Partly Interpolation Space in the Multidimensional Case

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Abstract. We give a necessary and sufficient condition for an intermediate Banach space A with respect to the Banach $N + 1$ -tuple $\vec{A}$ to be partly interpolation space. We also establish some properties on partly interpolation spaces.

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1 Introduction

It is well known that exist an intermediate space with respect to the triple (L_1, L_p, L_∞) , $1 < p < \infty$, which is partly interpolation space but not an interpolation space (see [3], [4],[5], [6]).

Working with Banach couples, Dimitriev [4] and Pustyl'nik [6] have been introduce and studied the functions ψ, ρ which describe the “position” of the intermediate space within the Banach couple. They show that a necessary and sufficient condition for an intermediate Banach space A with respect to the Banach couple $\vec{A}$ to be a party interpolation space is that for some $C > 0$ it holds

$$\psi(t) \leq C\rho(t), \quad \text{for all } t > 0. \quad (1.1)$$

This research was continued by Cobos and Romero [3] in the context of N -tuples of Banach spaces, by using the K -and J -functionals associated to a

convex polygon in the plane. They show that the corresponding inequality (1.1) is a characterization of partly interpolation spaces only if $N = 3$. Our aim is to study this problem in the context of $(N + 1)$ -tuples ($N \geq 2$) of Banach spaces, by using Sparr's [7] K -and J -functionals. In this case the corresponding inequality (1.1) is a characterization of partly interpolation space for every finite $N \geq 2$.

2 Preliminaries

Let $\vec{A} = (A_0, \dots, A_N)$ be a Banach $N + 1$ -tuple, that is, a family of $N + 1$ Banach spaces A_i all of them continuously embedded in a common Hausdorff topological vector space $\mathcal{U}$ (i.e. $A_i \hookrightarrow \mathcal{U}$). For each $\vec{t} = (t_1, \dots, t_N) > 0$, Sparr's K -and J -functionals are defined by

$$K(\vec{t}, a, \vec{A}) = \inf \left\{ \|a_0\|_{A_0} + \sum_{i=1}^N t_i \|a_i\|_{A_i} : a = \sum_{i=0}^N a_i, a_i \in A_i \right\},$$

for each $a \in \Sigma(\vec{A}) = A_0 + \dots + A_N$
and

$$J(\vec{t}, a, \vec{A}) = \max_{1 \leq i \leq N} (\|a\|_{A_0}, t_i \|a\|_{A_i})$$

for each $a \in \Delta(\vec{A}) = A_0 \cap \dots \cap A_N$.

A linear operator T in $\Sigma(\vec{A})$ is called a bounded linear operator on a Banach $N + 1$ -tuple $\vec{A}$ if T maps A_i into A_i ($i = 0, \dots, N$) boundedly. Let $\mathcal{L}(\vec{A}, \vec{A})$ stand for the set of all such operators. $\mathcal{L}(\vec{A}, \vec{A})$ is a Banach space with respect to the norm

$$\|T\|_{\vec{A}, \vec{A}} = \max_{0 \leq i \leq N} (\|T\|_{A_i, A_i}).$$

A Banach space A is said to be an intermediate space with respect to $\vec{A}$ if

$$\Delta(\vec{A}) \hookrightarrow A \hookrightarrow \Sigma(\vec{A}).$$

If B is an intermediate space with respect to $\vec{A}$ with $B \subset A$, then $B \hookrightarrow A$. An intermediate space A is an interpolation space if, given any $T \in \mathcal{L}(\vec{A}, \vec{A})$, the restriction of T to A defines a bounded operator from A into itself. Moreover, by the closed graph theorem, there exists a constant $C = C(A, \vec{A})$ such that

$$\|T\|_{A, A} \leq C \|T\|_{\vec{A}, \vec{A}}, \quad \text{for all } T \in \mathcal{L}(\vec{A}, \vec{A}). \quad (2.1)$$

We say that the intermediate space A is a partly interpolation space, or a rank-one interpolation space, if (2.1) holds for all $T \in \mathcal{L}(\vec{A}, \vec{A})$ of rank-one (i.e. of the special form: $T = f \otimes a$, where $f \in (\Sigma(\vec{A}))^*$ and $a \in \Delta(\vec{A})$).

