Decompositions of $\tau_{\mathcal{G}}$ -Continuity and Continuity

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Abstract. In this paper, we introduce and investigate the notion of weakly $\mathcal{G}$ -locally closed sets in a topological space with a grill. Furthermore, by using these sets, we obtain new decompositions of continuity.

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1 Introduction

The idea of grills on a topological space was first introduced by Choquet [6]. The concept of grills has shown to be a powerful supporting and useful tool like nets and filters, for getting a deeper insight into further studying some topological notions such as proximity spaces, closure spaces and the theory of compactifications and extension problems of different kinds (see [5], [4], [18] for details). In [17], Roy and Mukherjee defined and studied a typical topology associated rather naturally to the existing topology and a grill on a given topological space. Quite recently, Hatir and Jafari [9] defined new classes of sets and obtained a new decomposition of continuity in terms of grills. In this paper, we introduce and investigate the notion of weakly $\mathcal{G}$ -locally closed sets in a topological space with a grill. Furthermore, by using these sets, we obtain new decompositions of continuity.

2 Preliminaries

Let (X, τ) be a topological space with no separation properties assumed. For a subset A of a topological space (X, τ) , $Cl(A)$ and $Int(A)$ denote the closure and the interior of A in (X, τ) , respectively. The power set of X will be denoted by $\mathcal{P}(X)$. A subcollection $\mathcal{G}$ (not containing the empty set) of $\mathcal{P}(X)$ is called a grill [6] on X if $\mathcal{G}$ satisfies the following conditions:

1. $A \in \mathcal{G}$ and $A \subseteq B$ implies that $B \in \mathcal{G}$,
2. $A, B \subseteq X$ and $A \cup B \in \mathcal{G}$ implies that $A \in \mathcal{G}$ or $B \in \mathcal{G}$.

For any point x of a topological space (X, τ) , $\tau(x)$ denotes the collection of all open neighborhoods of x .

Definition 2.1. [17] Let (X, τ) be a topological space and $\mathcal{G}$ be a grill on X . A mapping $\Phi : \mathcal{P}(X) \rightarrow \mathcal{P}(X)$ is defined as follows: $\Phi(A) = \Phi_{\mathcal{G}}(A, \tau) = \{x \in X : A \cap U \in \mathcal{G} \text{ for all } U \in \tau(x)\}$ for each $A \in \mathcal{P}(X)$. The mapping Φ is called the operator associated with the grill $\mathcal{G}$ and the topology τ .

Proposition 2.1. [17] Let (X, τ) be a topological space and $\mathcal{G}$ be a grill on X . Then for all $A, B \subseteq X$:

1. $A \subseteq B$ implies that $\Phi(A) \subseteq \Phi(B)$,
2. $\Phi(A \cup B) = \Phi(A) \cup \Phi(B)$,
3. $\Phi(\Phi(A)) \subseteq \Phi(A) = Cl(\Phi(A)) \subseteq Cl(A)$.

Let G be a grill on a space X . Then in [17] a map $\Psi : \mathcal{P}(X) \rightarrow \mathcal{P}(X)$ is defined by $\Psi(A) = A \cup \Phi(A)$ for all $A \in \mathcal{P}(X)$. The map Ψ satisfies a Kuratowski closure axiom. Thus a subset A of X is $\tau_{\mathcal{G}}$ -closed if $\Psi(A) = A$ or equivalently $\Phi(A) \subseteq A$. Corresponding to a grill $\mathcal{G}$ on a topological space (X, τ) , there exists a unique topology $\tau_{\mathcal{G}}$ on X given by $\tau_{\mathcal{G}} = \{U \subseteq X : \Psi(X - U) = X - U\}$, where for any $A \subseteq X$, $\Psi(A) = A \cup \Phi(A) = \tau_{\mathcal{G}}-Cl(A)$. For any grill $\mathcal{G}$ on a topological space (X, τ) , $\tau \subseteq \tau_{\mathcal{G}}$. If (X, τ) is a topological space with a grill $\mathcal{G}$ on X , then we call it a grill topological space and denote it by $(X, \tau, \mathcal{G})$.

Corollary 2.2. [17] Let $(X, \tau, \mathcal{G})$ be a grill topological space and suppose $A, B \subseteq X$ with $B \notin \mathcal{G}$. Then $\Phi(A \cup B) = \Phi(A) = \Phi(A - B)$.

