On Uniform Exponential Dichotomy of Evolution Operators

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Abstract. The present paper presents three distinct concepts of uniform exponential dichotomy for evolution operators in Banach spaces. Characterizations and relationships between them and some illustrative counterexamples are given.

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1 Preliminaries

Let X be a real or complex Banach space, and $\mathcal{B}(X)$ the algebra of bounded linear operators acting on X . We denote by $\|\cdot\|$ the norm on X and on $\mathcal{B}(X)$, and let Δ be the set of all pairs of real nonnegative numbers (t, s) satisfying $t \geq s$.

Definition 1.1. A map $U : \Delta \rightarrow \mathcal{B}(X)$ is called an *evolution operator* on X if the following conditions hold:

$e_1)$ $U(t, t) = I$, for all $t \geq 0$.

$e_2)$ $U(t, s)U(s, t_0) = U(t, t_0)$, for all $(t, s), (s, t_0) \in \Delta$.

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In addition,

(i) if there exist $M \geq 1$ and $\omega > 0$ such that

$$\|U(t, s)x\| \leq Me^{\omega(t-s)}\|x\| \quad \forall (t, s, x) \in \Delta \times X$$

then we say that $U : \Delta \rightarrow \mathcal{B}(X)$ is an evolution operator with **uniform exponential growth** (and we write $U : \Delta \rightarrow \mathcal{B}(X)$ has (ueg));

(ii) if for all $(t, s) \in \Delta$ the operator $U(t, s)$ is bijective then we say that $U : \Delta \rightarrow \mathcal{B}(X)$ is **reversible**.

Definition 1.2. A map $P : \mathbb{R}_+ \rightarrow \mathcal{B}(X)$ is called a **family of projections on X** if

$$P(t)^2 = P(t), \quad \forall t \geq 0.$$

If $P : \mathbb{R}_+ \rightarrow \mathcal{B}(X)$ is a family of projections on X then $Q : \mathbb{R}_+ \rightarrow \mathcal{B}(X)$ defined, for all $t \geq 0$ by

$$Q(t) = I - P(t)$$

is also a family of projections on X , called the *complementary projection* of P .

Definition 1.3. We say that a family of projections $P : \mathbb{R}_+ \rightarrow \mathcal{B}(X)$ is **compatible** with an evolution operator $U : \Delta \rightarrow \mathcal{B}(X)$ if the following condition holds

$$U(t, s)P(s) = P(t)U(t, s), \quad \forall (t, s) \in \Delta.$$

Remark 1.1. Let $U : \Delta \rightarrow \mathcal{B}(X)$ be an evolution operator and $P : \mathbb{R}_+ \rightarrow \mathcal{B}(X)$ a family of projections compatible with U . Then we have that

(i) for all $(t, s) \in \Delta$, $U(t, s)Q(s) = Q(t)U(t, s)$.

(ii) In addition, if $U : \Delta \rightarrow \mathcal{B}(X)$ is reversible, then for all $(t, s) \in \Delta$ we have that

$$Q(s)U(t, s)^{-1} = U(t, s)^{-1}Q(t).$$

In what follows, for an evolution operator $U : \Delta \rightarrow \mathcal{B}(X)$ and a family of projections $P : \mathbb{R}_+ \rightarrow \mathcal{B}(X)$, we will use the following notations:

$$U_P(t, s) = U(t, s)P(s), \quad U_Q(t, s) = U(t, s)Q(s), \quad (t, s) \in \Delta.$$

In this paper, three notions of uniform exponential dichotomy for evolution operators are introduced. The classical concept of exponential dichotomy for

nonautonomous linear differential equations has been introduced by Perron [9] in the late 1920s. In this sequel, many authors developed the theory. For fundamental books we refer to Coppel [3], Massera and Schäffer [5], Daleckii and Krein [4], Barreira and Valls [1].

The first concept considered in this paper is the strong uniform exponential dichotomy. It is inspired by the [1] and [2]. The second, the uniform exponential dichotomy, is considered in [3], [4], [5], [6], [7] and [10].

We introduce a new concept of uniform exponential dichotomy which will be named weak uniform exponential dichotomy. It is implied by both of the preceding concepts and it is different from them.

The implications between these concepts are studied.

