On Radial Continuity of Metric Projections

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Abstract. Some new continuity concepts called Outer Radially Lower(ORL), Outer Radially Upper (ORU) and Inner Radially Lower (IRL), for set-valued metric projections were introduced in Banach spaces by B. Brosowski and F. Deutsch [Bull. Amer. Math. Soc. 78(1972), 974-978] to characterize suns and Chebyshev sets. In this paper we extend these concepts together with the concept of Inner Radially Upper (IRU) continuity to convex metric spaces and prove some results including those of Brosowski and Deutsch in such spaces.

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1 Introduction

Let V be a non-empty subset of a metric space (X, d) . The metric projection onto V is the mapping P_V , which takes each element x of

X into the set of (global) best approximations to x in V i.e. $P_V(x) = \{v \in V : d(x, v) = \text{dist}(x, V)\}$. V is called *proximal* if $P_V(x) \neq \emptyset$ for each $x \in X$ and *Chebyshev* if $P_V(x)$ contains exactly one point for each $x \in X$. P_V is called *lower semi-continuous* (*upper semi-continuous*) at x if for each open set W with $P_V(x) \cap W \neq \emptyset$ ($P_V(x) \subset W$) there exists a neighbourhood U of x such that $P_V(y) \cap W \neq \emptyset$ ($P_V(y) \subset W$) for all $y \in U$. Upper semi-continuity and lower semi-continuity of the metric projections have played a key role in discussing the structure of the set V and the geometry of the space (see [4] and references cited therein). Brosowski and Deutsch [3], [4] introduced in normed linear spaces some simple and more general 'radial' continuity criteria (called Outer Radial Lower, Inner Radial Lower and Outer Radial Upper), which require that the restriction of the metric projection to certain prescribed line segments be lower semi continuous (lsc) or upper semi continuous (usc). It turned out (see [3], [4]) that these criteria, which are much weaker than lsc or usc, were strong enough to generalize a number of known results and weak enough so that many of the known theorems have valid converses (which they did not have under the strong hypotheses of lsc or usc). In this paper we extend these concepts and also the concept of Inner Radially Upper (IRU) continuity introduced in normed linear spaces by Panda [10] to convex metric spaces and prove some results including those of Brosowski and Deutsch in such spaces. We shall see that for suns, the metric projection is ORL-continuous, for Chebyshev sets the metric projection is IRL and IRU-continuous and for closed convex set whose metric projection is compact-valued, the metric projection is ORU-continuous. A set having both IRL and ORU continuous metric projection is shown to be boundedly connected and have a connected-valued metric projection. The proofs of some of the lemmas and subsequent theorems given in this paper follow the line of arguments presented in [4] and [8] with suitable modifications.

2 Notations and Definitions

In this section we give some notations and recall few definitions. Let (X, d) be a metric space and $x, y, z \in X$. We say that z is *between* x

and y if $d(x, z) + d(z, y) = d(x, y)$. For any two points x, y of X , the set $\{z \in X : d(x, z) + d(z, y) = d(x, y)\}$ is called the *metric segment* and is denoted by $G[x, y]$. The set $G[x, y, -]$ denotes the *ray* starting from x and passing through y . The set $G_1(x, y, -) \equiv G[x, y, -] \setminus G[x, y]$ i.e. it is the set of all those points on the ray starting from x and passing through y which do not lie between x and y .

A continuous mapping $W : X \times X \times [0, 1] \rightarrow X$ is said to be a *convex structure* on X if for all $x, y \in X$ and $\lambda \in [0, 1]$

$$d(u, W(x, y, \lambda)) \leq \lambda d(u, x) + (1 - \lambda)d(u, y)$$

holds for all $u \in X$. A metric space (X, d) together with a convex structure is called a *convex metric space* [12].

A convex metric space (X, d) is said to satisfy *Property(I)* [6] if for all $x, y \in X$ and $\lambda \in [0, 1]$,

$$d(W(x, p, \lambda), W(y, p, \lambda)) \leq \lambda d(x, y),$$

where p is arbitrary but fixed point of X .

A convex metric space (X, d) is called *strongly convex* [11] or an *M-space* [7] if for every two points $x, y \in X$ and every $r \in [0, 1]$ there exists unique point $z_r = W(x, y, r) \in X$ such that $d(x, z_r) = (1-r)d(x, y)$ and $d(y, z_r) = rd(x, y)$ i.e. unique point z_r of the metric segment $G[x, y]$.

A metric space (X, d) is called *externally convex* [7] if for all distinct points x, y such that $d(x, y) = \lambda$, and $r > \lambda$ there exists a unique z of X such that $d(x, y) + d(y, z) = d(x, z) = r$.

A subset K of a convex metric space (X, d) is said to be *convex* [12] if $W(x, y, \lambda) \in K$ for all $x, y \in K$ and $\lambda \in [0, 1]$ i.e. $G[x, y] \subset K$ for all $x, y \in K$.

