - \gamma h_3''' * e^{-\gamma t} \end{aligned}$$

and

$$\begin{aligned} f_3^{(4)}(t) &= h_3^{(4)}(t) + \gamma^4 e^{-\gamma t} - \gamma^3 h_3'(0) e^{-\gamma t} + \gamma^2 h_3''(0) e^{-\gamma t} - \gamma h_3'''(0) e^{-\gamma t} - \\ &\quad - \gamma h_3^{(4)} * e^{-\gamma t}. \end{aligned}$$

By taking $\gamma = \gamma_3 = a_{30}/5$, inequalities (2.18), (2.19), (2.20) and

$$\begin{aligned} |f_3^{(4)}(t)| &\leq \frac{1}{625}(-1031a_{30}^4 + 2400a_{30}^2a_{31} + 10000a_{31}^2 + \\ &\quad + 10750a_{30}a_{32} + 10000a_{33}) \\ &= b_{34} \end{aligned} \tag{2.24}$$

are obtained for all t . On the other hand, differentiation of the equivalent equation (2.23) leads to

$$f_3^{(4)}(t) = \gamma^4 e^{-\gamma t} - L_3(0)f_3'''(t) - L_3'(0)f_3''(t) - L_3''(0)f_3'(t) - L_3'''(0)f_3(t) - L_3^{(4)} * f_3.$$

Taking $\gamma = \gamma_3 = a_{30}/5$, the inequality

$$\begin{aligned} |f_3^{(4)}(t)| &\leq \frac{1}{625}(-995a_{30}^4 + 2725a_{30}^2a_{31} + 9875a_{30}a_{32} + \\ &\quad + 10000a_{31}^2 + 10000a_{33}) \\ &= c_{34} \end{aligned} \tag{2.25}$$

is obtained for all t from (2.18), (2.19), (2.20) and $|f_3(t)| \leq 8$. Thus, from (2.24) and (2.25), we have (2.21) for all t by conditions (4) and (3) of Theorem 3. From (2.17),

$$f_3^{(5)}(t) = -a_{30}f_3^{(4)}(t) - a_{31}f_3'''(t) - a_{32}f_3''(t) - a_{33}f_3'(t) - a_{34}f_3(t) - K_3^{(5)} * f_3.$$

By inequalities (2.18), (2.19), (2.20), (2.21) obtained above and $|f_3(t)| \leq 8$, we get (2.22) for all t and this concludes the proof of Theorem 3. $\square$

Theorem 4. Consider the equation

$$f_4(t) = 1 - \int_0^t K_4(t - \tau)f_4(\tau)d\tau = 1 - K_4 * f_4. \tag{2.26}$$

If

- (1) $K_4 \in C^6[0, \infty)$, $K_4(t) > 0$, $K_4'(t) > 0, \dots, K_4^{(5)}(t) > 0$ and $K_4^{(6)}(t) \leq 0$ for all t ,
- (2) $a_{40}^2 < \frac{5}{2}a_{41}$,
- (3) $\frac{12}{5}a_{41} \leq a_{40}^2$,
- (4) $5a_{40}^3 - 18a_{40}a_{41} + 27a_{42} \leq 0$,
- (5) $-5a_{40}^4 + 24a_{40}^2a_{41} - 72a_{40}a_{42} + 144a_{43} \leq 0$,
- (6) $7a_{40}^3 - 36a_{40}a_{41} + 108a_{42} > 0$,
- (7) $a_{40}^5 - 6a_{40}^3a_{41} + 27a_{40}^2a_{42} - 108a_{40}a_{43} + 324a_{44} \leq 0$,
- (8) $5a_{40}^3 - 36a_{40}a_{41} + 216a_{42} > 0$,
- (9) $-5a_{40}^4 + 36a_{40}^2a_{41} - 216a_{40}a_{42} + 1296a_{43} > 0$,
- (10) $-5a_{40}^6 + 36a_{40}^4a_{41} - 216a_{40}^3a_{42} + 1296a_{40}^2a_{43} - 7776a_{40}a_{44} + 46656a_{45} \leq 0$,
- then

$$|f_4'(t)| \leq \frac{21a_{40}}{2}, \quad (2.27)$$

$$|f_4''(t)| \leq \frac{1}{9}(-58a_{40}^2 + 288a_{41}), \quad (2.28)$$

$$|f_4'''(t)| \leq \frac{1}{54}(-293a_{40}^3 + 936a_{40}a_{41} + 1728a_{42}), \quad (2.29)$$

$$|f_4^{(4)}(t)| \leq \frac{1}{648}(-1079a_{40}^4 - 3744a_{40}^2a_{41} + 20736a_{41}^2 + 16848a_{40}a_{42} + 20736a_{43}), \quad (2.30)$$

