Decompositions of $\tau_\mathcal{G}$-Continuity and Continuity

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Abstract. In this paper, we introduce and investigate the notion of weakly $\mathcal{G}$-locally closed sets in a topological space with a grill. Furthermore, by using these sets, we obtain new decompositions of continuity.

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1 Introduction

The idea of grills on a topological space was first introduced by Choquet [6]. The concept of grills has shown to be a powerful supporting and useful tool like nets and filters, for getting a deeper insight into further studying some topological notions such as proximity spaces, closure spaces and the theory of compactifications and extension problems of different kinds (see [5], [4], [18] for details). In [17], Roy and Mukherjee defined and studied a typical topology associated rather naturally to the existing topology and a grill on a given topological space. Quite recently, Hatir and Jafari [9] defined new classes of sets and obtained a new decomposition of continuity in terms of grills. In this paper, we introduce and investigate the notion of weakly $\mathcal{G}$-locally closed sets in a topological space with a grill. Furthermore, by using these sets, we obtain new decompositions of continuity.
2 Preliminaries

Let \((X, \tau)\) be a topological space with no separation properties assumed. For a subset \(A\) of a topological space \((X, \tau)\), \(Cl(A)\) and \(Int(A)\) denote the closure and the interior of \(A\) in \((X, \tau)\), respectively. The power set of \(X\) will be denoted by \(\mathcal{P}(X)\). A subcollection \(\mathcal{G}\) (not containing the empty set) of \(\mathcal{P}(X)\) is called a grill [6] on \(X\) if \(\mathcal{G}\) satisfies the following conditions:

1. \(A \in \mathcal{G}\) and \(A \subseteq B\) implies that \(B \in \mathcal{G}\),
2. \(A, B \subseteq X\) and \(A \cup B \in \mathcal{G}\) implies that \(A \in \mathcal{G}\) or \(B \in \mathcal{G}\).

For any point \(x\) of a topological space \((X, \tau)\), \(\tau(x)\) denotes the collection of all open neighborhoods of \(x\).

Definition 2.1. [17] Let \((X, \tau)\) be a topological space and \(\mathcal{G}\) be a grill on \(X\). A mapping \(\Phi : \mathcal{P}(X) \rightarrow \mathcal{P}(X)\) is defined as follows: 
\[\Phi(A) = \Phi_{\mathcal{G}}(A, \tau) = \{x \in X : A \cap U \in \mathcal{G} \text{ for all } U \in \tau(x)\}\] for each \(A \in \mathcal{P}(X)\). The mapping \(\Phi\) is called the operator associated with the grill \(\mathcal{G}\) and the topology \(\tau\).

Proposition 2.1. [17] Let \((X, \tau)\) be a topological space and \(\mathcal{G}\) be a grill on \(X\). Then for all \(A, B \subseteq X\):

1. \(A \subseteq B\) implies that \(\Phi(A) \subseteq \Phi(B)\),
2. \(\Phi(A \cup B) = \Phi(A) \cup \Phi(B)\),
3. \(\Phi(\Phi(A)) \subseteq \Phi(A) = Cl(\Phi(A)) \subseteq Cl(A)\).

Let \(G\) be a grill on a space \(X\). Then in [17] a map \(\Psi : \mathcal{P}(X) \rightarrow \mathcal{P}(X)\) is defined by \(\Psi(A) = A \cup \Phi(A)\) for all \(A \in \mathcal{P}(X)\). The map \(\Psi\) satisfies a Kuratowski closure axiom. Thus a subset \(A\) of \(X\) is \(\tau_{\mathcal{G}}\)-closed if \(\Psi(A) = A\) or equivalently \(\Phi(A) \subseteq A\). Corresponding to a grill \(\mathcal{G}\) on a topological space \((X, \tau)\), there exists a unique topology \(\tau_{\mathcal{G}}\) on \(X\) given by \(\tau_{\mathcal{G}} = \{U \subseteq X : \Psi(X - U) = X - U\}\), where for any \(A \subseteq X\), \(\Psi(A) = A \cup \Phi(A) = \tau_{\mathcal{G}} - Cl(A)\).