2 Concepts of uniform exponential dichotomy

A first concept of uniform exponential dichotomy is introduced by

Definition 2.1. Let $U : \Delta \rightarrow \mathcal{B}(X)$ be an evolution operator and $P : \mathbb{R}_+ \rightarrow \mathcal{B}(X)$ a family of projections. We say that U is ***P-strongly uniformly exponentially dichotomic*** (and we write *P-s.u.e.d*) if there exist constants $N \geq 1$, $\nu > 0$ such that for all $(t, s) \in \Delta$ the following hold:

$$(sued1) \quad \|U_P(t, s)\| \leq N e^{-\nu(t-s)};$$

$$(sued2) \quad N \|U_Q(t, s)\| \geq e^{\nu(t-s)}.$$

Remark 2.1. If $U : \Delta \rightarrow \mathcal{B}(X)$ is P-s.u.e.d then

$$1 \leq \|P(t)\| \leq N, \quad 1 \leq \|Q(t)\| \leq 2N \quad \forall t \geq 0.$$

The analogus of Definition 2.1 in the strong topology cannon give a proper dichotomy concept, the decomposition of the state space always being trivial as we will show below. Assume that there exist an evolution operator $U : \Delta \rightarrow \mathcal{B}(X)$ and a family of projections $P : \mathbb{R}_+ \rightarrow \mathcal{B}(X)$ such that there exist $N \geq 1$ and $\nu > 0$ verifying

$$N \|U_Q(t, s)x\| \geq e^{\nu(t-s)} \|x\|, \quad \forall (t, s, x) \in \Delta \times X.$$

Let $t \geq 0$ and $s = t$ in the above estimation. It follows that

$$N \|Q(t)x\| \geq \|x\|, \quad \forall x \in X$$

which shows us that the projection $Q(t)$ is injective, hence

$$\text{Ran}P(t) = \text{Ker}Q(t) = \{0\}, \quad \forall t \geq 0$$

from where we get that $P(t)$ is the null operator for all $t \geq 0$ which means that the state space X is not decomposed into two proper closed subspaces hence this concept characterizes a uniform exponential instability of the evolution operator $U : \Delta \rightarrow \mathcal{B}(X)$.

Definition 2.2. Let $U : \Delta \rightarrow \mathcal{B}(X)$ be an evolution operator and $P : \mathbb{R}_+ \rightarrow \mathcal{B}(X)$ a family of projections. We say that U is ***P-uniformly exponentially dichotomic*** (and we write *P-u.e.d*) if there exist constants $N \geq 1$, $\nu > 0$ such that for all $(t, s, x) \in \Delta \times X$ the following hold:

$$(ued1) \quad \|U_P(t, s)x\| \leq Ne^{-\nu(t-s)}\|P(s)x\|;$$

$$(ued2) \quad N\|U_Q(t, s)x\| \geq e^{\nu(t-s)}\|Q(s)x\|.$$

Remark 2.2. Let $U : \Delta \rightarrow \mathcal{B}(X)$ be an evolution operator and $P : \mathbb{R}_+ \rightarrow \mathcal{B}(X)$ a family of projections. We have that U is P-u.e.d if and only if there exist $N \geq 1$ and $\nu > 0$ such that for all $(t, s, x) \in \Delta \times X$ the following condition holds:

$$\|U_P(t, s)x\| + e^{\nu(t-s)}\|Q(s)x\| \leq N(e^{-\nu(t-s)}\|P(s)x\| + \|U_Q(t, s)x\|).$$

Proposition 2.1. If $U : \Delta \rightarrow \mathcal{B}(X)$ is a reversible evolution operator and $P : \mathbb{R}_+ \rightarrow \mathcal{B}(X)$ is a family of projections compatible with U , then U is P-u.e.d if and only if there exist $N \geq 1$ and $\nu > 0$ such that the following conditions hold

$$(i) \quad \|U_P(t, s)x\| \leq Ne^{-\nu(t-s)}\|P(s)x\|$$

$$(ii) \quad \|U(t, s)^{-1}Q(t)x\| \leq Ne^{-\nu(t-s)}\|Q(t)x\|$$

for all $(t, s, x) \in \Delta \times X$.

