Some Convexity Properties for a New Integral Operator

Laura Filofteia Stanciu

Abstract. In this paper we study some properties for a new integral operator in the classes of the univalent functions denoted by $\mathcal{SP}$, $\mathcal{S}(p)$ and $\mathcal{B}(\mu, \alpha)$.

AMS Subject Classification (2000). 30C45, 30C75.

Keywords. Analytic functions; Integral Operator; Starlike functions; Convex functions; General Schwarz Lemma.

1 Introduction

Let $\mathcal{A}$ denote the class of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n,$$

which are analytic in the open unit disk

$$\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$$

and satisfy the following usual normalization condition

$$f(0) = f'(0) - 1 = 0.$$

We denote by $\mathcal{S}$ the subclass of $\mathcal{A}$ consisting of functions f which are univalent in $\mathbb{U}$.

A function $f \in \mathcal{A}$ is the starlike function of order α , $0 \leq \alpha < 1$ if f satisfies the inequality

$$\operatorname{Re} \left(\frac{zf'(z)}{f(z)} \right) > \alpha \quad (z \in \mathbb{U}).$$

We denote this class by $\mathcal{S}^*(\alpha)$.

A function $f \in \mathcal{A}$ is the convex function of order α , $0 \leq \alpha < 1$ if f satisfies the inequality

$$\operatorname{Re} \left(\frac{zf''(z)}{f'(z)} + 1 \right) > \alpha \quad (z \in \mathbb{U}).$$

We denote this class by $\mathcal{K}(\alpha)$.

A function $f \in \mathcal{A}$ is in the class $\mathcal{R}(\alpha)$ if and only if

$$\operatorname{Re} (f'(z)) > \alpha \quad (z \in \mathbb{U}).$$

Let $\mathcal{N}(\beta)$ be the subclass of $\mathcal{A}$ consisting of the functions f which satisfy the inequality

$$\operatorname{Re} \left(\frac{zf''(z)}{f'(z)} + 1 \right) < \beta \quad (z \in \mathbb{U}; \beta > 1).$$

This class was studied by Owa and Srivastava [11].

In [4], Frasin and Jahangiri introduced the class $\mathcal{B}(\mu, \alpha)$ defined as follows.

Definition 1.1. ([4]). A function $f \in \mathcal{A}$ is said to be a member of the class $\mathcal{B}(\mu, \alpha)$ if and only if

$$\left| f'(z) \left(\frac{z}{f(z)} \right)^\mu - 1 \right| < 1 - \alpha \quad (1.1)$$

$(z \in \mathbb{U}; 0 \leq \alpha < 1; \mu \geq 0)$.

Note that the condition (1.1) is equivalent to

$$\operatorname{Re} \left(f'(z) \left(\frac{z}{f(z)} \right)^\mu \right) > \alpha$$

for $(z \in \mathbb{U}; 0 \leq \alpha < 1; \mu \geq 0)$.

This family is a comprehensive class of analytic functions that contains other new classes of analytic univalent functions as well as some very well-known ones, such as $\mathcal{B}(1, \alpha) = \mathcal{S}^*(\alpha)$ and $\mathcal{B}(0, \alpha) = \mathcal{R}(\alpha)$. Frasin and Darus in [5] (see also [6]) has been introduced another interesting subclass $\mathcal{B}(2, \alpha) = \mathcal{B}(\alpha)$.

In the paper [13], F. Ronning introduced the class of univalent functions, $\mathcal{SP}$ defined as:

Definition 1.2. ([13]). A function $f \in \mathcal{A}$ is said to be in the class $\mathcal{SP}$, if f satisfies the following inequality

$$\left| \frac{zf'(z)}{f(z)} - 1 \right| < \operatorname{Re} \frac{zf'(z)}{f(z)} \quad (z \in \mathbb{U}). \quad (1.2)$$

In the paper [15] is defined the class $\mathcal{S}(p)$. For $0 < p \leq 2$, let $\mathcal{S}(p)$ denote the class of function $f \in \mathcal{A}$ which satisfies the conditions

$$f(z) \neq 0 \quad (0 < |z| < 1)$$

and

$$\left| \left(\frac{z}{f(z)} \right)'' \right| \leq p \quad (z \in \mathbb{U}).$$

In [14], Singh has shown the following condition for a function $f \in \mathcal{S}(p)$.