We shall denote the set of partly interpolation spaces for the $N+1$ -tuple $\vec{A}$ by $R1 - \text{Int}(\vec{A})$.

Let A be an intermediate space with respect to $\vec{A} = (A_0, \dots, A_N)$. For each $\bar{t} = (t_1, \dots, t_N) > 0$ set

$$\psi(\bar{t}) = \psi(\bar{t}; A, \vec{A}) := \sup\{K(\bar{t}, a, \vec{A}) : \|a\|_A = 1\}$$

and

$$\rho(\bar{t}) = \rho(\bar{t}; A, \vec{A}) := \inf\{J(\bar{t}, a, \vec{A}) : a \in \Delta(\vec{A}), \|a\|_A = 1\}.$$

To each intermediate space A we associate the Gagliardo completion of A (denoted by $\tilde{A}^{\Sigma(\vec{A})}$), consists of all those $a \in \Sigma(\vec{A})$ for which there exists a bounded sequence $(a_n)_{n \in \mathbb{N}}$ in A which converges to a in $\Sigma(\vec{A})$. It is normed by

$$\|a\|_{\tilde{A}^{\Sigma(\vec{A})}} = \inf_{(a_n)} \left\{ \sup_{n \in \mathbb{N}} (\|a_n\|_A) \right\}.$$

It is easy to show (see [1]) that

$$\|a\|_{\tilde{A}_0^{\Sigma(\vec{A})}} = \lim_{\bar{t} \rightarrow \infty} K(\bar{t}, a, \vec{A}).$$

3 Partly interpolation spaces

Theorem 3.1. *Let $\vec{A} = (A_0, \dots, A_N)$ be a Banach $N+1$ -tuple and let A be an intermediate spaces with respect to $\vec{A}$. Then A is partly interpolation space if and only if there exists a positive constant $C = C(A, \vec{A})$ such that*

$$\psi(\bar{t}) \leq C\rho(\bar{t}), \text{ for all } \bar{t} = (t_1, \dots, t_N) > 0. \quad (3.1)$$

Proof. Assume that A is partly interpolation space and let C be the constant from (2.1) for one-dimensional operators. We prove that, for any $a \in A$ and any $b \in \Delta(\vec{A})$, we have

$$\frac{K(\bar{t}, a, \vec{A})}{\|a\|_A} \leq C \frac{J(\bar{t}, b, \vec{A})}{\|b\|_A}, \text{ for all } \bar{t} > 0. \quad (3.2)$$

This can be done using ideas of [3].

For a given $a \in A$ we take the functional $f \in \left(\Sigma(\vec{A})\right)^*$ such that $f(a) = K(\bar{t}, a, \vec{A})$ and $|f(x)| \leq K(\bar{t}, x, \vec{A})$ for any $x \in \Sigma(\vec{A})$. Then, we have

$$\|f\|_{A_0^*} = \sup_{x \in A_0} \frac{|f(x)|}{\|x\|_{A_0}} \leq \sup_{x \in A_0} \frac{K(\bar{t}, x, \vec{A})}{\|x\|_{A_0}} \leq \sup_{x \in A_0} \frac{\|x\|_{A_0}}{\|x\|_{A_0}} = 1$$

and, for $i = \overline{1, N}$

$$\|f\|_{A_i^*} = \sup_{x \in A_i} \frac{|f(x)|}{\|x\|_{A_i}} \leq \sup_{x \in A_i} \frac{K(\bar{t}, x, \vec{A})}{\|x\|_{A_i}} \leq \sup_{x \in A_i} \frac{t_i \|x\|_{A_i}}{\|x\|_{A_i}} = t_i.$$

Take any $b \in \Delta(\vec{A})$ and consider the one-dimensional operator $Tx = f(x)b$. Then, for $i \in \overline{0, N}$, we have

$$\|T\|_{A_i, A_i} = \sup_{x \in A_i} \frac{\|Tx\|_{A_i}}{\|x\|_{A_i}} = \|f\|_{A_i^*} \|b\|_{A_i}.$$

Since A is partly interpolator space, from (2.1) we get that

$$\begin{aligned} \|Ta\|_A &\leq C \max_{0 \leq i \leq N} (\|T\|_{A_i, A_i}) \|a\|_A = C \max_{0 \leq i \leq N} (\|f\|_{A_i^*} \|b\|_{A_i}) \|a\|_A \leq \\ &\leq C \max (\|b\|_{A_0}, t_1 \|b\|_{A_1}, \dots, t_N \|b\|_{A_N}) \|a\|_A = C J(\bar{t}, a, \vec{A}) \|a\|_A. \end{aligned}$$

But, from the definition of T we get

$$\|Ta\|_A = |f(a)| \|b\|_A = K(\bar{t}, a, \vec{A}) \|b\|_A.$$

So, we obtain (3.2). From (3.2) follows (3.1).