Proof. Because (i) and (ued1) are the same, we only have to prove the equivalence (ued2) $\Leftrightarrow$ (ii). Assume that (ued2) holds. Then, for $(t, s, x) \in \Delta \times X$ we have that

$$\begin{aligned} \|U(t, s)^{-1}Q(t)x\| &= \|Q(s)U(t, s)^{-1}x\| \leq Ne^{-\nu(t-s)}\|U_Q(t, s)U(t, s)^{-1}x\| = \\ &= Ne^{-\nu(t-s)}\|Q(t)x\|. \end{aligned}$$

Conversely, for $(t, s, x) \in \Delta \times X$ we have, taking into account (ii), that

$$\|U(t, s)^{-1}Q(t)U_Q(t, s)x\| = \|Q(s)x\| \leq Ne^{-\nu(t-s)}\|U_Q(t, s)x\|$$

(ued2) easily follows. □

Proposition 2.2. *Let $U : \Delta \rightarrow \mathcal{B}(X)$ be an evolution operator and $P : \mathbb{R}_+ \rightarrow \mathcal{B}(X)$ a family of projections compatible with U . Then U is P -u.e.d if and only if there exist $N \geq 1$ and $\nu > 0$ such that for all $t \geq s \geq t_0 \geq 0$ and for all $x \in X$ we have that*

$$(i) \quad \|U_P(t, t_0)x\| \leq N e^{-\nu(t-s)} \|U_P(s, t_0)x\|;$$

$$(ii) \quad N \|U_Q(t, t_0)x\| \geq e^{\nu(t-s)} \|U_Q(s, t_0)x\|.$$

Proof. Necessity. Let $N \geq 1$ and $\nu > 0$ given by Definition 2.2. For $t \geq s \geq t_0 \geq 0$ and $x \in X$ we have that

$$\begin{aligned} \|U_P(t, t_0)x\| &= \|U_P(t, s)U_P(s, t_0)x\| \leq \\ &\leq N e^{-\nu(t-s)} \|U_P(s, t_0)x\|; \\ N \|U_Q(t, t_0)x\| &= N \|U_Q(t, s)U_Q(s, t_0)x\| \geq \\ &\geq e^{\nu(t-s)} \|U_Q(s, t_0)x\|. \end{aligned}$$

The *sufficiency* easily follows by putting $s = t_0$ in (i) and (ii). $\square$

Remark 2.3. It is well known (see for example [8]) that if an evolution operator $U : \Delta \rightarrow \mathcal{B}(X)$ has a uniform exponential growth and it is P -u.e.d then

$$\sup_{t \geq 0} \|P(t)\| < +\infty.$$

Definition 2.3. *Let $U : \Delta \rightarrow \mathcal{B}(X)$ be an evolution operator and $P : \mathbb{R}_+ \rightarrow \mathcal{B}(X)$ a family of projections. We say that U is **P -weakly exponentially dichotomic** (and we write P -w.u.e.d) if there exist constants $N \geq 1$, $\nu > 0$ such that for all $(t, s) \in \Delta$ the following hold:*

$$(wued1) \quad \|U_P(t, s)\| \leq N e^{-\nu(t-s)} \|P(s)\|;$$

$$(wued2) \quad N \|U_Q(t, s)\| \geq e^{\nu(t-s)} \|Q(s)\|.$$

Proposition 2.3. *Let $U : \Delta \rightarrow \mathcal{B}(X)$ be an evolution operator with (ueg) and $P : \mathbb{R}_+ \rightarrow \mathcal{B}(X)$ a family of projections. If U is P -w.u.e.d then*

$$\sup_{t \geq 0} \|P(t)\| < +\infty.$$

Proof. Let N, ν given by the P -w.u.e.d property of $U : \Delta \rightarrow \mathcal{B}(X)$, and M, ω given by the uniform exponential growth. Let $s \geq 0$, $t \geq s$. It follows that

$$\begin{aligned} &\left(\frac{1}{2N} e^{\nu(t-s)} - N e^{-\nu(t-s)} \right) \|P(s)\| \leq \\ &\leq \frac{1}{N} e^{\nu(t-s)} \|Q(s)\| - N e^{-\nu(t-s)} \|P(s)\| \leq \\ &\leq \|U_Q(t, s)\| - \|U_P(t, s)\| \leq \|U(t, s)\| \leq M e^{\omega(t-s)}. \end{aligned}$$