Definition 1.3. ([14]). A function $f \in \mathcal{A}$ is said to be in the class $\mathcal{S}(p)$, if f satisfies the following inequality

$$\left| \frac{z^2 f'(z)}{[f(z)]^2} - 1 \right| \leq p |z|^2 \quad (z \in \mathbb{U}). \quad (1.3)$$

For $f_i(z), g_i(z) \in \mathcal{A}$ and $\delta_i > 0, \gamma_i > 0$ ($i \in \{1, 2, \dots, n\}$), we define the integral operator $F_n(z)$ given by

$$F_n(z) = \int_0^z \prod_{i=1}^n \left(\frac{f_i(t)}{t} \right)^{\delta_i} (e^{g_i(t)})^{\gamma_i} dt. \quad (1.4)$$

Remark 1.4. For $n = 1, \delta_1 = 1, \gamma_1 = \gamma_2 = \dots = \gamma_n = 0$ and $f \in \mathcal{A}$ we obtain Alexander integral operator introduced in 1915 in [1]

$$I(z) = \int_0^z \frac{f(t)}{t} dt \quad (z \in \mathbb{U}).$$

Remark 1.5. For $n = 1, \delta_1 = \alpha, \gamma_1 = \gamma_2 = \dots = \gamma_n = 0$ and $f \in \mathcal{A}$ we obtain the integral operator

$$I_\alpha(z) = \int_0^z \left(\frac{f(t)}{t} \right)^\alpha dt \quad (z \in \mathbb{U})$$

studied in [9], (see also [7], [8], [12]).

Remark 1.6. For $\delta_i = \alpha_i$, $\alpha_i > 0$, $\gamma_1 = \gamma_2 = \dots = \gamma_n = 0$ and $f_i \in \mathcal{A}$ for $i \in \{1, 2, \dots, n\}$, we obtain the integral operator

$$I_n(z) = \int_0^z \prod_{i=1}^n \left(\frac{f_i(t)}{t} \right)^{\alpha_i} dt$$

studied in [2], [3].

In order to prove our main results, we recall the following lemma.

Lemma 1.7. (General Schwarz Lemma) (see [10]). *Let the function f be regular in the disk $\mathbb{U}_R = \{z \in \mathbb{C} : |z| < R\}$, with $|f(z)| < M$ for fixed M . If f has in $z = 0$, one zero with multiplicity order larger than m , then*

$$|f(z)| \leq \frac{M}{R^m} |z|^m \quad (z \in \mathbb{U}_R).$$

The equality can hold only if

$$f(z) = e^{i\theta} \frac{M}{R^m} z^m,$$

where θ is constant.

2 Main results

Theorem 2.1. *Let the functions $f_i, g_i \in \mathcal{A}$, where g_i be in the class $\mathcal{S}(p_i)$ and satisfy the inequality (1.3) with $0 < p_i \leq 2$ for all $i \in \{1, 2, \dots, n\}$. For any given $M_i \geq 1$, $N_i \geq 1$ and $0 < p_i \leq 2$ satisfying the conditions*

$$\left| \frac{f'_i(z)}{f_i(z)} \right| \leq M_i \quad (z \in \mathbb{U}), \quad |g_i(z)| \leq N_i \quad (z \in \mathbb{U})$$

there exist numbers $\delta_i, \gamma_i \in \mathbb{R}$ with $\delta_i > 0$, $\gamma_i > 0$ such that

$$\mu = 1 + \sum_{i=1}^n \delta_i (M_i + 1) + \sum_{i=1}^n \gamma_i N_i^2 (p_i + 1)$$

for all $i \in \{1, 2, \dots, n\}$.

