Interpolation of Locally Convex Triples

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Abstract. In this paper some interpolation results are extended from locally convex couples to locally convex triples.

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1 Introduction

The theory of interpolation usually deals with interpolation of two Banach spaces. In this case we mention, in particular the K and J - methods (see e.g. [9],[1],[2],[3]). The paper by J.L. Lions and E. Magenes [8] incited to extend the theory of interpolation in the case of two locally convex spaces (see [4],[5],[6],[7]). In 1972 G. Sparr [10] extended the theory of interpolation to several (more than two) Banach spaces. The purpose of this paper is to extend some interpolation results from locally convex couples to locally convex triples.

2 Preliminaries

Let $\mathcal{C}_2$ be the category of locally convex triples. The objects of $\mathcal{C}_2$ are all triples $\vec{A} = (A_0, A_1, A_2)$ of Hausdorff locally convex spaces A_0, A_1, A_2 so that each $A_i (i = 0, 1, 2)$ is linearly and continuously embedded in a Hausdorff

topological vector space $\mathcal{A}(A_i \hookrightarrow \mathcal{A}, i = 0, 1, 2)$. Let $\vec{A} = (A_0, A_1, A_2)$ and $\vec{B} = (B_0, B_1, B_2)$ be two such triples then, by a morphism

$$T : \vec{A} \rightarrow \vec{B} \quad (T \in \mathcal{L}(\vec{A}, \vec{B}))$$

we mean a linear mapping $T : A_0 + A_1 + A_2 \rightarrow B_0 + B_1 + B_2$ so that $T|_{A_i}$ (the restriction of T to A_i) is continuous from A_i into B_i ($i = 0, 1, 2$). Thus $\mathcal{C}_0$ is the category of Hausdorff locally convex spaces with morphisms defined by continuous linear mappings between such spaces. The set of all subspaces of a vector space forms, in a natural way, a lattice. Thus we can speak of the sum: $\Sigma(\vec{A}) := A_0 + A_1 + A_2$ and the intersection: $\Delta(\vec{A}) := A_0 \cap A_1 \cap A_2$. The topology of $\Sigma(\vec{A})$ is defined by the inductive limit of the A_k (by $j_k : A_k \rightarrow \Sigma(\vec{A}), k = 0, 1, 2$). On $\Delta(\vec{A})$ we consider the projective topology, with respect to canonical imbeddings $i_k : \Delta(\vec{A}) \rightarrow A_k (k = 0, 1, 2)$. Thus, $\Sigma(\vec{A})$ and $\Delta(\vec{A})$ are objects in $\mathcal{C}_0$. Now, we define two basic covariants functors: $F_\Sigma := \Sigma$ (sum) and $F_\Delta := \Delta$ (intersection) from $\mathcal{C}_2$ to $\mathcal{C}_0$ as follows:

$$\vec{A} = (A_0, A_1, A_2) \rightarrow F_\Sigma(\vec{A}) := A_0 + A_1 + A_2$$

$$T \in \mathcal{L}(\vec{A}, \vec{B}) \rightarrow F_\Sigma(T) = T : A_0 + A_1 + A_2 \rightarrow B_0 + B_1 + B_2$$

and

$$\vec{A} = (A_0, A_1, A_2) \rightarrow F_\Delta(\vec{A}) = A_0 \cap A_1 \cap A_2$$

$$T \in \mathcal{L}(\vec{A}, \vec{B}) \rightarrow F_\Delta(T) = T : A_0 \cap A_1 \cap A_2 \rightarrow B_0 \cap B_1 \cap B_2$$

Indeed, if $T \in \mathcal{L}(\vec{A}, \vec{B})$ then T is continuous from A_i into $B_0 + B_1 + B_2$ ($i = 0, 1, 2$), and from the definition of the inductive limit topology, it is continuous from $A_0 + A_1 + A_2$ into $B_0 + B_1 + B_2$. Analogously, $T|_{A_0 \cap A_1 \cap A_2}$ is continuous from $A_0 \cap A_1 \cap A_2$ into $B_0 \cap B_1 \cap B_2$.