Proposition 2.3. [17] Let $(X, \tau, \mathcal{G})$ be a grill topological space and $A \subseteq X$ with $A \subseteq \Phi(A)$. Then $Cl(A) = \Psi(A) = Cl(\Phi(A)) = \Phi(A)$.

Lemma 2.4. [17] *Let $(X, \tau, \mathcal{G})$ be a grill topological space with $\tau - \phi \subseteq \mathcal{G}$. Then for all $U \in \tau$, $U \subseteq \Phi(U)$.*

Definition 2.2. *Let $(X, \tau, \mathcal{G})$ be a grill topological space. A subset A in X is said to be*

1. Φ -open [9] if $A \subseteq \text{Int}(\Phi(A))$,
2. $\mathcal{G}$ -preopen [9] if $A \subseteq \text{Int}(\Psi(A))$.

3 weakly $\mathcal{G}$ -locally closed sets

A subset A of a topological space (X, τ) is said to be locally closed [3] if A is the intersection of an open set and a closed set. Locally closed sets are further investigated by Ganster and Reilly in [7]. It is easy to see that all open sets as well as all closed sets are locally closed. Recently Mandal and Mukherjee [13] introduced the notion of $\mathcal{G}$ -locally closed sets as a new type of locally closed sets.

Definition 3.1. [13] *A subset A of a grill topological space $(X, \tau, \mathcal{G})$ is said to be $\mathcal{G}$ -locally closed if $A = U \cap \Phi(A)$ for some $U \in \tau$.*

We now introduce a new type of locally closed sets called weakly $\mathcal{G}$ -locally closed as follows:

Definition 3.2. *A subset A of a grill topological space $(X, \tau, \mathcal{G})$ is said to be weakly $\mathcal{G}$ -locally closed (briefly weakly- $\mathcal{G}$ -LC) if $A = U \cap V$, where U is open and V is $\tau_{\mathcal{G}}$ -closed.*

Remark 3.1. 1. [13] Every $\mathcal{G}$ -locally closed set in a grill topological space $(X, \tau, \mathcal{G})$ is locally closed. But the converse is false.

2. Every locally closed set in a grill topological space $(X, \tau, \mathcal{G})$ is weakly $\mathcal{G}$ -locally closed. But the converse is false as is shown below.

Example 3.1. Let $X = \{a, b, c, d\}$, $\tau = \{\phi, \{d\}, \{a, c\}, \{a, c, d\}, X\}$ and $\mathcal{G} = \{\{a, b\}, \{a, b, c\}, \{a, b, d\}, X\}$. Then $A = \{a, b\}$ is weakly $\mathcal{G}$ -locally closed but it is not locally closed.

Proposition 3.1. *Let $(X, \tau, \mathcal{G})$ be a grill topological space and A a subset of X . Then the following properties hold:*

1. If A is open, then A is weakly- $\mathcal{G}$ -LC.

2. If A is $\tau_{\mathcal{G}}$ -closed, then A is weakly- $\mathcal{G}$ -LC.

Proof. The proof is obvious. $\square$

The converses of the statements in Proposition 3.1 need not be true as shown in the following example.

Example 3.2. Let $X = \{a, b, c\}$, $\tau = \{\phi, \{a\}, \{c\}, \{a, c\}, X\}$ and $\mathcal{G} = \{\{a\}, \{c\}, \{a, c\}, \{a, b\}, \{b, c\}, X\}$. Then

1. $A = \{b\}$ is a weakly- $\mathcal{G}$ -LC set but it is not open.
2. $A = \{a\}$ is a weakly- $\mathcal{G}$ -LC set but it is not $\tau_{\mathcal{G}}$ -closed.

Theorem 3.2. For a subset A of a grill topological space $(X, \tau, \mathcal{G})$, the following are equivalent:

1. A is open.
2. A is weakly- $\mathcal{G}$ -LC and $\mathcal{G}$ -preopen.

Proof. (1) $\Rightarrow$ (2): It is obvious since X is $\tau_{\mathcal{G}}$ -closed.