In these conditions, the integral operator $F_n(z)$ defined by (1.4) is in $\mathcal{N}(\mu)$.

Proof. We calculate for $F_n(z)$ the derivatives of the first and second order.
From (1.4) we obtain

$$F'_n(z) = \prod_{i=1}^n \left(\frac{f_i(z)}{z} \right)^{\delta_i} (e^{g_i(z)})^{\gamma_i}$$

and

$$\begin{aligned} F''_n(z) = & \sum_{i=1}^n \left[\delta_i \left(\frac{f_i(z)}{z} \right)^{\delta_i-1} \left(\frac{zf'_i(z) - f_i(z)}{z^2} \right) (e^{g_i(z)})^{\gamma_i} \right] \prod_{\substack{k=1 \\ k \neq i}}^n \left(\frac{f_k(z)}{z} \right)^{\delta_k} (e^{g_k(z)})^{\gamma_k} + \\ & + \sum_{i=1}^n \left[\left(\frac{f_i(z)}{z} \right)^{\delta_i} \gamma_i (e^{g_i(z)})^{\gamma_i-1} g'_i(z) e^{g_i(z)} \right] \prod_{\substack{k=1 \\ k \neq i}}^n \left(\frac{f_k(z)}{z} \right)^{\delta_k} (e^{g_k(z)})^{\gamma_k}. \end{aligned}$$

After the calculus, we obtain that

$$\frac{zF''_n(z)}{F'_n(z)} = \sum_{i=1}^n \left[\delta_i \left(\frac{zf'_i(z)}{f_i(z)} - 1 \right) + \gamma_i z g'_i(z) \right] \quad (2.1)$$

It follows from (2.1) that

$$\begin{aligned} \left| \frac{zF''_n(z)}{F'_n(z)} \right| & \leq \sum_{i=1}^n \delta_i \left| \frac{zf'_i(z)}{f_i(z)} - 1 \right| + \sum_{i=1}^n \gamma_i |z g'_i(z)| \\ & \leq \sum_{i=1}^n \delta_i \left(\left| \frac{zf'_i(z)}{f_i(z)} \right| + 1 \right) + \sum_{i=1}^n \gamma_i |z g'_i(z)| \\ & \leq \sum_{i=1}^n \delta_i \left(\left| \frac{zf'_i(z)}{f_i(z)} \right| + 1 \right) + \sum_{i=1}^n \gamma_i \left| \frac{z^2 g'_i(z)}{[g_i(z)]^2} \right| \left| \frac{[g_i(z)]^2}{z} \right|. \end{aligned} \quad (2.2)$$

Since

$$\left| \frac{f'_i(z)}{f_i(z)} \right| \leq M_i \quad (z \in \mathbb{U}), \text{ and } |g_i(z)| \leq N_i \quad (z \in \mathbb{U}),$$

applying the General Schwarz Lemma for the functions g_i for all $i \in \{1, 2, \dots, n\}$, we have

$$|g_i(z)| \leq N_i |z| \quad (z \in \mathbb{U}).$$

The functions $g_i \in \mathcal{S}(p_i)$, $0 < p_i \leq 2$, so

$$\left| \frac{z^2 g'_i(z)}{[g_i(z)]^2} - 1 \right| \leq p_i |z|^2 \quad (z \in \mathbb{U})$$

for all $i \in \{1, 2, \dots, n\}$.

