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Numerical Solution of the Singular Integral Equations of the First Kind on the Curve

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Abstract. In this work we present a numerical solution for singular integral equations of the first kind on the oriented smooth contour with Cauchy type kernel. For this one we use an adapted quadratic approximation constructed by the author for this goal, based on the Simpson rule. Many examples are treated in order to prove the efficiency of this approximation.

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1 Introduction

A singular integral equation of the first kind with Cauchy kernel has the form

$$A\varphi(t_0) = \frac{1}{\pi i}\int_{\Gamma} \frac{b_0(t_0)\varphi(t)}{t-t_0}dt + \frac{1}{\pi i}\int_{\Gamma} k(t,t_0)\varphi(t)dt = f(t_0), \tag{1}$$

where under Γ we designate an oriented smooth contour in the complex plane of the variable $t = x(s) + iy(s)$, t and t_0 are complex points on Γ, $\varphi(t)$ is the unknown function and $f(t), b_0(t)$ and $k(t,t_0)$ are given functions on Γ, where $f(t)$ is called the right hand side of the equation (1). The first integral of the left hand side must be exist in the sense of the Cauchy principal value for a given density $\varphi(t)$, for this one, we will need more than mere continuity. In other words, the density $\varphi(t)$ has to satisfy the Hölder condition $H(\mu)$[2]. So we note that, singular integral equations of the first kind with Cauchy kernel have an index zero. In particular, injective singular integral operators of the first kind are bijective and have a bounded inverse.

2 The Quadrature

We denote by t the parametric complex function $t(s)$ of the curve Γ defined by

$$t(s) = x(s) + iy(s), \quad a \leq s \leq b,$$

where $x(s)$ and $y(s)$ are continuous functions on the finite interval of definition $[a, b]$ and have continuous first derivatives $x'(s)$ and $y'(s)$ never simultaneously null. Let N be an arbitrary natural number, generally we take it large enough and divide the interval $[a, b]$ into N equal subintervals $I_1, I_2, ..., I_N$ by the points

$$s_\sigma = a + \sigma\frac{l}{N} \ , \ l = b - a \ , \ \sigma = 0, 1, 2, ..., N.$$

Further, we fix a natural number $M > 1$, and divide each of segments $[s_\sigma, s_{\sigma+1}]$ by the equidistant points

$$s_{\sigma k} = s_\sigma + k\frac{h}{2M}, \quad h = \frac{l}{N}, \quad k = 0, 1, ..., 2M.$$

In other words, we have for each subinterval $[s_\sigma, s_{\sigma+1}]$ the following subdivision

$$s_\sigma = s_{\sigma 0} < s_{\sigma 1} < ..... < s_{\sigma 2M} = s_{\sigma+1}.$$

We introduce the notation

$$t_\sigma = t(s_\sigma), \ t_{\sigma k} = t(s_{\sigma k}); \quad \sigma = 0, 1, 2, ..., N; \quad k = 0, 1, ..., 2M.$$

Assuming that, for the indices $\sigma, \nu = 0, 1, 2, ..., N - 1$, the points t and t_0 belong respectively to the arcs $\overset{\frown}{t_\sigma t_{\sigma+1}}$ and $\overset{\frown}{t_\nu t_{\nu+1}}$ where $\overset{\frown}{t_\alpha t_{\alpha+1}}$ designates the smallest arc with ends t_α and $t_{\alpha+1}$ [3],[5],[6] and [7].
Following [6], we define the approximation $\psi_{\sigma\nu}(\varphi; t, t_0)$ for the density $\varphi(t)$ by the following expression

$$\begin{aligned}\psi_{\sigma\nu}(\varphi;t,t_0) &= \varphi(t_0)+\beta_{\sigma\nu}(\varphi;t,t_0)\\ &= \varphi(t_0)+U(\varphi;t,\sigma)-V(\varphi;t_0,\sigma,\nu)\end{aligned} \tag{1}$$

where the expression $\psi_{\sigma\nu}(\varphi;t_0,t)$, designates the approximation of the function density $\varphi(t)$ on the subinterval $[t_\sigma,t_{\sigma+1}]$ of the curve Γ [6], destined for the first integral of the left hand side of the equation (1).
Indeed, for $t_{\sigma k}\le t\le t_{\sigma,k+2}$ we put