For any grill \(\mathcal{G}\) on a topological space \((X, \tau)\), \(\tau \subseteq \tau_{\mathcal{G}}\). If \((X, \tau)\) is a topological space with a grill \(\mathcal{G}\) on \(X\), then we call it a grill topological space and denote it by \((X, \tau, \mathcal{G})\).

Corollary 2.2. [17] Let \((X, \tau, \mathcal{G})\) be a grill topological space and suppose \(A, B \subseteq X\) with \(B \notin \mathcal{G}\). Then \(\Phi(A \cup B) = \Phi(A) = \Phi(A - B)\).

Proposition 2.3. [17] Let \((X, \tau, \mathcal{G})\) be a grill topological space and \(A \subseteq X\) with \(A \subseteq \Phi(A)\). Then \(Cl(A) = \Psi(A) = Cl(\Phi(A)) = \Phi(A)\).
Lemma 2.4. [17] Let \((X, \tau, \mathcal{G})\) be a grill topological space with \(\tau - \phi \subseteq \mathcal{G}\). Then for all \(U \in \tau\), \(U \subseteq \Phi(U)\).

Definition 2.2. Let \((X, \tau, \mathcal{G})\) be a grill topological space. A subset \(A\) in \(X\) is said to be

1. \(\Phi\)-open [9] if \(A \subseteq \text{Int}(\Phi(A))\),
2. \(\mathcal{G}\)-preopen [9] if \(A \subseteq \text{Int}(\Psi(A))\).

3 weakly \(\mathcal{G}\)-locally closed sets

A subset \(A\) of a topological space \((X, \tau)\) is said to be locally closed [3] if \(A\) is the intersection of an open set and a closed set. Locally closed sets are further investigated by Ganster and Reilly in [7]. It is easy to see that all open sets as well as all closed sets are locally closed. Recently Mandal and Mukherjee [13] introduced the notion of \(\mathcal{G}\)-locally closed sets as a new type of locally closed sets.

Definition 3.1. [13] A subset \(A\) of a grill topological space \((X, \tau, \mathcal{G})\) is said to be \(\mathcal{G}\)-locally closed if \(A = U \cap \Phi(A)\) for some \(U \in \tau\).

We now introduce a new type of locally closed sets called weakly \(\mathcal{G}\)-locally closed as follows:

Definition 3.2. A subset \(A\) of a grill topological space \((X, \tau, \mathcal{G})\) is said to be weakly \(\mathcal{G}\)-locally closed (briefly weakly- \(\mathcal{G}\)-LC) if \(A = U \cap V\), where \(U\) is open and \(V\) is \(\tau_{\mathcal{G}}\)-closed.

Remark 3.1. 1. [13] Every \(\mathcal{G}\)-locally closed set in a grill topological space \((X, \tau, \mathcal{G})\) is locally closed. But the converse is false.

2. Every locally closed set in a grill topological space \((X, \tau, \mathcal{G})\) is weakly \(\mathcal{G}\)-locally closed. But the converse is false as is shown below.

Example 3.1. Let \(X = \{a, b, c, d\}\), \(\tau = \{\phi, \{d\}, \{a, c\}, \{a, c, d\}, X\}\) and \(\mathcal{G} = \{\{a, b\}, \{a, b, c\}, \{a, b, d\}, X\}\). Then \(A = \{a, b\}\) is weakly \(\mathcal{G}\)-locally closed but it is not locally closed.

Proposition 3.1. Let \((X, \tau, \mathcal{G})\) be a grill topological space and \(A\) a subset of \(X\). Then the following properties hold:

1. If \(A\) is open, then \(A\) is weakly-\(\mathcal{G}\)-LC.
2. If \( A \) is \( \tau_3 \)-closed, then \( A \) is weakly-\( \mathcal{G} \)-LC.