If V is a proximal subset of an externally convex M-space (X, d) , a point $v_0 \in V$ is called a *solar point* of V if $v_0 \in P_V(x)$ implies $v_0 \in P_V(x_\lambda)$ for every $x_\lambda \in G_1(v_0, x, -)$. The set V is called a *sun* [5] if for each $x \in X \setminus V$, every $v_0 \in P_V(x)$ is a solar point of V . A point $v_0 \in V$ is called a *lunar point* if $v_0 \in P_V(x)$ and $K(v_0, x) \cap V \neq \emptyset$ imply $v_0 \in \overline{K(v_0, x) \cap V}$ where $K(v_0, x) \equiv \bigcup_z B(z, d(z, v_0))$, $z \in G_1(v_0, x, -)$ and $B(z, r)$ stands for open ball with centre z and radius r . The set V is called a *moon* [1] if each of its point is lunar.

A point $v_0 \in V$ is called a *local best approximation* (LBA) to x in V if there is an $r > 0$ such that $v_0 \in P_{V \cap B(v_0, r)}(x)$. Clearly each best

approximation to x in V is a LBA but not conversely (see e.g. Braess [2], p.38).

3 Outer Radial Lower (ORL) Continuity

First generalization of lower semi continuity (lsc) is the following:

Let V be a non-empty subset of an externally convex M-space (X, d) and $x_0 \in X$. We say that P_V is *ORL-continuous* at x_0 if for each $v_0 \in P_V(x_0)$ and each open set W for which $P_V(x_0) \cap W \neq \emptyset$ there exists a neighbourhood U of x_0 such that $P_V(x) \cap W \neq \emptyset$ for all $x \in U \cap G_1(v_0, x_0, -)$. P_V is called ORL-continuous if it is ORL-continuous at each point.

The following lemma, which was proved for normed linear spaces by Brosowski and Deutsch ([4] - Lemma 2.2) and holds for externally convex M-spaces will be used in proving our first theorem:

Lemma 3.1. *Let V be a non-empty subset of an externally convex M-space (X, d) and $x_0 \in X$ then the following are equivalent:*

1. P_V is ORL-continuous at x_0 ,
2. for each $v_0, v_1 \in P_V(x_0)$ and each $\epsilon > 0$ there exists a $\delta > 0$ such that $P_V(x) \cap B(v_1, \epsilon) \neq \emptyset$ for all $x \in \{z : d(z, x_0) < \delta \text{ and } z \in G_1(v_0, x_0, -)\}$,
3. for each $v_0, v_1 \in P_V(x_0)$ and every sequence z_n in $G_1(v_0, x_0, -)$ with $z_n \rightarrow x_0$ there exists $v_n \in P_V(z_n)$ such that $v_n \rightarrow v_1$.

Theorem 3.1. *For a non-empty subset V of an externally convex M-space (X, d) , consider the following:*

1. V is a sun,
2. The metric projection P_V is ORL-continuous,
3. for each $x \in X$, every local best approximation is a global best approximation,
4. V is a moon,

then (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (4).

Proof. (1) $\Rightarrow$ (2) Suppose V is a sun. Let $x_0 \in X$, $v_0, v_1 \in P_V(x_0)$ and $\epsilon > 0$ be given. In view of Lemma 3.1, it is sufficient to show that $P_V(z) \cap B(v_1, \epsilon) \neq \emptyset$ for all $z \in G_1(v_0, x_0, -)$. As V is a sun, $v_0 \in P_V(z)$ for each $z \in G_1(v_0, x_0, -)$ i.e. for each z satisfying $d(z, v_0) = d(z, x_0) + d(x_0, v_0)$. Consider

$$\begin{aligned} d(z, v_1) &\leq d(z, x_0) + d(x_0, v_1) \\ &= d(z, v_0) - d(x_0, v_0) + d(x_0, v_1) \\ &= d(z, v_0) - d(x_0, V) + d(x_0, V) \\ &= d(z, v_0) \\ &\leq d(z, v_1) \text{ as } v_0 \in P_V(z). \end{aligned}$$

This gives $d(z, v_0) = d(z, v_1)$ and so $v_1 \in P_V(z)$ and hence $P_V(z) \cap B(v_1, \epsilon) \neq \emptyset$.

(2) $\Rightarrow$ (3) Let v_0 be a LBA to $x_0 \in X$ i.e. $v_0 \in P_{V \cap B(v_0, r)}(x_0)$ for some $r > 0$. Suppose v_0 is not GBA to x_0 . Therefore $d(x_0, v_0) \leq d(x_0, v)$ for all $v \in B(v_0, r) \cap V$ but $d(x_0, v_0) > d(x_0, V)$. Let x_1 be the last point on $G[v_0, x_0]$ which has v_0 as a GBA in V (such a point clearly exists since $v_0 \in P_V(v_0)$ and $v_0 \notin P_V(x_0)$ but v_0 is a LBA to x_0). Thus $d(x, V) < d(x, v_0)$ for all $x \in G[x_1, x_0] \setminus \{x_1\}$. Now $P_V(x_1) \cap B(v_0, r) \neq \emptyset$, and so by ORL continuity we can find a neighbourhood $B(x_1, \delta)$ such that $P_V(x) \cap B(v_0, r) \neq \emptyset$ for all $x \in B(x_1, \delta) \cap G_1(v_0, x_1, -)$. If $x_\lambda \in B(x_1, \delta) \cap G_1(v_0, x_1, -)$ and $v \in P_V(x_\lambda) \cap B(v_0, r)$ then

$$d(x_\lambda, v) = d(x_\lambda, V) < d(x_\lambda, v_0).$$

Also $d(v, v_0) < r$. Consider

$$d(x_0, v_0) = d(x_0, x_\lambda) + d(x_\lambda, v_0) > d(x_0, x_\lambda) + d(x_\lambda, v) \geq d(x_0, v).$$

Therefore v_0 is not a LBA to x_0 , which is a contradiction. Hence v_0 is a GBA to x_0 .