$$|f_4^{(5)}(t)| \leq \frac{1}{7776}(-5117a_{40}^5 - 88992a_{40}^3a_{41} + 300672a_{40}a_{41}^2 + 71712a_{40}^2a_{42} + 266976a_{40}a_{43} + 497664a_{41}a_{42} + 248832a_{44}) \quad (2.31)$$

and

$$|f_4^{(6)}(t)| \leq \frac{1}{7776}(-5117a_{40}^6 - 101940a_{40}^4a_{41} + 255744a_{40}^2a_{41}^2 + 29520a_{40}^3a_{42} + 216864a_{40}^2a_{43} + 834624a_{40}a_{41}a_{42} + 330480a_{40}a_{44} + 248832a_{41}^3 + 497664a_{41}a_{43} + 248832a_{42}^2 + 248832a_{45}) \quad (2.32)$$

for all t , where $a_{40} = K_4(0)$, $a_{41} = K_4'(0)$, $a_{42} = K_4''(0)$, $a_{43} = K_4'''(0)$, $a_{44} = K_4^{(4)}(0)$ and $a_{45} = K_4^{(5)}(0)$.

Proof. It is easily seen by Theorem 4 of [8] that f_4 satisfies the inequality $|f_4(t)| \leq 16$ for all t . By choosing $g(t) = e^{-\gamma t}$ and $\gamma = \gamma_4 = a_{40}/6$ in the Equivalence Theorem, we get the equivalent equation to (2.26) as

$$f_4(t) = e^{-\gamma t} - \int_0^t L_4(t - \tau) f_4(\tau) d\tau = e^{-\gamma t} - L_4 * f_4, \quad (2.33)$$

where

$$L_4(t) = (a_{40} - \gamma)e^{-\gamma t} + K_4' * e^{-\gamma t}.$$

Thus,

$$\begin{aligned} L_4'(t) &= (\gamma^2 - a_{40}\gamma + a_{41})e^{-\gamma t} + K_4'' * e^{-\gamma t}, \\ L_4''(t) &= (-\gamma^3 + a_{40}\gamma^2 - a_{41}\gamma + a_{42})e^{-\gamma t} + K_4''' * e^{-\gamma t}, \\ L_4'''(t) &= (\gamma^4 - a_{40}\gamma^3 + a_{41}\gamma^2 - a_{42}\gamma + a_{43})e^{-\gamma t} + K_4^{(4)} * e^{-\gamma t}, \\ L_4^{(4)}(t) &= (-\gamma^5 + a_{40}\gamma^4 - a_{41}\gamma^3 + a_{42}\gamma^2 - a_{43}\gamma + a_{44})e^{-\gamma t} + K_4^{(5)} * e^{-\gamma t} \end{aligned}$$

and

$$L_4^{(5)}(t) = (\gamma^6 - a_{40}\gamma^5 + a_{41}\gamma^4 - a_{42}\gamma^3 + a_{43}\gamma^2 - a_{44}\gamma + a_{45})e^{-\gamma t} + K_4^{(6)} * e^{-\gamma t}.$$

It is known that L_4 satisfies all of the conditions of Theorem 3 from the proof of Theorem 4 in [8]. Hence, the first five derivatives of the solution of the equation $h_4(t) = 1 - L_4 * h_4$ verify the inequalities

$$|h_4'(t)| \leq \frac{31L_4(0)}{5} = \frac{31a_{40}}{6},$$

$$|h_4''(t)| \leq \frac{1}{25} [-52(L_4(0))^2 + 400L_4'(0)] = \frac{1}{3}(-11a_{40}^2 + 48a_{41}),$$

$$\begin{aligned} |h_4'''(t)| &\leq \frac{1}{125} [-316(L_4(0))^3 + 1500L_4(0)L_4'(0) + 2000L_4''(0)] \\ &= \frac{1}{27}(-67a_{40}^3 + 198a_{40}a_{41} + 432a_{42}), \end{aligned}$$

$$\begin{aligned} |h_4^{(4)}(t)| &\leq \frac{1}{625} [-1031(L_4(0))^4 + 2400(L_4(0))^2L_4'(0) + \\ &\quad + 10750L_4(0)L_4''(0) + 10000(L_4'(0))^2 + 10000L_4'''(0)] \\ &= \frac{1}{1296}(-761a_{40}^4 - 4824a_{40}^2a_{41} + 15120a_{40}a_{42} + \\ &\quad + 20736a_{41}^2 + 20736a_{43}) \end{aligned}$$

and

$$\begin{aligned}
 |h_4^{(5)}(t)| &\leq \frac{1}{625}[-1031(L_4(0))^5 + 820(L_4(0))^3 L_4'(0) + 17500L_4(0)(L_4'(0))^2 + \\
 &\quad + 9450(L_4(0))^2 L_4''(0) + 13875L_4(0)L_4'''(0) + 20000L_4'(0)L_4''(0) + \\
 &\quad + 10000L_4^{(4)}(0)] \\
 &= \frac{1}{648}(-155a_{40}^5 - 3597a_{40}^3 a_{41} + 11664a_{40}a_{41}^2 + 2214a_{40}^2 a_{42} + \\
 &\quad + 10260a_{40}a_{43} + 20736a_{41}a_{42} + 10368a_{44})
 \end{aligned}$$