Conversely, suppose that (3.1) holds. Take any one-dimensional operator $Tx = f(x)b$ (i.e. take any $f \in \left(\Sigma(\vec{A})\right)^*$ and any $b \in \Delta(\vec{A})$). For any $a \in A$, using (3.1) we get

$$\begin{aligned} \|Ta\|_A &= |f(a)| \|b\|_A \leq |f(a)| \frac{J(\bar{t}, b, \vec{A})}{\rho(\bar{t})} \leq C |f(a)| \frac{J(\bar{t}, b, \vec{A})}{\psi(\bar{t})} \leq \\ &\leq C |f(a)| \frac{J(\bar{t}, b, \vec{A})}{K(\bar{t}, a, \vec{A})} \|a\|_A. \end{aligned}$$

Let $a = \sum_{i=0}^N a_i$, with $a_i \in A_i$ a decomposition of a such that:

$$\|a_0\|_{A_0} + \sum_{i=1}^N t_i \|a_i\|_{A_i} \leq 2K(\bar{t}, a, \vec{A}).$$

Then

$$\begin{aligned}
\|Ta\|_A &\leq 2C \frac{\sum_{i=0}^N |f(a_i)|}{\|a_0\|_{A_0} + \sum_{i=1}^N t_i \|a_i\|_{A_i}} J(\bar{t}, b, \vec{A}) \|a\|_A \leq \\
&\leq 2C(N+1) \max \left(\frac{|f(a_0)|}{\|a_0\|_{A_0}}, \frac{|f(a_1)|}{t_1 \|a_1\|_{A_1}}, \dots, \frac{|f(a_N)|}{t_N \|a_N\|_{A_N}} \right) J(\bar{t}, b, \vec{A}) \|a\|_A \leq \\
&\leq 2C(N+1) \max \left(\|f\|_{A_0^*}, \frac{\|f\|_{A_1^*}}{t_1}, \dots, \frac{\|f\|_{A_N^*}}{t_N} \right) \cdot \\
&\quad \cdot \max (\|b\|_{A_0}, t_1 \|b\|_{A_1}, \dots, t_N \|b\|_{A_N}) \|a\|_A.
\end{aligned}$$

Now, we put $t_i = \frac{\|f\|_{A_i^*}}{\|f\|_{A_0^*}}$, $i = 1, 2, \dots, N$. Then

$$\|Ta\|_A \leq 2C(N+1) \max_{0 \leq i \leq N} (\|f\|_{A_i^*} \|b\|_{A_i}) \|a\|_A = 2(N+1)C \|T\|_{\vec{A}, \vec{A}} \|a\|_A.$$

□

Proposition 3.2. Let $\vec{A} = (A_0, \dots, A_N)$ be a Banach $(N+1)$ -tuple and let A be a partly interpolation space with respect to $\vec{A}$.

- a) If $\lim_{\bar{t} \rightarrow \infty} \frac{1}{\rho(\bar{t})} > 0$ then $A \hookrightarrow \tilde{A}_0^{\Sigma(\vec{A})}$.
b) If $\lim_{\bar{t} \rightarrow \vec{0}} \psi(\bar{t}) > 0$ then $\overline{\Delta(\vec{A})}^{A_0} \hookrightarrow A$.