Definition 2.1. By an interpolation functor on $\mathcal{C}_2$ we mean a functor

$$F : \mathcal{C}_2 \rightarrow \mathcal{C}_0$$

so that:

(i) $\Delta(\vec{A}) \hookrightarrow F(\vec{A}) \hookrightarrow \Sigma(\vec{A})$ for any object in $\mathcal{C}_2$.

(ii) $F(T) = T|_{F(\vec{A})}$ for any $T \in \mathcal{L}(\vec{A}, \vec{B})$ in $\mathcal{C}_2$.

Definition 2.2. Let $\vec{A} = (A_0, A_1, A_2)$ be a given triple in $\mathcal{C}_2$. Then a space A in $\mathcal{C}_0$ will be called an interpolation space with respect to $\vec{A}$ if there is an interpolation functor F on $\mathcal{C}_2$ so that $F(\vec{A}) = A$.

If in Definition 2.1 we consider only one-dimensional operators T i.e.

$$Ta = f(a)b, b \in \Delta(\vec{A}), f \in \left(\Sigma(\vec{A})\right)^*$$

then the space A is called partly interpolation.

Now we will define a suitable "function norm". Let $\mathcal{M}$ be the set of all numerical (finite or infinite) functions $\varphi = \varphi(t_1, t_2)$ on $(0, \infty) \times (0, \infty)$ measurable with respect to the measure $\frac{dt_1}{t_1} \frac{dt_2}{t_2}$. We denote by $\mathcal{M}_+$ the set of positive functions in $\mathcal{M}$. Generally, by a function norm on $\mathcal{M}_+$ we mean a functional Φ defined for all $\varphi \in \mathcal{M}_+$ with values in $[0, \infty]$ which has the following properties:

N1. $\Phi(\varphi) = 0$ if and only if $\varphi = 0$ a.e. ,

N2. $\Phi(\varphi) < \infty$ implies $\varphi < \infty$ a.e. ,

N3. $\Phi(c\varphi) = c\Phi(\varphi)$, $(c > 0)$,

N4.

$$\varphi \leq \sum_{i=0}^{\infty} \varphi_i \quad \text{a.e. implies} \quad \Phi(\varphi) \leq \sum_{i=0}^{\infty} \Phi(\varphi_i)$$

But our intention is to define interpolation functors $\mathcal{K}_\Phi$ which will be done by applying a suitable chosen function norm Φ on the Peetre $\mathcal{K}$ -functional, so we need the following condition

N5. $\Phi(\min(1, t_1, t_2)) < \infty$

Example 2.1. Let $1 \leq p \leq \infty$, $\bar{\theta} = (\theta_1, \theta_2) \in (0, 1) \times (0, 1)$ with $\theta_1 + \theta_2 < 1$. We define a function norm Φ (denoted in this case $\Phi_{\bar{\theta}, p}$) by

$$\Phi_{\bar{\theta}, p}(\varphi) = \left(\iint_{\mathbb{R}_+^2} (t_1^{-\theta_1} t_2^{-\theta_2} \varphi(t_1, t_2))^p \frac{dt_1}{t_1} \frac{dt_2}{t_2} \right)^{\frac{1}{p}}.$$

If we put $R_+^2 = D_1 \cup D_2 \cup D_3 \cup D_4$ where: $D_1 = (0, 1) \times (0, 1)$, $D_2 = (0, 1) \times [1, \infty]$, $D_3 = [1, \infty] \times [1, \infty]$ and $D_4 = [1, \infty] \times (0, 1)$, then

$$\Phi_{\bar{\theta}, p}(\min(1, t_1, t_2)) = \left(\frac{1}{p^2 \theta_1 \theta_2 (1 - \theta_1 - \theta_2)} \right)^{\frac{1}{p}} < \infty$$

3 Interpolation results

Let $\vec{A} = (A_0, A_1, A_2)$ be an object in $\mathcal{C}_2$. The topology of A_0 is generated by the family of semi-norms $(p_i)_{i \in I}$, that of A_1 by $(q_j)_{j \in J}$ and that of A_2 by $(\rho_k)_{k \in K}$. We suppose that the families $(p_i)_{i \in I}$, $(q_j)_{j \in J}$ and $(\rho_k)_{k \in K}$ are filled completed. For $\alpha \in A_0 + A_1 + A_2$, and $t_1, t_2 \in (0, \infty)$ we consider the Peetre $\mathcal{K}$ -functional, $\mathcal{K}_{ijk}$ asociated with the triple $\vec{A}$, i.e.