(3) $\Rightarrow$ (4) Suppose V is not a moon. Then there exists $v_0 \in V$ and $x \in P_V^{-1}(v_0)$ with $K(v_0, x) \cap V \neq \emptyset$ such that $v_0 \notin \overline{K(v_0, x) \cap V}$ i.e. there exists $r > 0$ such that

$$B(v_0, r) \cap [K(v_0, x) \cap V] = \emptyset \text{ i.e. } B(v_0, r) \cap K(v_0, x) \subseteq X \setminus V. \quad (3.1)$$

Now, by the definition of $K(v_0, x)$,

$B(z, d(z, v_0)) \subseteq K(v_0, x)$ for every $z \in G_1(v_0, x, -)$. So, (3.1) implies

$$B(v_0, r) \cap B(z, d(z, v_0)) \subseteq X \setminus V \text{ for every } z \in G_1(v_0, x, -) \quad (3.2)$$

As $K(v_0, x) \cap V \neq \emptyset$, there exists $u \in K(v_0, x) \cap V$ and so $u \in B(z, d(z, v_0))$ for some $z \in G_1(v_0, x, -)$ i.e. $d(z, u) < d(z, v_0)$ and so v_0 can never be a GBA to z in V .

Now we show that v_0 is a LBA to z so chosen. Suppose z has v' other than v_0 as a LBA in the r -neighbourhood of v_0 in V i.e. $v' \in P_{V \cap B(v_0, r)}(z)$ and so $d(z, v') < d(z, v_0)$ which implies $v' \in B(z, d(z, v_0))$. Also $v' \in B(v_0, r)$ and so $v' \in B(z, d(z, v_0)) \cap B(v_0, r)$. But then (3.2) gives $v' \in X \setminus V$, a contradiction. Therefore v_0 is a LBA but not a GBA to z , contradicting the hypothesis. Hence V is a moon. $\square$

Note 3.1. For normed linear spaces above theorem was given by Brosowski and Deutsch [4]- Theorem 2.3 (see also Mhaskar and Pai [8]- Theorem 7, p.468). In general, none of the implications in Theorem 3.1 is reversible (see [4]).

We call a space X an *MS-space* [4] if every moon in X is a sun. In such a space all the conditions of the above theorem are equivalent. In particular, we have

Corollary 3.1. Let X be an MS-space and $V \subset X$. Then V is a sun if and only if P_V is ORL-continuous.

4 Inner Radial Lower(IRL) Continuity

A second generalization of lsc is:

Let V be a non-empty subset of a convex metric space (X, d) and $x_0 \in X$. We say that P_V is *IRL-continuous* at x_0 if for each $v_0 \in P_V(x_0)$ and each open set W with $P_V(x_0) \cap W \neq \emptyset$, there exists a neighbourhood U of x_0 such that $P_V(x) \cap W \neq \emptyset$ for every $x \in U \cap G[v_0, x_0]$. P_V is called IRL-continuous if it is IRL-continuous at each point of X .

Each lsc metric projection is IRL continuous but converse is not true (see [4]).

The following lemma, which was proved for normed linear spaces by Brosowski and Deutsch [4] and holds for convex metric spaces, will be used in our next theorem:

Lemma 4.1. *Let V be a non-empty subset of a convex metric space (X, d) and $x_0 \in X$ then the following are equivalent:*

1. P_V is IRL-continuous at x_0 ,
2. for each $v_0, v_1 \in P_V(x_0)$, and each $\epsilon > 0$ there exists a $\delta > 0$ such that $P_V(x) \cap B(v_1, \epsilon) \neq \emptyset$ for all x in $G[v_0, x_0]$ with $d(x, x_0) < \delta$,
3. for each $v_0, v_1 \in P_V(x_0)$, and every sequence x_n in $G[v_0, x_0]$ with $x_n \rightarrow x_0$, $d(v_1, P_V(x_n)) \rightarrow 0$ (i.e. there exists $v_n \in P_V(x_n)$ such that $v_n \rightarrow v_1$).