Proof. a) Suppose that $\lim_{\bar{t} \rightarrow \infty} \frac{1}{\rho(\bar{t})} > 0$. Then, by (3.1) we have $\lim_{\bar{t} \rightarrow \infty} \psi(\bar{t}) = C_1 < \infty$. It follows that for each $a \in A$ we have

$$\|a\|_{\tilde{A}_0^{\Sigma(\vec{A})}} = \lim_{\bar{t} \rightarrow \infty} K(\bar{t}, a, \vec{A}) \leq \lim_{\bar{t} \rightarrow \infty} \psi(\bar{t}) \|a\|_A = C_1 \|a\|_A$$

and so $A \hookrightarrow \tilde{A}_0^{\Sigma(\vec{A})}$.

b) Suppose that $\lim_{\bar{t} \rightarrow \vec{0}} \psi(\bar{t}) > 0$. Then, by (3.1) we have $\lim_{\bar{t} \rightarrow \vec{0}} \rho(\bar{t}) = C_2 > 0$. It

follows that $C_2 \leq \frac{J(\bar{t}, a, \vec{A})}{\|a\|_A}$, for all $\bar{t} = (t_1, \dots, t_N) > 0$, and all $a \in \Delta(\vec{A})$.

Therefore

$$\|a\|_A \leq \frac{1}{C_2} \max (\|a\|_{A_0}, t_1 \|a\|_{A_1}, \dots, t_N \|a\|_{A_N}).$$

Now, we put $t_i = \frac{\|a\|_{A_0}}{\|a\|_{A_i}}$, ($i = 1, 2, \dots, N$). This implies

$$\|a\|_A \leq \frac{1}{C_2 \|a\|_{A_0}}, \text{ for all } a \in \Delta(\vec{A})$$

and, consequently $\overline{\Delta(\vec{A})}^{A_0} \hookrightarrow A$. $\square$

Proposition 3.3. Let $\vec{A} = (A_0, \dots, A_N)$ and $\vec{B} = (B_0, \dots, B_N)$ be two Banach $N + 1$ -tuples. Then

$$R1 - \text{Int}(\vec{A}) \subset R1 - \text{Int}(\vec{B}) \Leftrightarrow A_i \in R1 - \text{Int}(\vec{B}), \quad (i = 0, \dots, N).$$

Proof. The implication $\Rightarrow$ is trivial. Conversely, suppose that $A_i \in R1 - \text{Int}(\vec{B})$, ($i = 0, \dots, N$). Let $A \in R1 - \text{Int}(\vec{A})$ and the one-dimensional operator $T \in \mathcal{L}(\vec{B}, \vec{B})$. Then, by (2.1) we have

$$\|T\|_{A_i, A_i} \leq C \|T\|_{\vec{B}, \vec{B}} \quad (i = -, 1, \dots, N).$$

It follows that

$$\|T\|_{A, A} \leq C \|T\|_{\vec{A}, \vec{A}} \leq C \|T\|_{\vec{B}, \vec{B}}.$$

Then $A \in R1 - \text{Int}(\vec{B})$. $\square$

Proposition 3.4. Let $\vec{A} = (A_0, \dots, A_N)$ be a Banach $N + 1$ -tuple and let A be a party interpolation space with respect to $\vec{A}$.

a) If $A \not\subset \overline{A_i}$ then $A_0 \cap \dots \cap A_{i-1} \cap A_{i+1} \cap \dots \cap A_N \subset A$, ($i = 0, \dots, N$).

b) If $A \not\subset \overline{\sum_{\substack{k=0 \\ k \neq i}}^N A_k}$ then $A_i \subset A$, ($i = 0, \dots, N$).

Proof. a) If $A \not\subset \overline{A_i}$, there exists a such that:

$$0 \neq a \in A, \quad a \notin \overline{A_i} \quad (i = 0, \dots, N).$$

Then, according to the Hahn–Banach theorem, there exists a linear functional $f \in \left(\Sigma(\vec{A})\right)^*$ for which

$$f|_{A_i} = 0, \quad f(a) = 1.$$

Let y be an arbitrary fixed element of $A_0 \cap \dots \cap A_{i-1} \cap A_{i+1} \cap \dots \cap A_N$. We put $T := f \otimes y$. Then $T \in \mathcal{L}(\vec{A}, \vec{A})$ and $y = T(a) \in A$.

The proof (b) is similar. $\square$

Corollary 3.5. If for every $i \in \{0, 1, \dots, N\}$ $A \not\subset \overline{\sum_{\substack{k=0 \\ k \neq i}}^N A_k}$ then $A = \Sigma(\vec{A})$.

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