**Proof.** The proof is obvious. \( \square \)

The converses of the statements in Proposition 3.1 need not be true as shown in the following example.

**Example 3.2.** Let \( X = \{a,b,c\} \), \( \tau = \{\emptyset, \{a\}, \{c\}, \{a,c\}, X\} \) and \( \mathcal{G} = \{\{a\}, \{c\}, \{a,c\}, \{a,b\}, \{b,c\}, X\} \). Then

1. \( A = \{b\} \) is a weakly-\( \mathcal{G} \)-LC set but it is not open.
2. \( A = \{a\} \) is a weakly-\( \mathcal{G} \)-LC set but it is not \( \tau_3 \)-closed.

**Theorem 3.2.** For a subset \( A \) of a grill topological space \((X, \tau, \mathcal{G})\), the following are equivalent:

1. \( A \) is open.
2. \( A \) is weakly-\( \mathcal{G} \)-LC and \( \mathcal{G} \)-preopen.

**Proof.** (1) \( \Rightarrow \) (2): It is obvious since \( X \) is \( \tau_3 \)-closed.

(2) \( \Rightarrow \) (1): Let \( A \) be a weakly-\( \mathcal{G} \)-LC set and \( \mathcal{G} \)-preopen. Then, we have \( A \subseteq \text{Int} (\Psi (A)) \) and \( A = U \cap V \), where \( U \in \tau \) and \( V \) is \( \tau_3 \)-closed, respectively. Therefore, we have

\[
A \subseteq \text{Int} (\Psi (A)) = \text{Int} (\Psi (U \cap V)) \subseteq \text{Int} (\Psi (U) \cap \Psi (V)) = \text{Int} (\Psi (U)) \cap \text{Int} (\Psi (V)) = \text{Int} (\Psi (U)) \cap \text{Int} (V).
\]

Since \( A = U \cap V \) and \( A \subseteq U \), we have

\[
A = A \cap U \subseteq [\text{Int} (\Psi (U)) \cap \text{Int} (V)] \cap U = [\text{Int} (\Psi (U)) \cap U] \cap \text{Int} (V) = \text{Int} [\Psi (U) \cap U] \cap \text{Int} (V) = \text{Int} [U \cap V] = \text{Int} (A).
\]

Hence \( A \) is an open set. \( \square \)

The notions of weakly-\( \mathcal{G} \)-LC sets and \( \mathcal{G} \)-preopen sets are independent as shown in the following examples.
Example 3.3. Let $X = \{a, b, c\}$, $\tau = \{\phi, \{a\}, \{c\}, \{a, c\}, X\}$ and $\mathcal{G} = \{\{a\}, \{c\}, \{a, c\}, \{a, b\}, \{b, c\}, X\}$. Then $A = \{b\}$ is a weakly-$\mathcal{G}$-LC set but it is not $\mathcal{G}$-preopen.

Example 3.4. Let $X = \{a, b, c, d\}$, $\tau = \{\phi, \{b\}, \{b, c, d\}, X\}$ and $\mathcal{G} = \{\{b\}, \{c\}, \{d\}, \{a, b\}, \{a, c\}, \{a, d\}, \{b, d\}, \{c, d\}, \{a, b, c\}, \{a, b, d\}, \{a, c, d\}, \{b, c, d\}, X\}$. Then $A = \{a, b\}$ is $\mathcal{G}$-preopen but it is not weakly-$\mathcal{G}$-LC.

Theorem 3.3. Let $(X, \tau, \mathcal{G})$ be a grill topological space and $A$ be a weakly-$\mathcal{G}$-LC subset of $X$. Then the following properties hold:

1. If $B$ is a $\tau_3$-closed set, then $A \cap B$ is a weakly-$\mathcal{G}$-LC set.

2. If $B$ is an open set, then $A \cap B$ is a weakly-$\mathcal{G}$-LC set.

3. If $B$ is a weakly-$\mathcal{G}$-LC set, then $A \cap B$ is a weakly-$\mathcal{G}$-LC set.

Proof. (1) Let $B$ be $\tau_3$-closed, then $A \cap B = (U \cap V) \cap B = U \cap (V \cap B)$, where $V \cap B$ is $\tau_3$-closed and $U$ is open. Hence $A \cap B$ is weakly-$\mathcal{G}$-LC.