Theorem 4.1. *Let (X, d) be a convex metric space satisfying Property (I) and $x \in X$. If $P_V(x)$ is convex then P_V is IRL-continuous at x .*

Proof. If $P_V(x) = \emptyset$, the result is trivially true. Let $v_0, v_1 \in P_V(x)$ and $x_n \in G[x, v_0]$ with $x_n \rightarrow x$. Thus $x_n = W(x, v_0, 1 - \epsilon_n)$ where $0 \leq \epsilon_n \leq 1$ and $\epsilon_n \rightarrow 0$. Also $d(x_n, x) = \epsilon_n d(v_0, x)$, $d(x_n, v_0) = (1 - \epsilon_n)d(x, v_0)$. Let $v_n = W(v_1, v_0, 1 - \epsilon_n)$. Then $v_n \in P_V(x)$ as $P_V(x)$ is convex and $v_n \rightarrow v_1$. Also

$$\begin{aligned} d(x_n, v_n) &= d(W(x, v_0, 1 - \epsilon_n), W(v_1, v_0, 1 - \epsilon_n)) \\ &\leq (1 - \epsilon_n)d(x, v_1) \\ &= (1 - \epsilon_n)d(x, v_0) \\ &= d(x_n, v_0) \\ &= d(x_n, V) \end{aligned}$$

as $x_n \in G[x, v_0]$ and $v_0 \in P_V(x)$ imply $v_0 \in P_V(x_n)$ [13]. This gives $v_n \in P_V(x_n)$ and hence P_V is IRL-continuous at x by the above lemma. $\square$

Remark 4.1. The converse of above theorem is not true (see [4]).

Remark 4.2. Since for convex sets V , $P_V(x)$ is convex [9] and for Chebyshev sets V , $P_V(x)$ being singleton is convex, we have

Corollary 4.1. If V is a convex or a Chebyshev set in a convex metric space (X, d) satisfying Property (I) then P_V is IRL-continuous.

Neither of the sufficient conditions of the corollary are necessary (see [4]).

5 Outer Radial Upper(ORU) Continuity

A first generalization of upper semi continuity is:

Let V be a non-empty subset of an externally convex M-space (X, d) and $x_0 \in X$. We say that P_V is *ORU-continuous* at x_0 if for each $v_0 \in P_V(x_0)$ and each open set $W \supset P_V(x_0)$ there exists a neighbourhood U of x_0 such that $P_V(x) \subset W$ for all $x \in U \cap G_1(v_0, x_0, -)$. P_V is called ORU-continuous if it is ORU-continuous at each point of X . The following lemma, which for normed linear spaces is given in [4]-Lemma 4.2, and holds in externally convex M-spaces deals with ORU-continuity.

Lemma 5.1. *Let V be a non-empty subset of an externally convex M-space (X, d) and $x_0 \in X$. Consider the following statements:*

1. P_V is ORU-continuous at x_0 .
2. For each $v_0 \in P_V(x_0)$ and each $\epsilon > 0$ there exists a $\delta > 0$ such that

$$\sup_{v \in P_V(x)} d(v, P_V(x_0)) < \epsilon$$

for every x in $G_1(v_0, x_0, -)$ with $d(x, x_0) < \delta$.

3. For each $v_0 \in P_V(x_0)$ and each sequence x_n in $G_1(v_0, x_0, -)$ with $x_n \rightarrow x_0$, $\sup_{v \in P_V(x_n)} d(v, P_V(x_0)) \rightarrow 0$.
4. For each $v_0 \in P_V(x_0)$ and each sequence x_n in $G_1(v_0, x_0, -)$ with $x_n \rightarrow x_0$ and each sequence v_n with $v_n \in P_V(x_n)$, $d(v_n, P_V(x_0)) \rightarrow 0$.

5. For each $v_0 \in P_V(x_0)$ and each sequence x_n in $G_1(v_0, x_0, -)$ with $x_n \rightarrow x_0$ and each sequence v_n with $v_n \in P_V(x_n)$ and $v_n \rightarrow v$, $v \in \overline{P_V(x_0)}$.

Then $(1) \Rightarrow (2) \Rightarrow (3) \Rightarrow (4) \Rightarrow (5)$.

Note 5.1. If in the above lemma, $P_V(x_0)$ is compact then $(4) \Rightarrow (1)$ and if V is compact then $(5) \Rightarrow (1)$.

Remark 5.1. In general, the implication $(4) \Rightarrow (1)$ and $(5) \Rightarrow (1)$ are false (see [4]).

The following theorem gives conditions under which metric projection is ORU continuous:

Theorem 5.1. If V is a sun in an externally convex M -space (X, d) such that $P_V(x)$ is compact for every $x \in X$ then P_V is ORU-continuous.

Before proving this theorem, we establish a lemma.

Lemma 5.2. If V is a sun in an externally convex M -space (X, d) , $v_0 \in P_V(x_0)$ and x_n is a sequence in $G_1(v_0, x_0, -)$ with $x_n \rightarrow x_0$ then $P_V(x_0) = \bigcap_{n=1}^{\infty} P_V(x_n)$.

Proof. Let $v \in P_V(x_0)$. Then for each n ,

$$\begin{aligned} d(x_n, v) &\leq d(x_n, x_0) + d(x_0, v) \\ &= d(x_n, x_0) + d(x_0, v_0) \\ &= d(x_n, v_0) \\ &= d(x_n, V) \end{aligned} \tag{5.1}$$

as $v_0 \in P_V(x_0)$ and V is a sun, so $v_0 \in P_V(x_n)$ [13]. Then (5.1) implies $v \in P_V(x_n)$ for each n and hence $P_V(x_0) \subset \bigcap_{n=1}^{\infty} P_V(x_n)$.