Let $\lambda_0 > 0$ be such that

$$c_0 := \frac{1}{2N} e^{\nu\lambda_0} - N e^{-\nu\lambda_0} > 0.$$

Let in the above estimations $t = s + \lambda_0$. It follows that

$$\|P(s)\| \leq \frac{M e^{\omega\lambda_0}}{c_0}, \quad \forall s \geq 0$$

from where the conclusion easily follows. $\square$

Proposition 2.4. *Let $U : \Delta \rightarrow \mathcal{B}(X)$ be an evolution operator and $P : \mathbb{R}_+ \rightarrow \mathcal{B}(X)$ a family of projections. If U is P -s.u.e.d then U is P -w.u.e.d.*

Proof. Let $N \geq 1$ and $\nu > 0$ be given by Definition 2.1. By Remark 2.1 it follows that

$$\|Q(t)\| \leq 2N, \quad \forall t \geq 0.$$

From (sued1) and (sued2), for $(t, s) \in \Delta$, we obtain that

$$\begin{aligned} \|U_P(t, s)\| &\leq N e^{-\nu(t-s)} \cdot 1 \leq N e^{-\nu(t-s)} \|P(s)\| \leq 2N^2 e^{-\nu(t-s)} \|P(s)\|; \\ 2N^2 \|U_Q(t, s)\| &\geq 2N e^{\nu(t-s)} \geq e^{\nu(t-s)} \|Q(s)\| \end{aligned}$$

from where we deduce that $U : \Delta \rightarrow \mathcal{B}(X)$ is P -w.u.e.d with constants $2N^2 \geq 1$ and $\nu > 0$. $\square$

Remark 2.4. Let $U : \Delta \rightarrow \mathcal{B}(X)$ be an evolution operator and $P : \mathbb{R}_+ \rightarrow \mathcal{B}(X)$ a family of projections. Then one has that if U is P -u.e.d then U is P -w.u.e.d

The converses of the above implications are not generally true, as we can see from the following examples.

Example 2.1. Consider the following evolution operators defined on $\mathbb{R}$ and $\mathbb{R}^2$ respectively, defined for all $(t, s) \in \Delta$ by

$$\begin{aligned} U_1(t, s) : \mathbb{R} &\rightarrow \mathbb{R}, \quad U_1(t, s)x = e^{s-t} \cdot x & \forall x \in \mathbb{R} \\ U_2(t, s) : \mathbb{R}^2 &\rightarrow \mathbb{R}^2, \quad U_2(t, s)y = \begin{cases} y & t = s \\ (e^{t-s}y_1, 0) & t > s \end{cases} & \forall y = (y_1, y_2) \in \mathbb{R}^2. \end{aligned}$$

On the Banach space $X = \mathbb{R} \oplus \mathbb{R}^2$ endowed with the norm

$$\|x \oplus y\| = (|x|^2 + \|y\|^2)^{1/2} \quad \forall x \oplus y \in X$$

we define the family of projections $P : \mathbb{R}_+ \rightarrow \mathcal{B}(X)$ by

$$P(t)x \oplus y = x \oplus 0 \quad \forall (t, x \oplus y) \in \mathbb{R}_+ \times X.$$

For $(t, s) \in \Delta$ we consider the evolution operator

$$U(t, s)x \oplus y = U_1(t, s)x \oplus U_2(t, s)y, \quad \forall x \oplus y \in X.$$

We have that for all $(t, s, x \oplus y) \in \Delta \times X$,

$$U_P(t, s)x \oplus y = U_1(t, s)x \oplus 0$$

$$U_Q(t, s)x \oplus y = 0 \oplus U_2(t, s)y.$$

It follows that for $(t, s) \in \Delta$ and $x \oplus y \in X$ we have that

$$\|U_P(t, s)x \oplus y\| = \|U_1(t, s)x\| = e^{s-t}\|x\| \leq e^{s-t}\|x \oplus y\|$$

from where we obtain that

$$\|U_P(t, s)\| \leq e^{s-t}\|P(s)\|, \quad \forall (t, s) \in \Delta$$

which shows us that (wued1) is fulfilled.