(2) $\Rightarrow$ (1): Let A be a weakly- $\mathcal{G}$ -LC set and $\mathcal{G}$ -preopen. Then, we have $A \subseteq \text{Int}(\Psi(A))$ and $A = U \cap V$, where $U \in \tau$ and V is $\tau_{\mathcal{G}}$ -closed, respectively. Therefore, we have

$$\begin{aligned}
 A &\subseteq \text{Int}(\Psi(A)) \\
 &= \text{Int}(\Psi(U \cap V)) \\
 &\subseteq \text{Int}(\Psi(U) \cap \Psi(V)) \\
 &= \text{Int}(\Psi(U)) \cap \text{Int}(\Psi(V)) \\
 &= \text{Int}(\Psi(U)) \cap \text{Int}(V).
 \end{aligned}$$

Since $A = U \cap V$ and $A \subseteq U$, we have

$$\begin{aligned}
 A &= A \cap U \\
 &\subseteq [\text{Int}(\Psi(U)) \cap \text{Int}(V)] \cap U \\
 &= [\text{Int}(\Psi(U)) \cap U] \cap \text{Int}(V) \\
 &= \text{Int}[\Psi(U) \cap U] \cap \text{Int}(V) \\
 &= \text{Int}[U \cap V] = \text{Int}(A).
 \end{aligned}$$

Hence A is an open set. $\square$

The notions of weakly- $\mathcal{G}$ -LC sets and $\mathcal{G}$ -preopen sets are independent as shown in the following examples.

Example 3.3. Let $X = \{a, b, c\}$, $\tau = \{\phi, \{a\}, \{c\}, \{a, c\}, X\}$ and $\mathcal{G} = \{\{a\}, \{c\}, \{a, c\}, \{a, b\}, \{b, c\}, X\}$. Then $A = \{b\}$ is a weakly- $\mathcal{G}$ -LC set but it is not $\mathcal{G}$ -preopen.

Example 3.4. Let $X = \{a, b, c, d\}$, $\tau = \{\phi, \{b\}, \{b, c, d\}, X\}$ and $\mathcal{G} = \{\{b\}, \{c\}, \{d\}, \{a, b\}, \{a, c\}, \{a, d\}, \{b, c\}, \{b, d\}, \{c, d\}, \{a, b, c\}, \{a, b, d\}, \{a, c, d\}, \{b, c, d\}, X\}$. Then $A = \{a, b\}$ is $\mathcal{G}$ -preopen but it is not weakly- $\mathcal{G}$ -LC.

Theorem 3.3. Let $(X, \tau, \mathcal{G})$ be a grill topological space and A be a weakly- $\mathcal{G}$ -LC subset of X . Then the following properties hold:

1. If B is a $\tau_{\mathcal{G}}$ -closed set, then $A \cap B$ is a weakly- $\mathcal{G}$ -LC set.
2. If B is an open set, then $A \cap B$ is a weakly- $\mathcal{G}$ -LC set.
3. If B is a weakly- $\mathcal{G}$ -LC set, then $A \cap B$ is a weakly- $\mathcal{G}$ -LC set.

Proof. (1) Let B be $\tau_{\mathcal{G}}$ -closed, then $A \cap B = (U \cap V) \cap B = U \cap (V \cap B)$, where $V \cap B$ is $\tau_{\mathcal{G}}$ -closed and U is open. Hence $A \cap B$ is weakly- $\mathcal{G}$ -LC.

(2) Let B be open, then $A \cap B = (U \cap V) \cap B = (U \cap B) \cap V$, where $U \cap B$ is open and V is $\tau_{\mathcal{G}}$ -closed. Hence $A \cap B$ is weakly- $\mathcal{G}$ -LC.

(3) Let B be weakly- $\mathcal{G}$ -LC, then $A \cap B = (U \cap V) \cap (F \cap G) = (U \cap F) \cap (V \cap G)$, where $U \cap F$ is open and $V \cap G$ is $\tau_{\mathcal{G}}$ -closed. Hence $A \cap B$ is weakly- $\mathcal{G}$ -LC. $\square$

Definition 3.3. [12] Let (X, τ) be a topological space and $\mathcal{G}$ be a grill on X . Then a subset A of X is said to be $\mathcal{G}$ - g -closed if $\Phi(A) \subseteq U$ whenever $A \subseteq U$ and U is open in X .

Theorem 3.4. A subset of a grill topological space $(X, \tau, \mathcal{G})$ is $\tau_{\mathcal{G}}$ -closed if and only if it is weakly- $\mathcal{G}$ -LC and $\mathcal{G}$ - g -closed.