for all t and $\gamma = \gamma_4$ by Theorem 3. By the Convolution Theorem, the solution f_4 of the equivalent equation (2.33) can be written by means of h_4 as $f_4(t) = h_4(t) - \gamma h_4 * e^{-\gamma t}$. Thus,

$$\begin{aligned}
 f_4'(t) &= h_4'(t) - \gamma e^{-\gamma t} - \gamma h_4' * e^{-\gamma t}, \\
 f_4''(t) &= h_4''(t) + \gamma^2 e^{-\gamma t} - \gamma h_4'(0)e^{-\gamma t} - \gamma h_4'' * e^{-\gamma t}, \\
 f_4'''(t) &= h_4'''(t) - \gamma^3 e^{-\gamma t} + \gamma^2 h_4'(0)e^{-\gamma t} - \gamma h_4''(0)e^{-\gamma t} - \gamma h_4''' * e^{-\gamma t}, \\
 f_4^{(4)}(t) &= h_4^{(4)}(t) + \gamma^4 e^{-\gamma t} - \gamma^3 h_4'(0)e^{-\gamma t} + \gamma^2 h_4''(0)e^{-\gamma t} - \gamma h_4'''(0)e^{-\gamma t} - \\
 &\quad - \gamma h_4^{(4)} * e^{-\gamma t}
 \end{aligned}$$

and

$$\begin{aligned}
 f_4^{(5)}(t) &= h_4^{(5)}(t) - \gamma^5 e^{-\gamma t} + \gamma^4 h_4'(0)e^{-\gamma t} - \gamma^3 h_4''(0)e^{-\gamma t} + \\
 &\quad + \gamma^2 h_4'''(0)e^{-\gamma t} - \gamma h_4^{(4)}(0)e^{-\gamma t} - \gamma h_4^{(5)} * e^{-\gamma t}.
 \end{aligned}$$

Therefore, by taking $\gamma = \gamma_4 = a_{40}/5$, inequalities (2.27), (2.28), (2.29), (2.30) and

$$\begin{aligned}
 |f_4^{(5)}(t)| &\leq \frac{1}{7776}(-5117a_{40}^5 - 88992a_{40}^3 a_{41} + 300672a_{40}a_{41}^2 + 71712a_{40}^2 a_{42} + \\
 &\quad + 266976a_{40}a_{43} + 497664a_{41}a_{42} + 248832a_{44}) \\
 &= b_{45}
 \end{aligned} \tag{2.34}$$

are obtained for all t .

On the other hand, differentiation of the equivalent equation (2.33) leads to

$$\begin{aligned}
 f_4^{(5)}(t) &= -\gamma^5 e^{-\gamma t} - L_4(0)f_4^{(4)}(t) - L_4'(0)f_4'''(t) - L_4''(0)f_4''(t) - \\
 &\quad - L_4'''(0)f_4'(t) - L_4^{(4)}(0)f_4(t) - L_4^{(5)} * f_4.
 \end{aligned}$$

Taking $\gamma = \gamma_4 = a_{40}/5$, the inequality

$$\begin{aligned}
 |f_4^{(5)}(t)| &\leq \frac{1}{7776}(-6244a_{40}^5 - 83124a_{40}^3 a_{41} + 300672a_{40}a_{41}^2 + 77112a_{40}^2 a_{42} + \\
 &\quad + 247536a_{40}a_{43} + 497664a_{41}a_{42} + 248832a_{44}) \\
 &= c_{45}
 \end{aligned} \tag{2.35}$$

is obtained for all t from inequalities (2.27), (2.28), (2.29), (2.30) and $|f_4(t)| \leq 16$. Hence, we have by (2.34) and (2.35) that inequality (2.31) holds for all t from conditions (5), (4) and (2) of Theorem 4.

Finally, from (2.26), we get

$$\begin{aligned} f_4^{(6)}(t) = & -a_{40}f_4^{(5)}(t) - a_{41}f_4^{(4)}(t) - a_{42}f_4'''(t) - a_{43}f_4''(t) - a_{44}f_4'(t) - \\ & -a_{45}f_4(t) - K_4^{(6)} * f_4. \end{aligned}$$

By inequalities (2.27), (2.28), (2.29), (2.30), (2.31) obtained above and $|f_4(t)| \leq 16$, we derive inequality (2.32) for all t and this completes the proof of Theorem 4. $\square$

By continuing this process for $n \in \mathbb{N}_2$, we have Theorem n which can be stated as follows:

Theorem n. *Let us consider the equation*

$$f_n(t) = 1 - \int_0^t K_n(t - \tau) f_n(\tau) d\tau = 1 - K_n * f_n \quad (2.36)$$

under the following assumptions:

- (1) $K_n \in C^{n+2}[0, \infty)$, $K_n(t) > 0$, $K_n'(t) > 0, \dots, K_n^{(n+1)}(t) > 0$ and $K_n^{(n+2)}(t) \leq 0$ for all t ,
- (2) $a_{n0}^2 < \frac{2(n+1)}{n} a_{n1}$ and
- (3) $\frac{2(n+2)}{n+1} a_{n1} \leq a_{n0}^2$.