From (2.2) we get

$$\begin{aligned} \left| \frac{zF_n''(z)}{F_n'(z)} \right| &\leq \sum_{i=1}^n \delta_i (M_i + 1) + \sum_{i=1}^n \gamma_i \left(\left| \frac{z^2 g_i'(z)}{[g_i(z)]^2} - 1 \right| + 1 \right) N_i^2 \\ &\leq \sum_{i=1}^n \delta_i (M_i + 1) + \sum_{i=1}^n \gamma_i N_i^2 (p_i |z|^2 + 1) \\ &\leq \sum_{i=1}^n \delta_i (M_i + 1) + \sum_{i=1}^n \gamma_i N_i^2 (p_i + 1) \\ &= \mu - 1. \end{aligned}$$

Hence $F_n(z) \in \mathcal{N}(\mu)$, $\mu = 1 + \sum_{i=1}^n \delta_i (M_i + 1) + \sum_{i=1}^n \gamma_i N_i^2 (p_i + 1)$. $\square$

Setting $n = 1$ in Theorem 2.1 we obtain

Corollary 2.2. *Let the functions $f, g \in \mathcal{A}$, where g be in the class $\mathcal{S}(p)$ and satisfies the inequality (1.3) with $0 < p \leq 2$. For any given $M \geq 1$, $N \geq 1$ and $0 < p \leq 2$ satisfying the conditions*

$$\left| \frac{f'(z)}{f(z)} \right| \leq M \quad (z \in \mathbb{U}), \quad |g(z)| \leq N \quad (z \in \mathbb{U})$$

there exist numbers $\delta, \gamma \in \mathbb{R}$ with $\delta > 0$, $\gamma > 0$ such that

$$\mu = 1 + [\delta (M + 1) + \gamma N^2 (p + 1)].$$

In these conditions, the integral operator

$$F(z) = \int_0^z \left(\frac{f(t)}{t} \right)^\delta (e^{g(t)})^\gamma dt$$

is in the class $\mathcal{N}(\mu)$.

Theorem 2.3. *Let the functions f_i be in the class $\mathcal{B}(\mu_i, \alpha_i)$, $\mu_i \geq 0$, $0 \leq \alpha_i < 1$ and $g_i \in \mathcal{SP}$ with $\operatorname{Re} \frac{z g_i'(z)}{g_i(z)} \leq 1$, $z \in \mathbb{U}$ for all $i \in \{1, 2, \dots, n\}$. For any given $\mu_i \geq 0$, $0 \leq \alpha_i < 1$, $M_i \geq 1$ and $N_i \geq 1$ satisfying the conditions*

$$|f_i(z)| \leq M_i \quad (z \in \mathbb{U}), \quad |g_i(z)| \leq N_i \quad (z \in \mathbb{U})$$

there exist numbers $\delta_i, \gamma_i \in \mathbb{R}$, $\delta_i > 0$ and $\gamma_i > 0$ such that

$$\lambda = 1 - \sum_{i=1}^n [\delta_i ((2 - \alpha_i) M_i^{\mu_i - 1} + 1) + 2\gamma_i N_i] > 0$$

and

$$\sum_{i=1}^n [\delta_i ((2 - \alpha_i) M_i^{\mu_i-1} + 1) + 2\gamma_i N_i] < 1$$

for all $i \in \{1, 2, \dots, n\}$.

In these conditions, the integral operator $F_n(z)$ defined in (1.4) is in the class $\mathcal{K}(\lambda)$.

Proof. If we make the similar operations as the proof of Theorem 2.1, we have

$$\frac{zF_n''(z)}{F_n'(z)} = \sum_{i=1}^n \left[\delta_i \left(\frac{zf_i'(z)}{f_i(z)} - 1 \right) + \gamma_i z g_i'(z) \right]. \quad (2.3)$$

From the relation (2.3), we obtain that

$$\begin{aligned} \left| \frac{zF_n''(z)}{F_n'(z)} \right| &\leq \sum_{i=1}^n \left[\delta_i \left(\left| \frac{zf_i'(z)}{f_i(z)} \right| + 1 \right) + \gamma_i |zg_i'(z)| \right] \\ &\leq \sum_{i=1}^n \left[\delta_i \left(\left| f_i'(z) \left(\frac{z}{f_i(z)} \right)^{\mu_i} \right| \left| \frac{f_i(z)}{z} \right|^{\mu_i-1} + 1 \right) + \gamma_i \left| \frac{zg_i'(z)}{g_i(z)} \right| |g_i(z)| \right]. \end{aligned} \quad (2.4)$$