(2) Let $B$ be open, then $A \cap B = (U \cap V) \cap B = (U \cap B) \cap V$, where $U \cap B$ is open and $V$ is $\tau_3$-closed. Hence $A \cap B$ is weakly-$\mathcal{G}$-LC.

(3) Let $B$ be weakly-$\mathcal{G}$-LC, then $A \cap B = (U \cap V) \cap (F \cap G) = (U \cap F) \cap (V \cap G)$, where $U \cap F$ is open and $V \cap G$ is $\tau_3$-closed. Hence $A \cap B$ is weakly-$\mathcal{G}$-LC.

Definition 3.3. [12] Let $(X, \tau)$ be a topological space and $\mathcal{G}$ be a grill on $X$. Then a subset $A$ of $X$ is said to be $\mathcal{G}$-$g$-closed if $\Phi(A) \subseteq U$ whenever $A \subseteq U$ and $U$ is open in $X$.

Theorem 3.4. A subset of a grill topological space $(X, \tau, \mathcal{G})$ is $\tau_3$-closed if and only if it is weakly-$\mathcal{G}$-LC and $\mathcal{G}$-$g$-closed.

Proof. Necessity is trivial. We prove only sufficiency. Let $A$ be weakly-$\mathcal{G}$-LC and $\mathcal{G}$-$g$-closed. Since $A$ is weakly-$\mathcal{G}$-LC, $A = U \cap V$, where $U$ is open and $V$ is $\tau_3$-closed. So, we have $A = U \cap V \subseteq U$. Since $A$ is $\mathcal{G}$-$g$-closed, $\Phi(A) \subseteq U$. Also $A = U \cap V \subseteq V$ and $V$ is $\tau_3$-closed, then $\Phi(A) \subseteq V$. Consequently, we have $\Phi(A) \subseteq U \cap V = A$ and hence $A$ is $\tau_3$-closed.

The notions of weakly-$\mathcal{G}$-LC sets and $\mathcal{G}$-$g$-closed sets are independent.

Example 3.5. Let $X = \{a, b, c, d\}$, $\tau = \{\phi, \{b\}, \{b, c, d\}, X\}$ and $\mathcal{G} = \{\{b\}, \{c\}, \{d\}, \{a, b\}, \{a, c\}, \{a, d\}, \{b, d\}, \{c, d\}, \{a, b, c\}, \{a, b, d\}, \{a, c, d\}, \{b, c, d\}, X\}$. Then

1. $A = \{a, b\}$ is $\mathcal{G}$-$g$-closed but it is not weakly-$\mathcal{G}$-LC.
2. \( A = \{c, d\} \) is a weakly-\( G \)-LC set but it is not \( G \)-closed.

**Theorem 3.5.** Let \((X, \tau, G)\) be a grill topological space and \( A \) a subset of \( X \). Then the following properties are equivalent:

1. \( A \) is weakly-\( G \)-LC;
2. \( A = U \cap \Psi(A) \) for some open set \( U \);
3. \( \Psi(A) - A = \Phi(A) - A \) is closed;
4. \( A \cup [X - \Phi(A)] = A \cup [X - \Psi(A)] \) is open;
5. \( A \subseteq \text{Int}[A \cup (X - \Phi(A))] \).

**Proof.** (1) \( \Rightarrow \) (2): If \( A \) is weakly-\( G \)-LC, then there exist an open set \( U \) and a \( \tau_G \)-closed set \( F \) such that \( A = U \cap F \). Clearly, \( A \subseteq U \cap \Psi(A) \). Since \( F \) is \( \tau_G \)-closed, \( \Psi(A) \subseteq \Psi(F) = F \) and so \( U \cap \Psi(A) \subseteq U \cap F = A \). Therefore, \( A = U \cap \Psi(A) \).