Conversely if $v \in \bigcap_{n=1}^{\infty} P_V(x_n)$ then $d(x_n, v) = d(x_n, V)$ for each n . Consider

$$\begin{aligned} d(x_0, v) &\leq d(x_0, x_n) + d(x_n, v) \\ &= d(x_0, x_n) + d(x_n, V) \\ &\rightarrow d(x_0, V). \end{aligned}$$

Therefore $v \in P_V(x_0)$. □

Proof of Theorem 5.1. Let $x_0 \in X$ and $v_0 \in P_V(x_0)$. Suppose x_n is a sequence in $G_1(v_0, x_0, -)$ with $x_n \rightarrow x_0$. As V is a sun, $v_0 \in P_V(x_n)$ for every n . Let W be an open set with $W \supset P_V(x_0)$. By Lemma 5.2, $P_V(x_0) = \bigcap_{n=1}^{\infty} P_V(x_n)$.

We claim that $P_V(x_n) \subset P_V(x_{n-1})$. Let $v \in P_V(x_n)$. Then $d(x_n, v) = d(x_n, V)$. Consider

$$\begin{aligned} d(x_{n-1}, v) &\leq d(x_{n-1}, x_n) + d(x_n, v) \\ &= d(x_{n-1}, x_n) + d(x_n, V) \text{ as } v \in P_V(x_n) \\ &= d(x_{n-1}, x_n) + d(x_n, v_0) \\ &= d(x_{n-1}, v_0) \text{ as } x_n \in G_1(v_0, x_0, -) \text{ for all } n \\ &= d(x_{n-1}, V) \text{ as } v_0 \in \bigcap_{n=1}^{\infty} P_V(x_n) \end{aligned}$$

This implies $v \in P_V(x_{n-1})$ and $P_V(x_n) \subset P_V(x_{n-1})$ for all n . Thus $P_V(x_n)$ is a decreasing sequence of compact sets. Let $v_n \in P_V(x_n)$, $n = 1, 2, \dots$. Then $v_n \in P_V(x_1)$ for all n . Since $P_V(x_1)$ is compact, v_n has a convergent subsequence $v_{n_i} \rightarrow v$. Now we claim that $v \in P_V(x_0) = \bigcap_{n=1}^{\infty} P_V(x_n)$. Let n be fixed but arbitrary. Consider

$$\begin{aligned} d(x_n, v) &= d(x_n, \lim v_{n_i}) \\ &= \lim_i d(x_n, v_{n_i}) \\ &\leq \lim_i [d(x_n, x_{n_i}) + d(x_{n_i}, v_{n_i})] \\ &= \lim_i [d(x_n, x_{n_i}) + d(x_{n_i}, V)] \\ &= \lim_i [d(x_n, x_{n_i}) + d(x_{n_i}, v_0)] \\ &= d(x_n, x_0) + d(x_0, v_0) \\ &= d(x_n, v_0) \\ &= d(x_n, V). \end{aligned}$$

This gives $v \in P_V(x_n)$ for all n i.e. $v \in \bigcap_{n=1}^{\infty} P_V(x_n) = P_V(x_0)$. Thus $v \in P_V(x_0) \subset W$. As W is open, there exists N such that $v_n \in W$ for all $n \geq N$. Thus $P_V(x_n) \subset W$ for all $n \geq N$. Therefore for some $\delta > 0$, $P_V(x) \subset W$ for all x in $G_1(v_0, x_0, -) \cap B(x_0, \delta)$ and so there exists a neighbourhood U of x_0 such that $P_V(x) \subset W$ for all $x \in G_1(v_0, x_0, -)$. Therefore P_V is ORU-continuous at x_0 and hence P_V is ORU-continuous. $\square$

Corollary 5.1. If V is a Chebyshev sun in an externally convex M-space (X, d) then P_V is ORU-continuous.

Note 5.2. For normed linear spaces, Theorem 5.1 is given in [4].

6 Inner Radial Upper (IRU) Continuity

A second generalization of usc is:

Let V be a non-empty subset of a convex metric space (X, d) and $x_0 \in X$. We say that P_V is *IRU-continuous* at x_0 if for each $v_0 \in P_V(x_0)$ and each open set $W \supset P_V(x_0)$ there exists a neighborhood U of x_0 such that $P_V(x) \subset W$ for all $x \in U \cap W(v_0, x_0, \lambda)$. P_V is called IRU-continuous if it is IRU-continuous at each point.

Concerning IRU-continuity in convex metric space we have:

Proposition 6.1. Let $V \subset X$ and $x_0 \in X$. Consider the following statements:

1. P_V is IRU-continuous at x_0 .
2. For each $v_0 \in P_V(x_0)$ and each $\epsilon > 0$ there exists a $\delta > 0$ such that

$$\sup_{v \in P_V(x)} d(v, P_V(x_0)) < \epsilon \text{ for every } x \text{ in } \{W(v_0, x_0, \lambda) : \lambda \in [0, 1]\}$$
with $d(x, x_0) < \delta$.
3. For each $v_0 \in P_V(x_0)$ and each sequence x_n in $W(v_0, x_0, \lambda)$ with $x_n \rightarrow x_0$, $\sup_{v \in P_V(x_n)} d(v, P_V(x_0)) \rightarrow 0$.
4. For each $v_0 \in P_V(x_0)$ and each sequence x_n in $W(v_0, x_0, \lambda)$ with $x_n \rightarrow x_0$ and each sequence v_n with $v_n \in P_V(x_n)$, $d(v_n, P_V(x_0)) \rightarrow 0$.
5. For each $v_0 \in P_V(x_0)$ and each sequence x_n in $W(v_0, x_0, \lambda)$ with $x_n \rightarrow x_0$ and each sequence v_n with $v_n \in P_V(x_n)$ and $v_n \rightarrow v$, $v \in \overline{P_V(x_0)}$.