Furthermore, conditions (4) th, (5) th, $\dots$, $\left[4 + \frac{(n-2)(n-1)}{2}\right]$ th of Theorem n are the inequalities obtained by taking $(a_{n0} - \gamma_n)$, $(\gamma_n^2 - a_{n0}\gamma_n + a_{n1})$, $\dots$, $((-1)^{n+1}\gamma_n^{n+1} + (-1)^n a_{n0}\gamma_n^n + (-1)^{n-1} a_{n1}\gamma_n^{n-1} + \dots - a_{n(n-1)}\gamma_n + a_{nn})$, respectively instead of the constants $a_{(n-1)0}$, $a_{(n-1)1}$, $\dots$, $a_{(n-1)n}$ in conditions (4) th, (5) th, $\dots$, $\left[4 + \frac{(n-2)(n-1)}{2}\right]$ th of Theorem $(n-1)$ and by writing $\gamma_n = \frac{a_{n0}}{n+2}$. Moreover, conditions $\left[5 + \frac{(n-2)(n-1)}{2}\right]$ th, $\left[6 + \frac{(n-2)(n-1)}{2}\right]$ th, $\dots$, $\left[5 + \frac{n(n-1)-4}{2}\right]$ th and $\left[5 + \frac{n(n-1)-2}{2}\right]$ th of Theorem n are also the inequalities $p_3(\gamma_n)$, $p_4(\gamma_n)$, $\dots$, $p_n(\gamma_n) > 0$ and $p_{n+2}(\gamma_n) \leq 0$, respectively. Besides, let us define the first constant x_{n1} as

$$x_{n1} = z_n a_{n0},$$

where

$$z_1 = \frac{7}{3}, \quad z_n = \frac{2(n+1)z_{n-1} + 1}{n+2}, \quad (n \geq 2).$$

The constants $x_{n2}, x_{n3}, \dots, x_{n(n+2)}$ are defined as follows:

$$x_{n2} = 2y_{n2} + \gamma_n^2 + \gamma_n y_{n1},$$

where y_{n1} has been obtained by taking $p_1(\gamma_n)$ instead of the constant $a_{(n-1)0}$ which is included at $x_{(n-1)1}$ which is the bound of f'_{n-1} in Theorem $(n-1)$ and y_{n2} has been got by taking $p_1(\gamma_n), p_2(\gamma_n)$, respectively instead of the constants $a_{(n-1)0}, a_{(n-1)1}$ which are included at $x_{(n-1)2}$ which is the bound of f''_{n-1} in Theorem $(n-1)$.

$$x_{n3} = 2y_{n3} + \gamma_n^3 + \gamma_n^2 y_{n1} + \gamma_n y_{n2},$$

where y_{n3} has been got by taking $p_1(\gamma_n), p_2(\gamma_n), p_3(\gamma_n)$, respectively instead of the constants $a_{(n-1)0}, a_{(n-1)1}, a_{(n-1)2}$ which are included at $x_{(n-1)3}$ which is the bound of f'''_{n-1} in Theorem $(n-1)$.

Similarly,

$$x_{nn} = 2y_{nn} + \gamma_n^n + \gamma_n^{n-1} y_{n1} + \gamma_n^{n-2} y_{n2} + \dots + \gamma_n y_{n(n-1)},$$

where y_{nn} has been obtained by taking $p_1(\gamma_n), p_2(\gamma_n), \dots, p_n(\gamma_n)$, respectively instead of the constants $a_{(n-1)0}, a_{(n-1)1}, \dots, a_{(n-1)(n-1)}$ which are included at $x_{(n-1)n}$ which is the bound of $f_{n-1}^{(n)}$ in Theorem $(n-1)$.