$$\begin{aligned}U(\varphi;t,\sigma) &= \frac{(t-t_{\sigma,k+1})(t-t_{\sigma,k+2})}{(t_{\sigma,k+1}-t_{\sigma k})(t_{\sigma,k+2}-t_{\sigma k})}\varphi(t_{\sigma k})\frac{t-t_0}{t_{\sigma k}-t_0}\\ &-\frac{(t-t_{\sigma k})(t-t_{\sigma,k+2})}{(t_{\sigma,k+1}-t_{\sigma k})(t_{\sigma,k+2}-t_{\sigma,k+1})}\varphi(t_{\sigma,k+1})\frac{t-t_0}{t_{\sigma,k+1}-t_0}\\ &+\frac{(t-t_{\sigma k})(t-t_{\sigma,k+1})}{(t_{\sigma,k+2}-t_{\sigma k})(t_{\sigma,k+2}-t_{\sigma,k+1})}\varphi(t_{\sigma,k+2})\frac{t-t_0}{t_{\sigma,k+2}-t_0},\end{aligned}$$

and the function $V(\varphi;t_0,\sigma,\nu)$ is given by

$$\begin{aligned}V(\varphi;t_0,\sigma,\nu) &= \frac{S_2(\varphi;t_0,\nu)(t-t_0)(t-t_{\sigma,k+1})(t-t_{\sigma,k+2})}{(t_{\sigma k}-t_0)(t_{\sigma,k+2}-t_{\sigma k})(t_{\sigma,k+1}-t_{\sigma k})}\\ &-\frac{S_2(\varphi;t_0,\nu)(t-t_0)(t-t_{\sigma k)})(t-t_{\sigma,k+2})}{(t_{\sigma,k+1}-t_0)(t_{\sigma,k+2}-t_{\sigma,k+1})(t_{\sigma,k+1}-t_{\sigma k})}\\ &+\frac{S_2(\varphi;t_0,\nu)(t-t_0)(t-t_{\sigma k)})(t-t_{\sigma,k+1})}{(t_{\sigma,k+2}-t_0)(t_{\sigma,k+2}-t_{\sigma,k+1})(t_{\sigma,k+2}-t_{\sigma k})},\end{aligned}$$

where the function $S_2(\varphi;t_0,\nu)$ represents the piecewise quadratic interpolating polynomial of the function density $\varphi(t_0)$ given by

$$\begin{aligned}S_2(\varphi;t,\nu) &= \frac{(t-t_{\nu,k+1})(t-t_{\nu,k+2})}{(t_{\nu,k+1}-t_{\nu k})(t_{\nu,k+2}-t_{\nu k})}\varphi(t_{\nu k})\\ &-\frac{(t-t_{\nu k})(t-t_{\nu,k+2})}{(t_{\nu,k+1}-t_{\nu k})(t_{\nu,k+2}-t_{\nu,k+1})}\varphi(t_{\nu,k+1})\\ &+\frac{(t-t_{\nu k})(t-t_{\nu,k+1})}{(t_{\nu,k+2}-t_{\nu k})(t_{\nu,k+2}-t_{\nu,k+1})}\varphi(t_{\nu,k+2}).\end{aligned}$$

For the second integral of the left hand side of the equation (1) we use the quadratic spline interpolation of the kernel $k(t_0,t)$ and of the density $\varphi(t)$.

This regular part of the singular integral equation is obtained as

$$\begin{aligned}
K\varphi(t_0) &= \frac{1}{\pi i}\int_\Gamma k(t,t_0)\varphi(t)dt \\
&\simeq \frac{1}{\pi i}\int_\Gamma \widetilde{k}(t,t_0)\widetilde{\varphi}(t)dt \\
&= \frac{1}{\pi i}\sum_{\sigma=0}^{N-1}\sum_{k=0}^{M-1}\int_{t_{\sigma 2k}}^{t_{\sigma,2k+2}} \frac{(t-t_{\sigma,2k+1})(t-t_{\sigma,2k+2})}{(t_{\sigma,2k+1}-t_{\sigma 2k})(t_{\sigma,2k+2}-t_{\sigma 2k})}k(t_{\sigma 2k},t_0)\varphi(t_{\sigma 2k}) \\
&\quad -\frac{(t-t_{\sigma 2k})(t-t_{\sigma,2k+2})}{(t_{\sigma,2k+1}-t_{\sigma 2k})(t_{\sigma,2k+2}-t_{\sigma,2k+1})}k(t_{\sigma,2k+1},t_0)\varphi(t_{\sigma,2k+1})dt \\
&\quad +\frac{(t-t_{\sigma 2k})(t-t_{\sigma,2k+1})}{(t_{\sigma,2k+2}-t_{\sigma 2k})(t_{\sigma,2k+2}-t_{\sigma,2k+1})}k(t_{\sigma,2k+2},t_0)\varphi(t_{\sigma,2k+2})dt. \\
&= \widetilde{K}\widetilde{\varphi}(t_0).
\end{aligned}$$