Then $(1) \Rightarrow (2) \Leftrightarrow (3) \Leftrightarrow (4) \Rightarrow (5)$. Moreover if $P_V(x_0)$ is compact, $(4) \Rightarrow (1)$ and the first four statements are equivalent. If V is compact then $(5) \Rightarrow (1)$ and hence all the five statements are equivalent.

Proof. (1) $\Rightarrow$ (2) Choose $v_0 \in P_V(x_0)$ and $W = \bigcup \{B(v, \frac{\epsilon}{2}) : v \in P_V(x_0)\} \supset P_V(x_0)$. By the IRU-continuity, there exists a neighbourhood $B(x_0, \delta)$ of x_0 such that $P_V(x) \subset W$ for every x in $B(x_0, \delta) \cap \{W(v_0, x_0, \lambda) : \lambda \in [0, 1]\}$.

Let $x \in B(x_0, \delta) \cap \{W(v_0, x_0, \lambda) : \lambda \in [0, 1]\}$ and $v \in P_V(x)$ then there exists $v' \in P_V(x_0)$ such that $d(v, v') < \frac{\epsilon}{2}$ and so $d(v, P_V(x_0)) < \frac{\epsilon}{2}$ as $d(v, P_V(x_0)) = \inf_{w \in P_V(x_0)} d(v, w)$. It follows that $\sup_{v \in P_V(x)} d(v, P_V(x_0)) \leq$

$$\frac{\epsilon}{2} < \epsilon.$$

(2) $\Rightarrow$ (3) Let $v_0 \in P_V(x_0)$ and x_n be a sequence in $W(v_0, x_0, \lambda)$ with $x_n \rightarrow x_0$. Therefore, for $\delta > 0$ there exists a positive integer N such that $d(x_n, x_0) < \delta$ for all $n \geq N$. If $n \geq N$, $x_n \in B(x_0, \delta) \cap W(v_0, x_0, \lambda)$ and so by the hypothesis, $\sup_{v \in P_V(x_n)} d(v, P_V(x_0)) < \epsilon$ for all $n \geq N$, which

implies $\sup_{v \in P_V(x_n)} d(v, P_V(x_0)) \rightarrow 0$.

(3) $\Rightarrow$ (2) Suppose (2) does not hold i.e. there exists $v_0 \in P_V(x_0)$ and $\epsilon > 0$ such that for every $\delta > 0$, $\sup_{v \in P_V(x)} d(v, P_V(x_0)) \geq \epsilon$ for all x in

$\{W(v_0, x_0, \lambda) : \lambda \in [0, 1]\}$ with $d(x, x_0) < \delta$. Let x_n be a sequence in $\{W(v_0, x_0, \lambda) : \lambda \in [0, 1]\}$ with $x_n \rightarrow x_0$. Then $\sup_{v \in P_V(x_n)} d(v, P_V(x_0)) \geq \epsilon$

for all n and so $\sup_{v \in P_V(x_n)} d(v, P_V(x_0))$ does not converge to 0, a contradiction.

(3) $\Rightarrow$ (4) Let $v_0 \in P_V(x_0)$ and x_n be a sequence in $W(v_0, x_0, \lambda)$ with $x_n \rightarrow x_0$ and v_n be a sequence in $P_V(x_n)$.

By hypothesis, $\sup_{v \in P_V(x_n)} d(v, P_V(x_0)) \rightarrow 0$ and so $d(v_n, P_V(x_0)) \rightarrow 0$.

(4) $\Rightarrow$ (3) Suppose (3) does not hold i.e. for some $v_0 \in P_V(x_0)$ there exists a sequence x_n in $W(v_0, x_0, \lambda)$ with $x_n \rightarrow x_0$ such that $\sup_{v \in P_V(x_n)} d(v, P_V(x_0))$ does not converge to 0. Let v_n be a sequence in $P_V(x_n)$.

So we have $\sup_{v_n \in P_V(x_n)} d(v_n, P_V(x_0))$ does not converge to 0. This implies $d(v_n, P_V(x_0))$ does not converge to 0 for some $v_n \in P_V(x_n)$, a contradiction.

(4) $\Rightarrow$ (5) Let $v_0 \in P_V(x_0)$, x_n be a sequence in $W(v_0, x_0, \lambda)$ with $x_n \rightarrow x_0$ and v_n be a sequence in $P_V(x_n)$ with $v_n \rightarrow v$. By hypothesis, $d(v_n, P_V(x_0)) \rightarrow 0$ i.e. $\lim d(v_n, P_V(x_0)) = 0$. This implies

$$d(v, P_V(x_0)) = 0 \Rightarrow v \in \overline{P_V(x_0)}.$$

Suppose $P_V(x_0)$ is compact. To prove (4) $\Rightarrow$ (1).