Since

$$|f_i(z)| \leq M_i (z \in \mathbb{U}), \quad |g_i(z)| \leq N_i (z \in \mathbb{U}),$$

applying the General Schwarz Lemma for the functions f_i ($i \in \{1, 2, \dots, n\}$), we obtain

$$|f_i(z)| \leq M_i |z| (z \in \mathbb{U}).$$

Because $f_i \in \mathcal{B}(\mu_i, \alpha_i)$, $\mu_i \geq 0$, $0 \leq \alpha_i < 1$, $g_i \in \mathcal{SP}$ for all $i \in \{1, 2, \dots, n\}$, we apply in the relation (2.4) the inequality (1.1) and the inequality (1.2), we obtain

$$\begin{aligned} \left| \frac{zF_n''(z)}{F_n'(z)} \right| &\leq \sum_{i=1}^n \left[\delta_i \left(\left| f_i'(z) \left(\frac{z}{f_i(z)} \right)^{\mu_i} - 1 \right| + 1 \right) M_i^{\mu_i-1} + 1 \right) \\ &\quad + \gamma_i \left(\left| \frac{zg_i'(z)}{g_i(z)} - 1 \right| + 1 \right) N_i \right] \\ &\leq \sum_{i=1}^n \left[\delta_i ((2 - \alpha_i) M_i^{\mu_i-1} + 1) + \gamma_i \left(\operatorname{Re} \left\{ \frac{zg_i'(z)}{g_i(z)} \right\} + 1 \right) N_i \right] \\ &\leq \sum_{i=1}^n \left[\delta_i ((2 - \alpha_i) M_i^{\mu_i-1} + 1) + 2\gamma_i N_i \right] \\ &= 1 - \lambda. \end{aligned}$$

So, the integral operator $F_n(z)$ defined in (1.4) is in the class $\mathcal{K}(\lambda)$. □

Setting $\mu_1 = \mu_2 = \dots = \mu_n = 1$ in Theorem 2.3 we obtain

Corollary 2.4. *Let the functions f_i be in the class $\mathcal{S}^*(\alpha_i)$, $0 \leq \alpha_i < 1$ and $g_i \in \mathcal{SP}$ with $\operatorname{Re} \frac{zg'_i(z)}{g_i(z)} \leq 1$, $z \in \mathbb{U}$ for all $i \in \{1, 2, \dots, n\}$. For any given $0 \leq \alpha_i < 1$, $M_i \geq 1$ and $N_i \geq 1$ satisfying the conditions*

$$|f_i(z)| \leq M_i (z \in \mathbb{U}), |g_i(z)| \leq N_i (z \in \mathbb{U})$$

there exist numbers $\delta_i, \gamma_i \in \mathbb{R}$, $\delta_i > 0$ and $\gamma_i > 0$ such that

$$\lambda = 1 - \sum_{i=1}^n [\delta_i (3 - \alpha_i) + 2\gamma_i N_i] > 0$$

and

$$\sum_{i=1}^n [\delta_i (3 - \alpha_i) + 2\gamma_i N_i] \leq 1$$

for all $i \in \{1, 2, \dots, n\}$.

In these conditions, the integral operator $F_n(z)$ defined in (1.4) is in the class $\mathcal{K}(\lambda)$.