Proof. Necessity is trivial. We prove only sufficiency. Let A be weakly- $\mathcal{G}$ -LC and $\mathcal{G}$ - g -closed. Since A is weakly- $\mathcal{G}$ -LC, $A = U \cap V$, where U is open and V is $\tau_{\mathcal{G}}$ -closed. So, we have $A = U \cap V \subseteq U$. Since A is $\mathcal{G}$ - g -closed, $\Phi(A) \subseteq U$. Also $A = U \cap V \subseteq V$ and V is $\tau_{\mathcal{G}}$ -closed, then $\Phi(A) \subseteq V$. Consequently, we have $\Phi(A) \subseteq U \cap V = A$ and hence A is $\tau_{\mathcal{G}}$ -closed. $\square$

The notions of weakly- $\mathcal{G}$ -LC sets and $\mathcal{G}$ - g -closed sets are independent.

Example 3.5. Let $X = \{a, b, c, d\}$, $\tau = \{\phi, \{b\}, \{b, c, d\}, X\}$ and $\mathcal{G} = \{\{b\}, \{c\}, \{d\}, \{a, b\}, \{a, c\}, \{a, d\}, \{b, c\}, \{b, d\}, \{c, d\}, \{a, b, c\}, \{a, b, d\}, \{a, c, d\}, \{b, c, d\}, X\}$. Then

1. $A = \{a, b\}$ is $\mathcal{G}$ - g -closed but it is not weakly- $\mathcal{G}$ -LC.

2. $A = \{c, d\}$ is a weakly- $\mathcal{G}$ -LC set but it is not $\mathcal{G}$ - g -closed .

Theorem 3.5. *Let $(X, \tau, \mathcal{G})$ be a grill topological space and A a subset of X . Then the following properties are equivalent:*

1. A is weakly- $\mathcal{G}$ -LC;
2. $A = U \cap \Psi(A)$ for some open set U ;
3. $\Psi(A) - A = \Phi(A) - A$ is closed;
4. $A \cup [X - \Phi(A)] = A \cup [X - \Psi(A)]$ is open;
5. $A \subseteq \text{Int}[A \cup (X - \Phi(A))]$.

Proof. (1) $\Rightarrow$ (2): If A is weakly- $\mathcal{G}$ -LC, then there exist an open set U and a $\tau_{\mathcal{G}}$ -closed set F such that $A = U \cap F$. Clearly, $A \subseteq U \cap \Psi(A)$. Since F is $\tau_{\mathcal{G}}$ -closed, $\Psi(A) \subseteq \Psi(F) = F$ and so $U \cap \Psi(A) \subseteq U \cap F = A$. Therefore, $A = U \cap \Psi(A)$.

(2) $\Rightarrow$ (3): Now $\Phi(A) - A = \Phi(A) \cap (X - A) = \Phi(A) \cap [X - (U \cap \Psi(A))] = \Phi(A) \cap (X - U)$. Therefore, $\Psi(A) - A = \Phi(A) - A$ is closed.

(3) $\Rightarrow$ (4): Since $X - (\Phi(A) - A) = (X - \Phi(A)) \cup A$, then $[X - \Phi(A)] \cup A$ is open. Clearly, $A \cup [X - \Phi(A)] = A \cup [X - \Psi(A)]$.

(4) $\Rightarrow$ (5): It is clear.

(5) $\Rightarrow$ (1): $X - \Phi(A) = \text{Int}(X - \Phi(A)) \subseteq \text{Int}[A \cup (X - \Phi(A))]$ which implies that $A \cup [X - \Phi(A)] \subseteq \text{Int}[A \cup (X - \Phi(A))]$ and so $A \cup [X - \Phi(A)]$ is open. Since $A = [A \cup [X - \Phi(A)]] \cap \Psi(A)$, A is weakly- $\mathcal{G}$ -LC. $\square$

Remark 3.2. In a grill topological space $(X, \tau, \mathcal{G})$, if $A \subseteq \Phi(A)$ for every subset A of X , then every weakly- $\mathcal{G}$ -LC set is $\mathcal{G}$ -locally closed.

4 Strongly $\mathcal{G}$ -locally closed sets

Definition 4.1. *A subset A of a grill topological space $(X, \tau, \mathcal{G})$ is said to be strongly $\mathcal{G}$ -locally closed (briefly strongly- $\mathcal{G}$ -LC) (resp. strongly-LC [10]) if $A = U \cap V$, where U is regular open and V is $\tau_{\mathcal{G}}$ -closed (resp. closed).*

Proposition 4.1. *Let $(X, \tau, \mathcal{G})$ be a grill topological space and A a subset of X . Then the following properties hold:*

1. If A is regular open, then A is strongly- $\mathcal{G}$ -LC.
2. If A is $\tau_{\mathcal{G}}$ -closed, then A is strongly- $\mathcal{G}$ -LC.