However, using our approximation for the singular integral of the equation (1) we obtain

$$\begin{aligned}
b_0(t_0)S\varphi(t_0) &= \frac{1}{\pi i}\int_\Gamma \frac{b_0(t_0)\varphi(t)}{t-t_0}dt \\
&= b_0(t_0)\varphi(t_0) + \frac{b_0(t_0)}{\pi i}\int_\Gamma \frac{\varphi(t)-\varphi(t_0)}{t-t_0}dt \\
&\simeq b_0(t_0)\widetilde{\varphi}(t_0) + \frac{b_0(t_0)}{\pi i}\int_\Gamma \frac{\beta_{\sigma\nu}(\varphi;t,t_0)}{t-t_0}dt \\
&= b_0(t_0)\widetilde{\varphi}(t_0) + b_0(t_0)\widetilde{S}_1\widetilde{\varphi}(t_0) \\
&= b_0(t_0)\widetilde{S}\widetilde{\varphi}(t_0).
\end{aligned}$$

Hence, the approximation of the left side hand of the equation (1) noted by$A\varphi(t_0)$ is given by

$$\begin{aligned}
A\varphi(t_0) &= b_0S\varphi(t_0) + K\varphi(t_0) \\
&= b_0(t_0)\varphi(t_0) + b_0(t_0)S_1\varphi(t_0) + K\varphi(t_0) \\
&\simeq b_0(t_0)\widetilde{\varphi}(t_0) + b_0(t_0)\widetilde{S}_1\widetilde{\varphi}(t_0) + \widetilde{K}\widetilde{\varphi}(t_0) \\
&= \widetilde{A}\widetilde{\varphi}(t_0)
\end{aligned}$$

where the function $\widetilde{\varphi}(t)$ denote the approximation solution of the equation (1) obtained by the equality of the functions $\widetilde{A}\widetilde{\varphi}(t_0)$ and $f(t_0)$ at the points $t_{\sigma k},\ \ \sigma = 0,1,..,N;\ k = 0,1,...,2M.$

3 Main Result

Theorem 1. *The singular integral equation* (1) *has a unique solution* $\varphi(t)$ *and its approximate solution* $\widetilde{\varphi}(t)$ *converges to the solution* $\varphi(t)$ *with the following estimation*

$$|\varphi(t) - \widetilde{\varphi}(t)| \le \frac{C_1 \ln(2MN)}{(2MN)^{\mu}} + \frac{C_2}{(MN)^2}, \quad M, N > 1,$$

where the constant C_1 and C_2 *depend only on the curve* Γ *and the Holder constant* μ *of the function* φ.

Proof
We can written the integral equation (1) as

$$A\varphi = (b_0 S + K)\varphi = f,$$

while as an approximating equation, we consider

$$\widetilde{A}\widetilde{\varphi} = (b_0 \widetilde{S} + \widetilde{K})\widetilde{\varphi} = f.$$

It follows from [6] that, for all $\varphi(t)$ in $H(\mu)$ we have

$$\left| S_1\varphi - \widetilde{S}_1\widetilde{\varphi} \right| \le \frac{C_1 \ln(2MN)}{(2MN)^{\mu}},$$

and also it is known that

$$\left| K\varphi - \widetilde{K}\widetilde{\varphi} \right| \le \frac{C_2}{(MN)^2},$$

for all K compact operator and $\varphi \in H(\mu)$.
Therefore, it is easy to see that

$$\begin{aligned} |\varphi - \widetilde{\varphi}| &= \left| (S_1\varphi - \widetilde{S}_1\widetilde{\varphi}) + \frac{1}{b(t_0)}(K\varphi - \widetilde{K}\widetilde{\varphi}) \right| \\ &\le \left| S_1\varphi - \widetilde{S}_1\widetilde{\varphi} \right| + \frac{1}{|b(t_0)|} \left| K\varphi - \widetilde{K}\widetilde{\varphi} \right| \\ |\varphi - \widetilde{\varphi}| &\le \frac{C_1 \ln(2MN)}{(2MN)^{\mu}} + \frac{C_3}{(MN)^2}, \quad M, N > 1. \end{aligned}$$