(2) \( \Rightarrow \) (3): Now \( \Phi(A) - A = \Phi(A) \cap (X - A) = \Phi(A) \cap [X - (U \cap \Psi(A))] = \Phi(A) \cap (X - U) \). Therefore, \( \Psi(A) - A = \Phi(A) - A \) is closed.

(3) \( \Rightarrow \) (4): Since \( X - (\Phi(A) - A) = (X - \Phi(A)) \cup A \), then \( [X - \Phi(A)] \cup A \) is open. Clearly, \( A \cup [X - \Phi(A)] = A \cup [X - \Psi(A)] \).

(4) \( \Rightarrow \) (5): It is clear.

(5) \( \Rightarrow \) (1): \( X - \Phi(A) = \text{Int}(X - \Phi(A)) \subseteq \text{Int}[A \cup (X - \Phi(A))] \) which implies that \( A \cup [X - \Phi(A)] \subseteq \text{Int}[A \cup (X - \Phi(A))] \) and so \( A \cup [X - \Phi(A)] \) is open. Since \( A = [A \cup [X - \Phi(A)] \cap \Psi(A) \), \( A \) is weakly-\( G \)-LC.

**Remark 3.2.** In a grill topological space \((X, \tau, G)\), if \( A \subseteq \Phi(A) \) for every subset \( A \) of \( X \), then every weakly-\( G \)-LC set is \( G \)-locally closed.

### 4 Strongly \( G \)-locally closed sets

**Definition 4.1.** A subset \( A \) of a grill topological space \((X, \tau, G)\) is said to be strongly \( G \)-locally closed (briefly strongly-\( G \)-LC) (resp. strongly-LC [10]) if \( A = U \cap V \), where \( U \) is regular open and \( V \) is \( \tau_G \)-closed (resp. closed).

**Proposition 4.1.** Let \((X, \tau, G)\) be a grill topological space and \( A \) a subset of \( X \). Then the following properties hold:

1. If \( A \) is regular open, then \( A \) is strongly-\( G \)-LC.
2. If \( A \) is \( \tau_G \)-closed, then \( A \) is strongly-\( G \)-LC.
3. If $A$ is strongly-$\mathcal{G}$-LC, then $A$ is weakly-$\mathcal{G}$-LC.

The converses of the statements in Proposition 4.1 need not be true as shown in the following example.

**Example 4.1.** Let $X = \{a, b, c\}$, $\tau = \{\emptyset, \{a\}, \{c\}, \{a, c\}, X\}$ and $\mathcal{G} = \{\{a\}, \{c\}, \{a, c\}, \{a, b\}, \{b, c\}, X\}$. Then

1. $A = \{b\}$ is a strongly-$\mathcal{G}$-LC set but it is not regular open.
2. $A = \{a\}$ is a strongly-$\mathcal{G}$-LC set but it is not $\tau_3$-closed.
3. $A = \{a, c\}$ is a weakly-$\mathcal{G}$-LC set but it is not strongly-$\mathcal{G}$-LC.

**Theorem 4.2.** Let $(X, \tau, \mathcal{G})$ be a grill topological space and $A$ be a strongly-$\mathcal{G}$-LC subset of $X$. Then the following properties hold:

1. If $B$ is a $\tau_3$-closed set, then $A \cap B$ is a strongly-$\mathcal{G}$-LC set.
2. If $B$ is a regular open set, then $A \cap B$ is a strongly-$\mathcal{G}$-LC set.
3. If $B$ is a strongly-$\mathcal{G}$-LC set, then $A \cap B$ is a strongly-$\mathcal{G}$-LC set.