Additionally, let us define the constant $x_{n(n+1)}$ by

$$x_{n(n+1)} = \min\{b_{n(n+1)}, c_{n(n+1)}\}$$

such that

$$b_{n(n+1)} = 2y_{n(n+1)} + \gamma_n^{n+1} + \gamma_n^n y_{n1} + \gamma_n^{n-1} y_{n2} + \dots + \gamma_n y_{nn},$$

where $y_{n(n+1)}$ has been got by taking $p_1(\gamma_n), p_2(\gamma_n), \dots, p_{n+1}(\gamma_n)$, respectively instead of the constants $a_{(n-1)0}, a_{(n-1)1}, \dots, a_{(n-1)n}$ which are included at $x_{(n-1)(n+1)}$ which is the bound of $f_{n-1}^{(n+1)}$ in Theorem $(n-1)$ and

$$c_{n(n+1)} = \gamma_n^{n+1} + p_1(\gamma_n)x_{nn} + p_2(\gamma_n)x_{n(n-1)} + \dots + p_n(\gamma_n)x_{n1} + 2^{n+1}p_{n+1}(\gamma_n).$$

Finally, the last constant $x_{n(n+2)}$ is defined by

$$x_{n(n+2)} = a_{n0}x_{n(n+1)} + a_{n1}x_{nn} + \dots + a_{nn}x_{n1} + 2^{n+1}a_{n(n+1)},$$

where

$$p_i(\gamma) = (-1)^i \gamma^i + (-1)^{i+1} a_{n0} \gamma^{i-1} + (-1)^{i+2} a_{n1} \gamma^{i-2} + \cdots - a_{n(i-2)} \gamma + a_{n(i-1)}$$

for $i \in \{1, 2, \dots, n+2\}$ and $a_{n0} = K_n(0)$, $a_{n1} = K'_n(0)$, $a_{n2} = K''_n(0)$, $\dots$, $a_{n(n+1)} = K_n^{(n+1)}(0)$. Then, the solution of equation (2.36) satisfies the inequalities

$$|f'_n(t)| \leq x_{n1}, |f''_n(t)| \leq x_{n2}, \dots, |f_n^{(n+1)}(t)| \leq x_{n(n+1)} \text{ and } |f_n^{(n+2)}(t)| \leq x_{n(n+2)}$$

for all t and $n \in \mathbb{N}_2$.

Proof. For this, we use the mathematical induction. It is conclude that Theorem n is true for $n = 2$. Let us suppose that Theorem m is true. Thus, we observe under the conditions of Theorem m that $|f'_m(t)| \leq x_{m1}$, $|f''_m(t)| \leq x_{m2}$, $\dots$, $|f_m^{(m+1)}(t)| \leq x_{m(m+1)}$ and $|f_m^{(m+2)}(t)| \leq x_{m(m+2)}$ for all t . In this case, we try to show that Theorem n is also true for $n = m+1$. Namely, by considering the conditions of Theorem $(m+1)$, we must show that the solution of the equation

$$f_{m+1}(t) = 1 - K_{m+1} * f_{m+1} \quad (2.37)$$

satisfies also the inequalities $|f'_{m+1}(t)| \leq x_{(m+1)1}$, $|f''_{m+1}(t)| \leq x_{(m+1)2}$, $\dots$, $|f_{m+1}^{(m+2)}(t)| \leq x_{(m+1)(m+2)}$ and $|f_{m+1}^{(m+3)}(t)| \leq x_{(m+1)(m+3)}$ for all t . We can immediately assert that $f_{m+1}(t)$ provides the inequality $|f_{m+1}(t)| \leq 2^{m+1}$ for all t from [8, Theorem n].

By choosing $\gamma = \gamma_{m+1} = a_{(m+1)0}/(m+3)$ and $g(t) = e^{-\gamma t}$ in the Equivalence Theorem, we get the equivalent equation to (2.37) of the form

$$f_{m+1}(t) = e^{-\gamma t} - L_{m+1} * f_{m+1}, \quad (2.38)$$

where

$$L_{m+1}(t) = (a_{(m+1)0} - \gamma)e^{-\gamma t} + K'_{m+1} * e^{-\gamma t}.$$

Thus,

$$\begin{aligned} L'_{m+1}(t) &= (\gamma^2 - a_{(m+1)0}\gamma + a_{(m+1)1})e^{-\gamma t} + K''_{m+1} * e^{-\gamma t}, \\ L''_{m+1}(t) &= (-\gamma^3 + a_{(m+1)0}\gamma^2 - a_{(m+1)1}\gamma + a_{(m+1)2})e^{-\gamma t} + K'''_{m+1} * e^{-\gamma t} \end{aligned}$$

and similarly,

$$\begin{aligned} L_{m+1}^{(m+1)}(t) &= ((-1)^{m+2}\gamma^{m+2} + (-1)^{m+1}a_{(m+1)0}\gamma^{m+1} + (-1)^m a_{(m+1)1}\gamma^m + \\ &\quad + \cdots - a_{(m+1)m}\gamma + a_{(m+1)(m+1)})e^{-\gamma t} + K_{m+1}^{(m+2)} * e^{-\gamma t} \end{aligned}$$

and

$$L_{m+1}^{(m+2)}(t) = ((-1)^{m+3}\gamma^{m+3} + (-1)^{m+2}a_{(m+1)0}\gamma^{m+2} + (-1)^{m+1}a_{(m+1)1}\gamma^{m+1} + \dots - a_{(m+1)(m+1)}\gamma + a_{(m+1)(m+2)})e^{-\gamma t} + K_{m+1}^{(m+3)} * e^{-\gamma t}.$$