3. If A is strongly- $\mathcal{G}$ -LC, then A is weakly- $\mathcal{G}$ -LC.

The converses of the statements in Proposition 4.1 need not be true as shown in the following example.

Example 4.1. Let $X = \{a, b, c\}$, $\tau = \{\phi, \{a\}, \{c\}, \{a, c\}, X\}$ and $\mathcal{G} = \{\{a\}, \{c\}, \{a, c\}, \{a, b\}, \{b, c\}, X\}$. Then

1. $A = \{b\}$ is a strongly- $\mathcal{G}$ -LC set but it is not regular open.
2. $A = \{a\}$ is a strongly- $\mathcal{G}$ -LC set but it is not $\tau_{\mathcal{G}}$ -closed.
3. $A = \{a, c\}$ is a weakly- $\mathcal{G}$ -LC set but it is not strongly- $\mathcal{G}$ -LC.

Theorem 4.2. Let $(X, \tau, \mathcal{G})$ be a grill topological space and A be a strongly- $\mathcal{G}$ -LC subset of X . Then the following properties hold:

1. If B is a $\tau_{\mathcal{G}}$ -closed set, then $A \cap B$ is a strongly- $\mathcal{G}$ -LC set.
2. If B is a regular open set, then $A \cap B$ is a strongly- $\mathcal{G}$ -LC set.
3. If B is a strongly- $\mathcal{G}$ -LC set, then $A \cap B$ is a strongly- $\mathcal{G}$ -LC set.

Definition 4.2. Let (X, τ) be a topological space and $\mathcal{G}$ be a grill on X . Then a subset A of X is said to be $\mathcal{G}$ -gr-closed if $\Phi(A) \subseteq U$ whenever $A \subseteq U$ and U is regular open in X .

Lemma 4.3. Let $(X, \tau, \mathcal{G})$ be a grill topological space and A a subset of X . If A is $\mathcal{G}$ -g-closed, then A is $\mathcal{G}$ -gr-closed.

Theorem 4.4. For a subset A of a grill topological space $(X, \tau, \mathcal{G})$, the following properties are equivalent:

1. A is $\tau_{\mathcal{G}}$ -closed;
2. A is strongly- $\mathcal{G}$ -LC and $\mathcal{G}$ -g-closed;
3. A is strongly- $\mathcal{G}$ -LC and $\mathcal{G}$ -gr-closed.

Proof. (1) $\Rightarrow$ (2): Obvious.

(2) $\Rightarrow$ (3): The proof follows from Lemma 4.3.

(3) $\Rightarrow$ (1): Let A be strongly- $\mathcal{G}$ -LC and $\mathcal{G}$ -gr-closed. Since A is strongly- $\mathcal{G}$ -LC, $A = U \cap V$, where U is regular open and V is $\tau_{\mathcal{G}}$ -closed. Since $A \subseteq U$ and A is $\mathcal{G}$ -gr-closed, $\Phi(A) \subseteq U$. Since $A \subseteq V$ and V is $\tau_{\mathcal{G}}$ -closed, $\Phi(A) \subseteq V$. Thus $\Phi(A) \subseteq U \cap V = A$. Hence A is $\tau_{\mathcal{G}}$ -closed. $\square$

Remark 4.1. 1. The notions of strongly- $\mathcal{G}$ -LC sets and $\mathcal{G}$ - g -closed sets are independent.

2. The notions of strongly- $\mathcal{G}$ -LC sets and $\mathcal{G}$ - gr -closed sets are independent.

Example 4.2. Let $X = \{a, b, c, d\}$, $\tau = \{\phi, \{b\}, \{b, c, d\}, X\}$ and $\mathcal{G} = \{\{b\}, \{c\}, \{d\}, \{a, b\}, \{a, c\}, \{a, d\}, \{b, c\}, \{b, d\}, \{c, d\}, \{a, b, c\}, \{a, b, d\}, \{a, c, d\}, \{b, c, d\}, X\}$. Then

1. $A = \{b\}$ is $\mathcal{G}$ - gr -closed but it is not strongly- $\mathcal{G}$ -LC.
2. $A = \{a, b, c\}$ is $\mathcal{G}$ - g -closed but it is not strongly- $\mathcal{G}$ -LC.