Moreover,

$$\|U_Q(t, s)x \oplus y\| = \|U_2(t, s)y\| \leq e^{t-s}\|y\| \leq e^{t-s}\|x \oplus y\|$$

for all $(t, s, x \oplus y) \in \Delta \times X$ from where we have

$$\|U_Q(t, s)\| \leq e^{t-s}.$$

Choosing $x \oplus y = 0 \oplus (1, 0) \in X$ with $\|x \oplus y\| = 1$, we have that

$$\|U_Q(t, s)x \oplus y\| = e^{t-s}\|x \oplus y\|$$

from where we finally get that

$$\|U_Q(t, s)\| = e^{t-s} = e^{t-s}\|Q(s)\|, \quad \forall (t, s) \in \Delta$$

which shows us that condition (wued2) is fulfilled hence $U : \Delta \rightarrow \mathcal{B}(X)$ is P-w.u.e.d.

Assume by a contradiction that $U : \Delta \rightarrow \mathcal{B}(X)$ is P-u.e.d. Then in particular there exist $N \geq 1$ and $\nu > 0$ such that

$$N\|U_Q(t, s)x \oplus y\| \geq e^{\nu(t-s)}\|Q(s)x \oplus y\|, \quad \forall (t, s, x \oplus y) \in \Delta \times X.$$

Let $t > s$ and $x \oplus y = 0 \oplus (0, 1)$. Then $\|Q(s)x \oplus y\| = 1$ and

$$e^{\nu(t-s)} \leq \|U_Q(t, s)x \oplus y\| = \|U(t, s)0 \oplus (0, 1)\| = 0$$

which is a contradiction.

We saw that in the general case, the concepts of P-u.e.d and P-w.u.e.d. do not coincide. We may ask ourselves whether in the case of reversible operators the converse implication from Remark 2.4 holds.

The answer is negative, and it is given by the following example.

Example 2.2. On $X = \mathbb{R}^3$ endowed with the max-norm

$$\|(x_1, x_2, x_3)\| = \max\{|x_1|, |x_2|, |x_3|\}, \quad \forall (x_1, x_2, x_3) \in \mathbb{R}^3$$

consider, for $(t, s) \in \Delta$ the operator $U(t, s) : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ defined by

$$U(t, s)(x_1, x_2, x_3) = (e^{s-t}x_1, e^{t-s}x_2, x_3) \quad \forall (x_1, x_2, x_3) \in \mathbb{R}^3.$$

Define $P : \mathbb{R}_+ \rightarrow \mathcal{B}(X)$ by

$$P(t)(x_1, x_2, x_3) = (x_1, 0, 0), \quad \forall (x_1, x_2, x_3) \in \mathbb{R}^3, \quad \forall t \geq 0.$$

We have that $U : \Delta \rightarrow \mathcal{B}(X)$ is a reversible evolution operator and $P : \mathbb{R}_+ \rightarrow \mathcal{B}(X)$ is compatible with U having

$$U(t, s)^{-1}(x_1, x_2, x_3) = (e^{t-s}x_1, e^{s-t}x_2, x_3) \quad \forall (x_1, x_2, x_3) \in \mathbb{R}^3.$$

We will prove that U is w.u.e.d with $N = \nu = 1$.

Let $(t, s) \in \Delta$. From

$$\|U_P(t, s)(x_1, x_2, x_3)\| = e^{s-t}P(s)(x_1, x_2, x_3), \quad \forall (x_1, x_2, x_3) \in \mathbb{R}^3$$

we obtain that

$$\|U_P(t, s)\| = e^{s-t}\|P(s)\| \tag{2.1}$$

which shows us that condition (wued1) is satisfied.

Moreover, from

$$\|U_Q(t, s)(x_1, x_2, x_3)\| = (0, e^{t-s}x_2, x_3)$$

it follows that

$$\|U_Q(t, s)(x_1, x_2, x_3)\| = \max\{e^{t-s}|x_2|, |x_3|\} \leq e^{t-s}\|Q(s)x\|$$

from where we get that

$$\|U_Q(t, s)\| \leq e^{t-s}\|Q(s)\|. \tag{2.2}$$

On the other hand, consider the vector $(0, 1, 0) \in \mathbb{R}^3$. It follows that

$$\|U_Q(t, s)(0, 1, 0)\| = e^{t-s} = e^{t-s}\|Q(s)x\|$$

from where we finally get that

$$\|U_Q(t, s)\| = e^{t-s} \|Q(s)\|$$

which shows us that condition (wued2) is fulfilled.