It is known that the kernel L_{m+1} of equation (2.38) satisfies all of the conditions of Theorem m by the proof of Theorem n of [8]. Hence, by using Theorem m , it is understood that the first derivative of the solution of the equation

$$h_{m+1}(t) = 1 - L_{m+1} * h_{m+1} \quad (2.39)$$

clearly satisfies the inequality

$$|h'_{m+1}(t)| \leq z_m L_{m+1}(0) = z_m(a_{(m+1)0} - \gamma_{m+1}) = \left(\frac{m+2}{m+3}\right) z_m a_{(m+1)0} = y_{(m+1)1}$$

for all t . On the other hand, the new constants $y_{(m+1)2}, y_{(m+1)3}, \dots, y_{(m+1)(m+1)}$ are defined as follows:

We derive the constant $y_{(m+1)2}$ by taking $L_{m+1}(0) = p_1(\gamma_{m+1})$ and $L'_{m+1}(0) = p_2(\gamma_{m+1})$, respectively instead of the constants a_{m0} and a_{m1} in x_{m2} which is the bound of f_m'' in Theorem m . We get the constant $y_{(m+1)3}$ by taking $L_{m+1}(0) = p_1(\gamma_{m+1})$, $L'_{m+1}(0) = p_2(\gamma_{m+1})$ and $L''_{m+1}(0) = p_3(\gamma_{m+1})$, respectively instead of the constants a_{m0}, a_{m1} and a_{m2} in x_{m3} which is the bound of f_m''' in Theorem m . Continuing this process, we have the last constant $y_{(m+1)(m+2)}$ by taking $L_{m+1}(0) = p_1(\gamma_{m+1})$, $L'_{m+1}(0) = p_2(\gamma_{m+1}), \dots, L_{m+1}^{(m)}(0) = p_{m+1}(\gamma_{m+1})$ and $L_{m+1}^{(m+1)}(0) = p_{m+2}(\gamma_{m+1})$, respectively instead of the constants $a_{m0}, a_{m1}, \dots, a_{mm}$ and $a_{m(m+1)}$ in $x_{m(m+2)}$ which is the bound of $f_m^{(m+2)}$ in Theorem m . Thus, h_{m+1} provides the inequalities

$$|h''_{m+1}(t)| \leq y_{(m+1)2}, |h'''_{m+1}(t)| \leq y_{(m+1)3}, \dots, |h^{(m+2)}_{m+1}(t)| \leq y_{(m+1)(m+2)}$$

for all t from Theorem m . By Theorem B , the solution f_{m+1} of the equivalent equation (2.38) can be written by means of the solution h_{m+1} of (2.39) of the form $f_{m+1}(t) = h_{m+1}(t) - \gamma h_{m+1} * e^{-\gamma t}$. Thus,

$$\begin{aligned} f'_{m+1}(t) &= h'_{m+1}(t) - \gamma e^{-\gamma t} - \gamma h'_{m+1} * e^{-\gamma t}, \\ f''_{m+1}(t) &= h''_{m+1}(t) + \gamma^2 e^{-\gamma t} - \gamma h'_{m+1}(0) e^{-\gamma t} - \gamma h''_{m+1} * e^{-\gamma t}, \\ f'''_{m+1}(t) &= h'''_{m+1}(t) - \gamma^3 e^{-\gamma t} + \gamma^2 h'_{m+1}(0) e^{-\gamma t} - \gamma h''_{m+1}(0) e^{-\gamma t} - \gamma h'''_{m+1} * e^{-\gamma t} \end{aligned}$$

and continuing this process, we get the equality

$$\begin{aligned} f_{m+1}^{(m+2)}(t) &= h_{m+1}^{(m+2)}(t) + (-1)^{m+2}\gamma^{m+2}e^{-\gamma t} + (-1)^{m+1}\gamma^{m+1}h'_{m+1}(0)e^{-\gamma t} + \\ &+ (-1)^m\gamma^m h''_{m+1}(0)e^{-\gamma t} + \dots - \gamma h_{m+1}^{(m+1)}(0)e^{-\gamma t} - \gamma h_{m+1}^{(m+2)} * e^{-\gamma t}. \end{aligned}$$