**Definition 4.2.** Let $(X, \tau)$ be a topological space and $\mathcal{G}$ be a grill on $X$. Then a subset $A$ of $X$ is said to be $\mathcal{G}$-gr-$\text{closed}$ if $\Phi(A) \subseteq U$ whenever $A \subseteq U$ and $U$ is regular open in $X$.

**Lemma 4.3.** Let $(X, \tau, \mathcal{G})$ be a grill topological space and $A$ a subset of $X$. If $A$ is $\mathcal{G}$-g-closed, then $A$ is $\mathcal{G}$-gr-closed.

**Theorem 4.4.** For a subset $A$ of a grill topological space $(X, \tau, \mathcal{G})$, the following properties are equivalent:

1. $A$ is $\tau_3$-closed;
2. $A$ is strongly-$\mathcal{G}$-LC and $\mathcal{G}$-g-closed;
3. $A$ is strongly-$\mathcal{G}$-LC and $\mathcal{G}$-gr-closed.

**Proof.** (1) $\Rightarrow$ (2): Obvious.
(2) $\Rightarrow$ (3): The proof follows from Lemma 4.3.
(3) $\Rightarrow$ (1): Let $A$ be strongly-$\mathcal{G}$-LC and $\mathcal{G}$-gr-closed. Since $A$ is strongly-$\mathcal{G}$-LC, $A = U \cap V$, where $U$ is regular open and $V$ is $\tau_3$-closed. Since $A \subseteq U$ and $A$ is $\mathcal{G}$-gr-closed, $\Phi(A) \subseteq U$. Since $A \subseteq V$ and $V$ is $\tau_3$-closed, $\Phi(A) \subseteq V$. Thus $\Phi(A) \subseteq U \cap V = A$. Hence $A$ is $\tau_3$-closed. \qed
Remark 4.1. 1. The notions of strongly-$\mathcal{G}$-LC sets and $\mathcal{G}$-$g$-closed sets are independent.

2. The notions of strongly-$\mathcal{G}$-LC sets and $\mathcal{G}$-$gr$-closed sets are independent.

Example 4.2. Let $X = \{a, b, c, d\}$, $\tau = \{\emptyset, \{b\}, \{b, c, d\}, X\}$ and $\mathcal{G} = \{\{b\}, \{c\}, \{d\}, \{a, b\}, \{a, c\}, \{a, d\}, \{b, c\}, \{b, d\}, \{c, d\}, \{a, b, c\}, \{a, b, d\}, \{a, c, d\}, \{b, c, d\}, X\}$. Then

1. $A = \{b\}$ is $\mathcal{G}$-$gr$-closed but it is not strongly-$\mathcal{G}$-LC.

2. $A = \{a, b, c\}$ is $\mathcal{G}$-$g$-closed but it is not strongly-$\mathcal{G}$-LC.

Example 4.3. Let $X = \{a, b, c, d\}$, $\tau = \{\emptyset, \{a\}, \{b\}, \{a, b\}, X\}$ and $\mathcal{G} = \{\{b\}, \{a\}, \{d\}, \{a, b\}, \{a, c\}, \{a, d\}, \{b, c\}, \{b, d\}, \{c, d\}, \{a, b, c\}, \{a, b, d\}, \{a, c, d\}, \{b, c, d\}, X\}$. Then

1. $A = \{a\}$ is strongly-$\mathcal{G}$-LC but it is not $\mathcal{G}$-$g$-closed.

2. $A = \{b\}$ is strongly-$\mathcal{G}$-LC but it is not $\mathcal{G}$-$gr$-closed.

5 Decompositions of $\tau_{\mathcal{G}}$-continuity and continuity

Definition 5.1. A function $f : (X, \tau, \mathcal{G}) \to (Y, \sigma)$ is said to be $\tau_{\mathcal{G}}$-continuous (resp. $\mathcal{G}$-$g$-continuous [12], $\mathcal{G}$-$gr$-continuous, weakly $\mathcal{G}$-LC-continuous, strongly $\mathcal{G}$-LC-continuous) if $f^{-1}(A)$ is a $\tau_{\mathcal{G}}$-closed (resp. $\mathcal{G}$-$g$-closed, $\mathcal{G}$-$gr$-closed, weakly-$\mathcal{G}$-LC, strongly-$\mathcal{G}$-LC) set in $(X, \tau, \mathcal{G})$ for every closed set $A$ of $(Y, \sigma)$.