4 Numerical Experiments

In this section we describe some of the numerical experiments performed in solving the singular integral equations (1). In all cases, the curve Γ designate the unit circle and we chose the right hand side $f(t)$ in such way that we know the exact solution. This exact solution is used only to show that the numerical solution obtained with our approximation is correct.
We apply the algorithms described in [3] and [6] to solve S.I.E. of the first kind and we present results concerning the accuracy of the calculations; in this numerical experiments it is easily to see that the matrix of the system of algebraic equation given by our approximation is invertible, confirmed in [3] and [7].
In each table, φ represents the exact solution given in the sense of the principal value of Cauchy and $\widetilde{\varphi}$ corresponds to the approximate solution produced by the approximation at points values interpolation [3] and [6].

Example 1. *We start with the easiest type, without the regular part*

$$\frac{(t_0+2)}{\pi i}\int_\Gamma \frac{\varphi(t)}{t-t_0}dt = t_0^3 + 2t_0^2 - t_0 - 2$$

where the function $f(t_0)$ is chosen so that the solution $\varphi(t)$ is given by

$$\varphi(t) = t^2 - 1$$

Table 1. The exact principal value of the singular integral, the approximate calculation of the integral and the error for N=10 in Example 1.

Values of points	Exact solution φ	Approximate solution $\widetilde{\varphi}$	Error
9.5106e-001+ 3.0902e-001i	-1.9098e-001+ 5.8779e-001i	-1.9098e-001+ 5.8779e-001i	6.4796e-016
5.8779e-001+ 8.0902e-001i	-1.3090e+000+ 9.5106e-001i	-1.3090e+000+ 9.5106e-001i	1.2561e-015
6.1232e-017+ 1.0000e+000i	-2.0000e+000+ +1.2246e-016i	-2.0000e+000 - -6.6613e-016i	1.3618e-015
-5.8779e-001+ 8.0902e-001i	-1.3090e+000 - 9.5106e-001i	-1.3090e+000 - 9.5106e-001i	4.4409e-016
-9.5106e-001+ 3.0902e-001i	-1.9098e-001 - 5.8779e-001i	-1.9098e-001- 5.8779e-001i	6.9389e-016

Example 2. *Consider the singular integral equation, without the regular part*

$$\frac{t_0}{\pi i}\int_\Gamma \frac{\varphi(t)}{t-t_0}dt = \frac{t_0}{t_0+3}$$

where the function $f(t_0)$ is chosen so that the solution $\varphi(t)$ is given by

$$\varphi(t) = \frac{1}{t+3}$$

Table 2. The exact principal value of the singular integral, the approximate calculation of the integral and the error for N=30 in Example 2.

Values of points	Exact solution φ	Approximate solution $\widetilde{\varphi}$	Error
8.6603e-001+ 5.0000e-001i	2.5441e-001 - 3.2903e-002i	2.5440e-001 - 3.2903e-002i	3.7716e-006
4.0674e-001+ 9.1355e-001i	2.7384e-001 - 7.3434e-002i	2.7384e-001 - 7.3438e-002i	5.2134e-006
-2.0791e-001+ 9.7815e-001i	3.1900e-001 - 1.1176e-001i	3.1901e-001 - 1.1176e-001i	9.8266e-006
-7.4314e-001+ 6.6913e-001i	4.0729e-001 - 1.2076e-001i	4.0729e-001 - 1.2073e-001i	2.3608e-005
-9.9452e-000+ 1.0453e-001i	4.9728e-001 - 2.5919e-002i	4.9726e-001 - 2.5961e-002i	4.7172e-005

5 Conclusion

We have considered the numerical solution of singular integral equations and have presented an efficient scheme to compute this singular integrals. The essential idea is to find a combination of functions of approximation for the function density where we can be used it to remove integrable singularities. The regular part where it is the remaining integrands are well behaved and pose no serious numerical problem. Typical examples taken from the literature with known closed form solutions, were used to illustrate the stability and convergence of the approach. The stability of the numerical solution was verified by comparing the analytical and numerical solutions which agree well.

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