Thus by taking $\gamma = \gamma_{m+1} = a_{(m+1)0}/(m+3)$, we have the inequalities

$$\begin{aligned} |f'_{m+1}(t)| &\leq |h'_{m+1}(t)| + \gamma + \gamma \int_0^t |h'_{m+1}(t-\tau)| e^{-\gamma\tau} d\tau \\ &\leq y_{(m+1)1} + \gamma + \gamma y_{(m+1)1} \int_0^t e^{-\gamma\tau} d\tau = 2y_{(m+1)1} + \gamma - y_{(m+1)1} e^{-\gamma t} \\ &\leq 2y_{(m+1)1} + \gamma = z_{m+1} a_{(m+1)0} = x_{(m+1)1}, \end{aligned}$$

$$\begin{aligned} |f''_{m+1}(t)| &\leq |h''_{m+1}(t)| + \gamma^2 + \gamma |h'_{m+1}(0)| + \gamma \int_0^t |h''_{m+1}(t-\tau)| e^{-\gamma\tau} d\tau \\ &\leq y_{(m+1)2} + \gamma^2 + \gamma y_{(m+1)1} + \gamma y_{(m+1)2} \int_0^t e^{-\gamma\tau} d\tau \\ &= 2y_{(m+1)2} + \gamma^2 + \gamma y_{(m+1)1} - y_{(m+1)2} e^{-\gamma t} \\ &\leq 2y_{(m+1)2} + \gamma^2 + \gamma y_{(m+1)1} = x_{(m+1)2}, \end{aligned}$$

$$\begin{aligned} |f'''_{m+1}(t)| &\leq |h'''_{m+1}(t)| + \gamma^3 + \gamma^2 |h'_{m+1}(0)| + \gamma |h''_{m+1}(0)| + \\ &\quad + \gamma \int_0^t |h'''_{m+1}(t-\tau)| e^{-\gamma\tau} d\tau \\ &\leq y_{(m+1)3} + \gamma^3 + \gamma^2 y_{(m+1)1} + \gamma y_{(m+1)2} + \gamma y_{(m+1)3} \int_0^t e^{-\gamma\tau} d\tau \\ &= 2y_{(m+1)3} + \gamma^3 + \gamma^2 y_{(m+1)1} + \gamma y_{(m+1)2} - y_{(m+1)3} e^{-\gamma t} \\ &\leq 2y_{(m+1)3} + \gamma^3 + \gamma^2 y_{(m+1)1} + \gamma y_{(m+1)2} = x_{(m+1)3} \end{aligned}$$

for all t and by continuing this process, we have

$$\begin{aligned} |f^{(m+2)}_{m+1}(t)| &\leq |h^{(m+2)}_{m+1}(t)| + \gamma^{m+2} + \gamma^{m+1} |h'_{m+1}(0)| + \gamma^m |h''_{m+1}(0)| + \\ &\quad + \cdots + \gamma |h^{(m+1)}_{m+1}(0)| + \gamma \int_0^t |h^{(m+2)}_{m+1}(t-\tau)| e^{-\gamma\tau} d\tau \\ &\leq y_{(m+1)(m+2)} + \gamma^{m+2} + \gamma^{m+1} y_{(m+1)1} + \gamma^m y_{(m+1)2} + \\ &\quad + \cdots + \gamma y_{(m+1)(m+1)} + \gamma y_{(m+1)(m+2)} \int_0^t e^{-\gamma\tau} d\tau \\ &= 2y_{(m+1)(m+2)} + \gamma^{m+2} + \gamma^{m+1} y_{(m+1)1} + \gamma^m y_{(m+1)2} + \\ &\quad + \cdots + \gamma y_{(m+1)(m+1)} - y_{(m+1)(m+2)} e^{-\gamma t} \\ &\leq 2y_{(m+1)(m+2)} + \gamma^{m+2} + \gamma^{m+1} y_{(m+1)1} + \gamma^m y_{(m+1)2} + \\ &\quad + \cdots + \gamma y_{(m+1)(m+1)} \\ &= b_{(m+1)(m+2)} \end{aligned} \tag{2.40}$$

for all t . On the other hand, differentiation of the equivalent equation (2.38) leads to

$$\begin{aligned} f_{m+1}^{(m+2)}(t) &= (-1)^{m+2} \gamma^{m+2} e^{-\gamma t} - L_{m+1}(0) f_{m+1}^{(m+1)}(t) - L'_{m+1}(0) f_{m+1}^{(m)}(t) - \\ &\quad - L''_{m+1}(0) f_{m+1}^{(m-1)}(t) - \cdots - L_{m+1}^{(m)}(0) f'_{m+1}(t) - \\ &\quad - L_{m+1}^{(m+1)}(0) f_{m+1}(t) - L_{m+1}^{(m+2)} * f_{m+1}. \end{aligned}$$