Definition 5.2. A function $f : (X, \tau, \mathcal{G}) \to (Y, \sigma)$ is said to be $g$-continuous [2] (resp. $gr$-continuous [15], strongly LC-continuous [10], LC-continuous [7]) if $f^{-1}(A)$ is a $g$-closed, (resp. $gr$-closed, strongly-LC, locally closed) set in $(X, \tau, \mathcal{G})$ for every closed set $A$ of $(Y, \sigma)$.

Definition 5.3. A function $f : (X, \tau, \mathcal{G}) \to (Y, \sigma)$ is said to be contra weakly $\mathcal{G}$-LC-continuous (resp. $\mathcal{G}$-precontinuous [9]) if $f^{-1}(A)$ is weakly-$\mathcal{G}$-LC (resp. $\mathcal{G}$-preopen) set in $(X, \tau, \mathcal{G})$ for every open set $A$ of $(Y, \sigma)$.

Theorem 5.1. For a function $f : (X, \tau, \mathcal{G}) \to (Y, \sigma)$, the following properties are equivalent:

1. $f$ is $\tau_{\mathcal{G}}$-continuous;

2. The inverse image of each open set in $Y$ is $\tau_{\mathcal{G}}$-open;
3. For each \( x \in X \) and each \( V \in \sigma \) containing \( f(x) \), there exists \( U \in \tau_3 \) containing \( x \) such that \( f(U) \subseteq V \);

4. \( f : (X, \tau_3) \rightarrow (Y, \sigma) \) is continuous.

**Theorem 5.2.** A function \( f : (X, \tau, \mathcal{G}) \rightarrow (Y, \sigma) \) is continuous if and only if it is contra weakly \( \mathcal{G} \)-LC-continuous and \( \mathcal{G} \)-precontinuous.

**Proof.** This is an immediate consequence of Theorem 3.2. \( \square \)

**Theorem 5.3.** A function \( f : (X, \tau, \mathcal{G}) \rightarrow (Y, \sigma) \) is \( \tau_3 \)-continuous if and only if it is weakly \( \mathcal{G} \)-LC-continuous and \( \mathcal{G} \)-g-continuous.

**Proof.** This is an immediate consequence of Theorem 3.4. \( \square \)

**Corollary 5.4.** [14] Let \((X, \tau, \mathcal{G})\) be a grill space and \( \mathcal{G} = \mathcal{P}(X) \setminus \{\phi\} \). A function \( f : (X, \tau, \mathcal{G}) \rightarrow (Y, \sigma) \) is continuous if and only if it is \( \mathcal{G} \)-continuous and \( g \)-continuous.

**Theorem 5.5.** For a function \( f : (X, \tau, \mathcal{G}) \rightarrow (Y, \sigma) \), the following properties are equivalent:

1. \( f \) is \( \tau_3 \)-continuous;

2. \( f \) is strongly \( \mathcal{G} \)-LC-continuous and \( \mathcal{G} \)-g-continuous;

3. \( f \) is strongly \( \mathcal{G} \)-LC-continuous and \( \mathcal{G} \)-gr-continuous.

**Proof.** This is an immediate consequence of Theorem 4.4. \( \square \)

**Corollary 5.6.** Let \((X, \tau, \mathcal{G})\) be a grill space and \( \mathcal{G} = \mathcal{P}(X) \setminus \{\phi\} \). For a function \( f : (X, \tau, \mathcal{G}) \rightarrow (Y, \sigma) \), the following properties are equivalent:

1. \( f \) is continuous;

2. \( f \) is strongly \( LC \)-continuous and \( g \)-continuous;

3. \( f \) is strongly \( LC \)-continuous and \( gr \)-continuous.