Setting $n = 1$ in Theorem 2.3 we obtain

Corollary 2.5. *Let the functions f be in the class $\mathcal{B}(\mu, \alpha)$, $\mu \geq 0$, $0 \leq \alpha < 1$ and $g \in \mathcal{SP}$ with $\operatorname{Re} \frac{zg'(z)}{g(z)} \leq 1$, $z \in \mathbb{U}$. For any given $\mu \geq 0$, $0 \leq \alpha < 1$, $M \geq 1$ and $N \geq 1$ satisfying the conditions*

$$|f(z)| \leq M (z \in \mathbb{U}), |g(z)| \leq N (z \in \mathbb{U})$$

there exist numbers $\delta, \gamma \in \mathbb{R}$, $\delta > 0$ and $\gamma > 0$ such that

$$\lambda = 1 - [\delta ((2 - \alpha) M^{\mu-1} + 1) + 2\gamma N] > 0$$

and

$$[\delta ((2 - \alpha) M^{\mu-1} + 1) + 2\gamma N] \leq 1.$$

In these conditions, the integral operator

$$F(z) = \int_0^z \left(\frac{f(t)}{t} \right)^\delta (e^{g(t)})^\gamma dt$$

is in the class $\mathcal{K}(\lambda)$.

Acknowledgement. This work was partially supported by the strategic project POSDRU 107/1.5/S/77265, inside POSDRU Romania 2007-2013 co-financed by the European Social Fund-Investing in People.

References

- [1] **J. W. Alexander**, Functions which map the interior of the unit circle upon simple regions, *The Annal. Math.*, **17 (1)**, (1915), 12-22.
- [2] **D. Breaz**, Certain integral operators on the classes $\mathcal{M}(\beta_i)$ and $\mathcal{N}(\beta_i)$, *J. Inequal. Appl.*, (2008, Art. I.D. 719354), 1-4.
- [3] **D. Breaz and N. Breaz**, Two integral operators, *Studia Univ. Babes-Bolyai Mathematica*, **3(3)**, Art. I.D. 42, 1-13.
- [4] **B. A. Frasin and J. Jahangiri**, A new and comprehensive class of analytic functions, *Anal. Univ. Ordea Fasc. Math.*, **XV(2008)**, 59-62.
- [5] **B. A. Frasin and M. Darus**, On certain analytic univalent functions, *Internat. J. Math. and Math. Sci.*, **5**, (2001), 305-310.
- [6] **B. A. Frasin**, A note on certain analytic and univalent functions, *Southeast Asian J. Math.*, **28**, (2004), 829-836.
- [7] **Y. J. Kim and E. P. Merkes**, On an integral of powers of a spiral like function, *Kyungpook Math. J.*, **12**, (1972), 249-252.
- [8] **E. P. Merkes and D. J. Wright**, On the univalence of a certain integral, *Proc. Amer. Math. Soc.*, **27 (1)**, (1971), 97-100.
- [9] **S. S. Miller, Mocanu. P. T., and M. O. Reade**, Starlike integral operators, *Pacific J. Math.*, **79 (19)**, (1978), 157-168.
- [10] **Z. Nehari**, *Conformal mapping*, McGraw-Hill Book Comp., New York, 1952.
- [11] **S. Owa and H. M. Srivastava**, Some generalized convolution properties associated with certain subclasses of analytic functions, **42**, (2003), 1-13.
- [12] **V. Pescar**, On some integral operators which preserve the univalence, *J. Math. The Punjab Univ.*, **30**, (1997), 1-10.
- [13] **F. Ronning**, Integral representations of bounded starlike functions, *Ann. Polon. Math.*, **LX(3)**, (1995), 289-297.
- [14] **V. Singh**, On a class of univalent functions, *Internat. J. Math. Math. Sci.*, **23**, (2000), 855-857.
- [15] **D. Yang and J. Liu**, On a class of univalent functions, *IJMMS*, **22(3)**, (1999), 605-610.

Laura Filofteia Stanciu

University of Pitești

Department of Mathematics

Târgul din Vale Str., No.1, 110040, Pitești, Argeș, România

E-mail: laura_stanciu_30@yahoo.com

Received: 4.04.2012

Accepted: 14.06.2012