Assume now that $U : \Delta \rightarrow \mathcal{B}(X)$ is P-u.e.d. From (ued2) it follows that there exist $N \geq 1$ and $\nu > 0$ such that

$$N \|U_Q(t, s)x\| \geq e^{\nu(t-s)} \|Q(s)x\| \quad (2.3)$$

for all $(t, s, x) \in \Delta \times X$. Having in mind that

$$Q(s)(0, 0, 1) = (0, 0, 1), \quad U_Q(t, s)(0, 0, 1) = (0, 0, 1)$$

taking in account (2.3) we get that

$$N \geq e^{\nu(t-s)}, \quad \forall (t, s) \in \Delta$$

which led us to a contradiction.

Remark 2.5. In the above example, having in mind that

$$\|P(t)\| = \|Q(t)\| = 1, \quad \forall t \geq 0$$

from the relations (2.1) and (2.2) we get that

$$\|U_P(t, s)\| = e^{s-t}, \quad \|U_Q(t, s)\| = e^{t-s}, \quad \forall (t, s) \in \Delta$$

hence the evolution operator corresponding to this example is P-s.u.e.d but fails to be P-u.e.d.

Seeing that P-s.u.e.d does not generally imply P-u.e.d, we may ask ourselves whether the P-u.e.d property implies P-s.u.e.d. The answer is negative, and we point out this fact through the following example.

Example 2.3. Let $X = \mathbb{R}^2$ endowed with the max-norm

$$\|x\| = \max\{|x_1|, |x_2|\}, \quad x = (x_1, x_2) \in \mathbb{R}^2.$$

For every $t \geq 0$, we define

$$P(t) : \mathbb{R}^2 \rightarrow \mathbb{R}^2, \quad P(t)(x_1, x_2) = (x_1 + tx_2, 0), \quad \forall (x_1, x_2) \in \mathbb{R}^2.$$

We have that $P : \mathbb{R}_+ \rightarrow \mathcal{B}(X)$ is a family of projections and

$$\|P(t)\| = 1 + t, \quad \forall t \geq 0.$$

For $(t, s) \in \Delta$ we define $U(t, s) : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ by

$$U(t, s)x = e^{s-t}P(s)x + e^{t-s}Q(t)x, \quad \forall x \in \mathbb{R}^2.$$

Having in mind that $P(t)P(s) = P(s)$, $Q(t)Q(s) = Q(t)$ and $Q(t)P(s) = 0$ for all $t, s \in \mathbb{R}_+$, it easily follows that $U : \Delta \rightarrow \mathcal{B}(X)$ is an evolution operator. We observe that for all $(t, s, x_1, x_2) \in \Delta \times \mathbb{R}^2$ we have

$$\|Q(s)(x_1, x_2)\| = \max\{s, 1\}|x_2| \leq \max\{t, 1\}|x_2| = \|Q(t)(x_1, x_2)\|.$$

Moreover, we have that for all $(t, s, x) \in \Delta \times X$,

$$\|U_P(t, s)x\| = e^{-(t-s)}\|P(s)x\|, \quad \|U_Q(t, s)x\| = e^{t-s}\|Q(t)x\| \geq e^{t-s}\|Q(s)x\| \quad (2.4)$$

which shows us that U is P-u.e.d.

Assuming by a contradiction that U is P-s.u.e.d, by Remark 2.1 we would have that there exists $K > 0$ such that

$$\|P(t)\| \leq K, \quad \forall t \geq 0$$

which obviously does not hold.

Remark 2.6. Considering the evolution operator $U : \Delta \rightarrow \mathcal{B}(X)$ and the family of projections from Example 2.3, the relations from (2.4) imply that

$$\|U_P(t, s)\| = e^{s-t}\|P(s)\|, \quad \|U_Q(t, s)\| \geq e^{t-s}\|Q(s)\| \quad \forall (t, s) \in \Delta$$

hence U is P-w.u.e.d. The same contradiction is obtained by assuming that U is P-s.u.e.d.

Remark 2.7. Having in mind all of the above, we can point out the diagram concerning the connections between the presented concepts:

$$\begin{array}{ccc} & P - s.u.e.d & \\ \nearrow \nwarrow & & \nwarrow \nearrow \\ P - u.e.d & \implies & P - w.u.e.d \\ & \Leftarrow & \end{array}$$

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