### 6 Additions

The concept of ideals in topological spaces is treated in the classic text by Kuratowski [11] and Vaidyanathaswamy [19]. Janković and Hamlett [8] investigated further properties of ideal spaces. An ideal \( \mathcal{I} \) on a topological...
space \((X, \tau)\) is a non-empty collection of subsets of \(X\) which satisfies the following properties: (1) \(A \in \mathcal{J}\) and \(B \subseteq A\) implies \(B \in \mathcal{I}\); (2) \(A \in \mathcal{J}\) and \(B \in \mathcal{I}\) implies \(A \cup B \in \mathcal{I}\). An ideal topological space or simply an ideal space is a topological space \((X, \tau)\) with an ideal \(\mathcal{I}\) on \(X\) and is denoted by \((X, \tau, \mathcal{I})\). For a subset \(A \subseteq X\), \(A^*(\mathcal{I}, \tau) = \{x \in X : A \cap U \notin \mathcal{I} \text{ for every } U \in \tau(x)\}\), where \(\tau(x) = \{U \in \tau : x \in U\}\), is called the local function of \(A\) with respect to \(\mathcal{I}\) and \(\tau\) [11]. We simply write \(A^*\) in case there is no chance for confusion. A Kuratowski closure operator \(Cl^*(.)\) for a topology \(\tau^*(\mathcal{I}, \tau)\) called the \(\tau\)-topology, finer than \(\tau\), is defined by \(Cl^*(A) = A \cup A^*\) [8].

The following lemma will be useful in the sequel.

**Lemma 6.1.** [16] Let \((X, \tau)\) be a topological space. Then the following hold.

1. \(\mathcal{G}\) is a grill on \(X\) if and only if \(\mathcal{J} = \mathcal{P}(X) - \mathcal{G}\) is an ideal on \(X\),

2. The operators \(Cl^*\) on \((X, \tau, \mathcal{J})\), where \(\mathcal{J} = \mathcal{P}(X) - \mathcal{G}\), and \(\Psi\) on \((X, \tau, \mathcal{G})\) are equal.

**Remark 6.1.** Let \((X, \tau, \mathcal{G})\) be a grill topological space and \(A\) a subset of \(X\).

1. Since \(\tau \subseteq \tau_\mathcal{G}\), then every strongly-LC set is strongly-\(\mathcal{G}\)-LC.

2. If \(\mathcal{G} = \mathcal{P}(X) \setminus \{\phi\}\), then \(\tau = \tau_\mathcal{G}\) and hence both the notions of strongly-\(\mathcal{G}\)-LC and strongly-LC are equal.

3. If \(A \subseteq \Phi(A)\), then \(Cl(A) = \tau_\mathcal{G}\)-\(Cl(A)\) and hence both the notions of strongly-\(\mathcal{G}\)-LC and strongly-LC are equal.

4. If \(\mathcal{G} = \{X\}\), then \(\Phi(A) = \phi\) for any subset \(A\) of \(X\) and \(\Psi(A) = A\). Then any subset \(A\) of \(X\) is strongly-\(\mathcal{G}\)-LC.

5. For any subset \(A\) of a space \(X\) and any grill \(\mathcal{G}\) on \(X\), \(\Phi(A)\) is \(\tau_\mathcal{G}\)-closed.

6. Let \(\tau\) be suitable for \(\mathcal{G}\), that is, \(A - \Phi(A) \notin \mathcal{G}\) for all \(A \subseteq X\) [ [17], Definition 3.1] and \(\tau - \{\phi\} \subseteq \mathcal{G}\). Then if \((X, \tau_\mathcal{G})\) is regular then \(\tau = \tau_\mathcal{G}\) by Theorem 3.8 of [17] and hence both notions of strongly-\(\mathcal{G}\)-LC and strongly-LC are equal.
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References


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