Example 4.3. Let $X = \{a, b, c, d\}$, $\tau = \{\phi, \{a\}, \{b\}, \{a, b\}, X\}$ and $\mathcal{G} = \{\{b\}, \{a\}, \{d\}, \{a, b\}, \{a, c\}, \{a, d\}, \{b, c\}, \{b, d\}, \{c, d\}, \{a, b, c\}, \{a, b, d\}, \{a, c, d\}, \{b, c, d\}, X\}$. Then

1. $A = \{a\}$ is strongly- $\mathcal{G}$ -LC but it is not $\mathcal{G}$ - g -closed.
2. $A = \{b\}$ is strongly- $\mathcal{G}$ -LC but it is not $\mathcal{G}$ - gr -closed.

5 Decompositions of $\tau_{\mathcal{G}}$ -continuity and continuity

Definition 5.1. A function $f : (X, \tau, \mathcal{G}) \rightarrow (Y, \sigma)$ is said to be $\tau_{\mathcal{G}}$ -continuous (resp. $\mathcal{G}$ - g -continuous [12], $\mathcal{G}$ - gr -continuous, weakly $\mathcal{G}$ -LC-continuous, strongly $\mathcal{G}$ -LC-continuous) if $f^{-1}(A)$ is a $\tau_{\mathcal{G}}$ -closed (resp. $\mathcal{G}$ - g -closed, $\mathcal{G}$ - gr -closed, weakly- $\mathcal{G}$ -LC, strongly- $\mathcal{G}$ -LC) set in $(X, \tau, \mathcal{G})$ for every closed set A of (Y, σ) .

Definition 5.2. A function $f : (X, \tau, \mathcal{G}) \rightarrow (Y, \sigma)$ is said to be g -continuous [2] (resp. gr -continuous [15], strongly LC-continuous [10], LC-continuous [7]) if $f^{-1}(A)$ is a g -closed, (resp. gr -closed, strongly-LC, locally closed) set in $(X, \tau, \mathcal{G})$ for every closed set A of (Y, σ) .

Definition 5.3. A function $f : (X, \tau, \mathcal{G}) \rightarrow (Y, \sigma)$ is said to be contra weakly $\mathcal{G}$ -LC-continuous (resp. $\mathcal{G}$ -precontinuous [9]) if $f^{-1}(A)$ is weakly- $\mathcal{G}$ -LC (resp. $\mathcal{G}$ -preopen) set in $(X, \tau, \mathcal{G})$ for every open set A of (Y, σ) .

Theorem 5.1. For a function $f : (X, \tau, \mathcal{G}) \rightarrow (Y, \sigma)$, the following properties are equivalent:

1. f is $\tau_{\mathcal{G}}$ -continuous;
2. The inverse image of each open set in Y is $\tau_{\mathcal{G}}$ -open;

3. For each $x \in X$ and each $V \in \sigma$ containing $f(x)$, there exists $U \in \tau_{\mathcal{G}}$ containing x such that $f(U) \subseteq V$;
4. $f : (X, \tau_{\mathcal{G}}) \rightarrow (Y, \sigma)$ is continuous.

Theorem 5.2. *A function $f : (X, \tau, \mathcal{G}) \rightarrow (Y, \sigma)$ is continuous if and only if it is contra weakly $\mathcal{G}$ -LC-continuous and $\mathcal{G}$ -precontinuous.*

Proof. This is an immediate consequence of Theorem 3.2. $\square$

Theorem 5.3. *A function $f : (X, \tau, \mathcal{G}) \rightarrow (Y, \sigma)$ is $\tau_{\mathcal{G}}$ -continuous if and only if it is weakly $\mathcal{G}$ -LC-continuous and $\mathcal{G}$ -g-continuous.*

Proof. This is an immediate consequence of Theorem 3.4. $\square$

Corollary 5.4. [14] *Let $(X, \tau, \mathcal{G})$ be a grill space and $\mathcal{G} = \mathcal{P}(X) \setminus \{\phi\}$. A function $f : (X, \tau, \mathcal{G}) \rightarrow (Y, \sigma)$ is continuous if and only if it is LC-continuous and g-continuous.*

Theorem 5.5. *For a function $f : (X, \tau, \mathcal{G}) \rightarrow (Y, \sigma)$, the following properties are equivalent:*

1. f is $\tau_{\mathcal{G}}$ -continuous;
2. f is strongly $\mathcal{G}$ -LC-continuous and $\mathcal{G}$ -g-continuous;
3. f is strongly $\mathcal{G}$ -LC-continuous and $\mathcal{G}$ -gr-continuous.