Taking $\gamma = \gamma_{m+1} = a_{(m+1)0}/(m+3)$, the inequality

$$\begin{aligned} |f_{m+1}^{(m+2)}(t)| &\leq \gamma^{m+2} + L_{m+1}(0) |f_{m+1}^{(m+1)}(t)| + L'_{m+1}(0) |f_{m+1}^{(m)}(t)| + \\ &\quad + L''_{m+1}(0) |f_{m+1}^{(m-1)}(t)| + \cdots + L_{m+1}^{(m)}(0) |f'_{m+1}(t)| + \\ &\quad + L_{m+1}^{(m+1)}(0) |f_{m+1}(t)| + \int_0^t |L_{m+1}^{(m+2)}(t-\tau)| |f_{m+1}(\tau)| d\tau \\ &\leq \gamma^{m+2} + p_1(\gamma) x_{(m+1)(m+1)} + p_2(\gamma) x_{(m+1)m} + \\ &\quad + p_3(\gamma) x_{(m+1)(m-1)} + \cdots + p_{m+1}(\gamma) x_{(m+1)1} + \\ &\quad + p_{m+2}(\gamma) 2^{m+1} - 2^{m+1} \int_0^t L_{m+1}^{(m+2)}(t-\tau) d\tau \\ &= \gamma^{m+2} + p_1(\gamma) x_{(m+1)(m+1)} + p_2(\gamma) x_{(m+1)m} + \\ &\quad + p_3(\gamma) x_{(m+1)(m-1)} + \cdots + p_{m+1}(\gamma) x_{(m+1)1} + \\ &\quad + 2^{m+2} p_{m+2}(\gamma) - 2^{m+1} L_{m+1}^{(m+1)}(t) \\ &\leq \gamma^{m+2} + p_1(\gamma) x_{(m+1)(m+1)} + p_2(\gamma) x_{(m+1)m} + \\ &\quad + p_3(\gamma) x_{(m+1)(m-1)} + \cdots + \\ &\quad + p_{m+1}(\gamma) x_{(m+1)1} + 2^{m+2} p_{m+2}(\gamma) \\ &= c_{(m+1)(m+2)} \end{aligned} \tag{2.41}$$

is held for all t . By (2.40) and (2.41), the bound of $f_{m+1}^{(m+2)}$ is obtained as

$$|f_{m+1}^{(m+2)}(t)| \leq \min \{c_{(m+1)(m+2)}, b_{(m+1)(m+2)}\} = x_{(m+1)(m+2)}$$

for all t . From (2.37),

$$\begin{aligned} f_{m+1}^{(m+3)}(t) &= -a_{(m+1)0} f_{m+1}^{(m+2)}(t) - a_{(m+1)1} f_{m+1}^{(m+1)}(t) - \cdots - \\ &\quad - a_{(m+1)(m+1)} f'_{m+1}(t) - a_{(m+1)(m+2)} f_{m+1}(t) - K_{m+1}^{(m+3)} * f_{m+1}. \end{aligned}$$

By the inequalities obtained above and $|f_{m+1}(t)| \leq 2^{m+1}$, we have

$$\begin{aligned}
|f_{m+1}^{(m+3)}(t)| &\leq a_{(m+1)0}x_{(m+1)(m+2)} + a_{(m+1)1}x_{(m+1)(m+1)} + \cdots + \\
&\quad + a_{(m+1)(m+1)}x_{(m+1)1} + 2^{m+1}a_{(m+1)(m+2)} + \\
&\quad + \int_0^t |K_{m+1}^{(m+3)}(t-\tau)| |f_{m+1}(\tau)| d\tau \\
&\leq a_{(m+1)0}x_{(m+1)(m+2)} + a_{(m+1)1}x_{(m+1)(m+1)} + \cdots + \\
&\quad + a_{(m+1)(m+1)}x_{(m+1)1} + 2^{m+1}a_{(m+1)(m+2)} - 2^{m+1} \int_0^t K_{m+1}^{(m+3)}(\tau) d\tau \\
&= a_{(m+1)0}x_{(m+1)(m+2)} + a_{(m+1)1}x_{(m+1)(m+1)} + \cdots + \\
&\quad + a_{(m+1)(m+1)}x_{(m+1)1} + 2^{m+2}a_{(m+1)(m+2)} - 2^{m+1}K_{m+1}^{(m+2)}(t) \\
&\leq a_{(m+1)0}x_{(m+1)(m+2)} + a_{(m+1)1}x_{(m+1)(m+1)} + \cdots + \\
&\quad + a_{(m+1)(m+1)}x_{(m+1)1} + 2^{m+2}a_{(m+1)(m+2)} \\
&= x_{(m+1)(m+3)}
\end{aligned}$$

for all t . Hence, it concludes that if Theorem n is true for $n = m$, it is also true for $n = m + 1$. That is to say that Theorem n is valid for all $n \in \mathbb{N}_2$. $\square$

The infinitely many numbers of kernels K_n are of the form

$$K_n(t) = c_0 t^{n+1} + c_1 t^n + \cdots + c_{n+2} e^{-t}$$

satisfying all of the conditions of Theorem n were obtained by the method in [8, Example n].

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