Suppose (1) does not hold i.e. there exists $v_0 \in P_V(x_0)$ and some open set $W \supset P_V(x_0)$ such that for every n there is an x_n in $\{W(v_0, x_0, \lambda) : 1 - \frac{1}{n} < \lambda \leq 1\}$ such that $P_V(x_n) \cap W^c \neq \emptyset$. Choose $v_n \in P_V(x_n) \cap W^c$. Then $x_n \rightarrow x_0$ and so $d(v_n, P_V(x_0)) \rightarrow 0$. Choose $y_n \in P_V(x_0)$ such that $d(v_n, y_n) \rightarrow 0$. As $P_V(x_0)$ is compact, there exists a subsequence $y_{n_i} \rightarrow y_0 \in P_V(x_0)$. Consider

$$d(v_{n_i}, y_0) \leq d(v_{n_i}, y_{n_i}) + d(y_{n_i}, y_0) \rightarrow 0.$$

This implies $v_{n_i} \rightarrow y_0$. Since $y_0 \in P_V(x_0) \subset W$, $y_0 \in W$ and W is open, $v_n \in W$ for large n , a contradiction.

Suppose V is compact. To prove (5) $\Rightarrow$ (1). We show that (5) $\Rightarrow$ (4) $\Rightarrow$ (1).

Let $v_0 \in P_V(x_0)$ and x_n be a sequence in $\{W(v_0, x_0, \lambda) : \lambda \in [0, 1]\}$ with $x_n \rightarrow x_0$ and v_n be a sequence with $v_n \in P_V(x_n)$. To prove $d(v_n, P_V(x_0)) \rightarrow 0$. Since $P_V(x_n)$ is a closed subset of the compact set V , it is compact and so v_n has a subsequence $v_{n_i} \rightarrow v$. By hypothesis, $v \in \overline{P_V(x_0)}$. Consider

$$d(v_{n_i}, P_V(x_0)) \leq d(v_{n_i}, v) + d(v, P_V(x_0)) \rightarrow 0$$

i.e. (4) is true. Since V is compact, $P_V(x_0)$ is compact and so (4) $\Rightarrow$ (1). Hence (5) $\Rightarrow$ (1). $\square$

For Chebyshev sets, we have

Theorem 6.1. *If V is Chebyshev set in a convex metric space (X, d) then P_V is IRU continuous.*

Proof. Let $x_0 \in X$ be arbitrary and $v_0 \in P_V(x_0)$. Let G be an open set with $G \supset P_V(x_0) = \{v_0\}$. Let $x \in \{W(x_0, v_0, \lambda) : \lambda \in [0, 1]\}$ then $v_0 \in P_V(x)[9]$. As V is Chebyshev, $P_V(x) = \{v_0\}$. Let U be any neighborhood of x_0 then for all $x \in U \cap \{W(x_0, v_0, \lambda) : \lambda \in [0, 1]\}$, $P_V(x) = \{v_0\} \subset G$. Hence P_V is IRU-continuous. $\square$

7 IRL and ORU Continuity

In this section, we discuss the structure of the set having both IRL and ORU continuous metric projection.

A subset V of a metric space (X, d) is said to be *boundedly connected* if $V \cap B(x, r)$ is connected for every $x \in X$ and $r > 0$.

Brosowski and Deutsch [4] proved that for Banach spaces, a set having both IRL and ORU-continuous metric projection is boundedly connected and have a connected valued metric projection. We now extend this result to externally convex M-spaces.

Theorem 7.1. *Let V be a proximal subset of an externally convex M-space (X, d) such that P_V is both IRU and ORU-continuous then V is boundedly connected and $P_V(x)$ is connected for each x in X .*

Proof. Suppose V is not boundedly connected. Then there exists some $x_0 \in X$ and a positive $r > d(x_0, V)$ such that $V \cap B(x_0, r)$ is not connected ($d(x_0, V) \geq r \Rightarrow V \cap B(x_0, r) = \emptyset$ and so there is nothing to prove). Let $V \cap B(x_0, r) = E \cup F$, where E and F are nonempty disjoint sets which are open in V . We claim that $P_V(x_0) \subseteq E \cup F$. Let $v_0 \in P_V(x_0)$ i.e. $v_0 \in V$ and $d(x_0, v_0) = d(x_0, V) < r$. Therefore $v_0 \in V \cap B(x_0, r) = E \cup F$ and so $P_V(x_0) \subseteq E \cup F$. We may assume $P_V(x_0) \cap E \neq \emptyset$ as $P_V(x_0) \subseteq E \cup F$, where E and F are non-empty disjoint sets open in V implies $P_V(x_0)$ will have some points common with E or F . Let $y \in F$ then there is a $\lambda_0 \in (0, 1)$ such that for every $\lambda \in [\lambda_0, 1]$, $P_V(W(y, x_0, \lambda)) \subseteq F$. Let