Proof. This is an immediate consequence of Theorem 4.4. $\square$

Corollary 5.6. *Let $(X, \tau, \mathcal{G})$ be a grill space and $\mathcal{G} = \mathcal{P}(X) \setminus \{\phi\}$. For a function $f : (X, \tau, \mathcal{G}) \rightarrow (Y, \sigma)$, the following properties are equivalent:*

1. f is continuous;
2. f is strongly LC-continuous and g-continuous;
3. f is strongly LC-continuous and gr-continuous.

6 Additions

The concept of ideals in topological spaces is treated in the classic text by Kuratowski [11] and Vaidyanathaswamy [19]. Janković and Hamlett [8] investigated further properties of ideal spaces. An ideal $\mathcal{I}$ on a topological

space (X, τ) is a non-empty collection of subsets of X which satisfies the following properties: (1) $A \in \mathcal{I}$ and $B \subseteq A$ implies $B \in \mathcal{I}$; (2) $A \in \mathcal{I}$ and $B \in \mathcal{I}$ implies $A \cup B \in \mathcal{I}$. An ideal topological space or simply an ideal space is a topological space (X, τ) with an ideal $\mathcal{I}$ on X and is denoted by $(X, \tau, \mathcal{I})$. For a subset $A \subseteq X$, $A^*(\mathcal{I}, \tau) = \{x \in X : A \cap U \notin \mathcal{I} \text{ for every } U \in \tau(x)\}$, where $\tau(x) = \{U \in \tau : x \in U\}$, is called the local function of A with respect to $\mathcal{I}$ and τ [11]. We simply write A^* in case there is no chance for confusion. A Kuratowski closure operator $Cl^*(.)$ for a topology $\tau^*(\mathcal{I}, \tau)$ called the $*$ -topology, finer than τ , is defined by $Cl^*(A) = A \cup A^*$ [8]. The following lemma will be useful in the sequel.

Lemma 6.1. [16] *Let (X, τ) be a topological space. Then the following hold.*

1. $\mathcal{G}$ is a grill on X if and only if $\mathcal{I} = \mathcal{P}(X) - \mathcal{G}$ is an ideal on X ,
2. The operators Cl^* on $(X, \tau, \mathcal{I})$, where $\mathcal{I} = \mathcal{P}(X) - \mathcal{G}$, and Ψ on $(X, \tau, \mathcal{G})$ are equal.

Remark 6.1. Let $(X, \tau, \mathcal{G})$ be a grill topological space and A a subset of X .

1. Since $\tau \subseteq \tau_{\mathcal{G}}$, then every strongly-LC set is strongly- $\mathcal{G}$ -LC.
2. If $\mathcal{G} = \mathcal{P}(X) \setminus \{\emptyset\}$, then $\tau = \tau_{\mathcal{G}}$ and hence both the notions of strongly- $\mathcal{G}$ -LC and strongly-LC are equal.
3. If $A \subseteq \Phi(A)$, then $Cl(A) = \tau_{\mathcal{G}}-Cl(A)$ and hence both the notions of strongly- $\mathcal{G}$ -LC and strongly-LC are equal.
4. If $\mathcal{G} = \{X\}$, then $\Phi(A) = \emptyset$ for any subset A of X and $\Psi(A) = A$. Then any subset A of X is strongly- $\mathcal{G}$ -LC.
5. For any subset A of a space X and any grill $\mathcal{G}$ on X , $\Phi(A)$ is $\tau_{\mathcal{G}}$ -closed. Then if every open set is regular open, then every $\mathcal{G}$ -locally closed set is strongly- $\mathcal{G}$ -LC.
6. Let τ be suitable for $\mathcal{G}$, that is, $A - \Phi(A) \notin \mathcal{G}$ for all $A \subseteq X$ [[17], Definition 3.1] and $\tau - \{\emptyset\} \subseteq \mathcal{G}$. Then if $(X, \tau_{\mathcal{G}})$ is regular then $\tau = \tau_{\mathcal{G}}$ by Theorem 3.8 of [17] and hence both notions of strongly- $\mathcal{G}$ -LC and strongly-LC are equal.

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