$$\beta = \inf\{\lambda \in [0, 1] : P_V(W(y, x_0, \lambda)) \subseteq F\}.$$

We first note that $P_V(W(y, x_0, \beta)) \subseteq F$. For if not, then $P_V(W(y, x_0, \beta)) \cap E \neq \emptyset$ (we have $P_V(W(y, x_0, \lambda)) \subseteq E \cup F$ for all $\lambda \in [0, 1]$ as $v_0 \in P_V(W(y, x_0, \lambda))$). This implies $d(W(y, x_0, \lambda), v_0) \leq d(W(y, x_0, \lambda), v)$ for all $v \in V$ and therefore $d(W(y, x_0, \lambda), v_0) \leq d(W(y, x_0, \lambda), y) = (1 - \lambda)d(y, x_0) < (1 - \lambda)r$ as $y \in F \subseteq E \cup F = V \cap B(x_0, r)$. Consider

$$d(x_0, v_0) \leq d(x_0, W(y, x_0, \lambda)) + d(W(y, x_0, \lambda), v_0) < \lambda r + (1 - \lambda)r = r$$

and so $v_0 \in V \cap B(x_0, r) = E \cup F$. Choose $v_0 \in P_V(W(y, x_0, \beta)) \cap E$. For any sequence $x_n \in G[W(y, x_0, \beta), y]$ such that $x_n \rightarrow W(y, x_0, \beta)$,

there exists (by IRL-continuity) $v_n \in P_V(x_n) \subseteq F$ ($x_n \in G[W(y, x_0, \beta), y] \Rightarrow P_V(x_n) \subseteq F$) such that $v_n \rightarrow v_0 \in E$. But this is not possible as E is open in V and $v_n \in F \setminus E$ for every n . Thus $P_V(W(y, x_0, \beta)) \subseteq F$. We now claim that best approximations to $W(y, x_0, \beta)$ lie on the line joining $W(y, x_0, \beta)$ and y . Suppose this is not true. Then there exists some $v \in F$ not on the line segment joining $W(y, x_0, \beta)$ and y such that v is a best approximation to $W(y, x_0, \beta)$ i.e.

$$d(W(y, x_0, \beta), v) \leq d(W(y, x_0, \beta), z) \text{ for all } z \in F. \quad (7.1)$$

Since $\beta = \inf\{\lambda \in [0, 1] : P_V(W(y, x_0, \lambda)) \subseteq F\}$, we can find a sequence β_n such that $P_V(W(y, x_0, \beta_n)) \subseteq F$ and $\beta_n \rightarrow \beta$. Then from (7.1), we get $d(W(y, x_0, \beta), v) < d(W(y, x_0, \beta), W(y, x_0, \beta_n))$ as ($W(y, x_0, \beta_n)$ lie on the line segment joining $W(y, x_0, \beta)$ and y and so cannot be a best approximation) $= d(W(y, x_0, \beta), y) - d(W(y, x_0, \beta_n), y) = (1 - \beta)d(y, x_0) - (1 - \beta_n)d(y, x_0) = (\beta_n - \beta)d(y, x_0) \rightarrow 0$.

So, $d(W(y, x_0, \beta), v) < 0$ as equality is not true in view of definition of v , a contradiction. Thus best approximation to $W(y, x_0, \beta)$ lie on the line segment joining $W(y, x_0, \beta)$ and y .

Let $W(y, x_0, \alpha) \in P_V(W(y, x_0, \beta)) \subseteq F$ with $\alpha > \beta$.

By the ORU-continuity of P_V , there exists a neighborhood $B_k(W(y, x_0, \beta))$ such that

$P_V(x) \subseteq F$ for every x in $B_k(W(y, x_0, \beta)) \cap \{G_1(W(y, x_0, \alpha), W(y, x_0, \beta), -)\}$. Choose $\epsilon < \frac{k}{d(x_0, y)}$. Consider

$$\begin{aligned} d(W(y, x_0, \beta), W(y, x_0, \beta - \epsilon)) &= d(x_0, W(y, x_0, \beta)) - \\ &\quad d(x_0, W(y, x_0, \beta - \epsilon)) \\ &= \beta d(x_0, y) - (\beta - \epsilon)d(x_0, y) \\ &= \epsilon d(x_0, y) \\ &< k. \end{aligned}$$

Therefore $W(y, x_0, \beta - \epsilon) \in B_k(W(y, x_0, \beta))$.

Also $W(y, x_0, \beta - \epsilon) \in \{G_1(W(y, x_0, \alpha), W(y, x_0, \beta), -)\}$ and so $P_V(W(y, x_0, \beta - \epsilon)) \subseteq F$, which contradicts the definition of β and hence V is boundedly connected. The proof that $P_V(x)$ is connected for each x is virtually the same.

□

Corollary 7.1. Let V be a Chebyshev set in an externally convex M-space satisfying Property (I) and P_V is ORU-continuous. Then V is boundedly connected.

Proof. Since V is Chebyshev, P_V is IRL-continuous (Corollary 4.1) and the proof follows. $\square$

Corollary 7.2. Let V be a Chebyshev sun in an externally convex M-space satisfying Property (I) then V is boundedly connected.

Proof. Since V is Chebyshev, P_V is IRL-continuous (Corollary 4.1). Also, $P_V(x)$ is compact as V is Chebyshev. Since V is a sun and $P_V(x)$ is compact, P_V is ORU-continuous (Theorem 5.1). Hence V is boundedly connected $\square$

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