$$\mathcal{K}_{ijk}(t_1, t_2, a) = \inf \{p_i(a_0) + t_1 q_j(a_1) + t_2 \rho_k(a_2) : a = a_0 + a_1 + a_2, a_i \in A_i, i = 0, 1, 2\}$$

we have the folowing inequalities

$$\min(1, t_1, t_2) \mathcal{K}_{ijk}(1, 1, a) \leq \mathcal{K}_{ijk}(t_1, t_2, a), \forall a \in A_0 + A_1 + A_2$$

$$\mathcal{K}_{ijk}(t_1, t_2, a) \leq \min(1, t_1, t_2) \max(p_i(a), t_1 q_j(a), t_2 \rho_k(a)), \forall a \in A_0 \cap A_1 \cap A_2$$

Let Φ be a function norm. We put

$$r_{ijk}(a) = \Phi(\mathcal{K}_{ijk}(t_1, t_2, a))$$

Theorem 3.1. *Let $\vec{A} = (A_0, A_1, A_2)$ be an object in $\mathcal{C}_2$ and Φ a function norm. Then*

$$\mathcal{K}_\Phi : \mathcal{C}_2 \rightarrow \mathcal{C}_0$$

defined by $\mathcal{K}_\Phi(\vec{A}) = \{a \in A_0 + A_1 + A_2 : r_{ijk}(a) < \infty, \forall i \in I, j \in J, k \in K\}$ is an interpolation functor on the category $\mathcal{C}_2$.

Proof. At first we show that $\mathcal{K}_\Phi(\vec{A})$ is an Hausdorff locally convex spaces so that

$$A_0 \cap A_1 \cap A_2 \hookrightarrow \mathcal{K}_\Phi(\vec{A}) \hookrightarrow A_0 + A_1 + A_2$$

Indeed the family $(r_{ijk})_{(i,j,k) \in I \times J \times K}$ defines a family of semi-norms on the space $\mathcal{K}_\Phi(\vec{A})$ and the topology corespondent is finer than that of $A_0 + A_1 + A_2$ and les fine that of $A_0 \cap A_1 \cap A_2$. Let Now $\vec{B} = (B_0, B_1, B_2)$ be an object in $\mathcal{C}_2$. The topologies of B_0, B_1 and B_2 are generated by the filtrant families of semi-norms $(p'_l)_{l \in L}$, $(q'_m)_{m \in M}$ and respective $(\rho'_s)_{s \in S}$. Let $a \in \mathcal{K}_\Phi(\vec{A})$, $T \in \mathcal{L}(\vec{A}, \vec{B})$ and $(l, m, s) \in L \times M \times S$. We put

$$r'_{lms}(Ta) = \Phi(\mathcal{K}_{lms}(t_1, t_2, Ta))$$

But

$$\begin{aligned}\mathcal{K}_{lms}(t_1, t_2, Ta) &= \inf\{p'_l(b_0) + t_1 q'_m(b_1) + t_2 \rho'_s(b_2), Ta = b_0 + b_1 + b_2, b_i \in B_i, \\ &\quad i = 0, 1, 2\} \\ &\leq \inf\{p'_l(Ta_0) + t_1 q'_m(Ta_1) + t_2 \rho'_s(Ta_2), a = a_0 + a_1 + a_2, a_i \in A_i, \\ &\quad i = 0, 1, 2\}\end{aligned}$$

Since T is continuous from A_n into B_n , $n = 0, 1, 2$ it follows that $\exists i(l) \in I, j(m) \in J, k(s) \in K$ and $M > 0$ so that

$$p'_l(Ta_0) \leq Mp_{i(l)}(a_0)$$

$$q'_m(Ta_1) \leq Mq_{j(m)}(a_1)$$

$$\rho'_s(Ta_2) \leq M\rho_{k(s)}(a_2)$$

Then

$$\mathcal{K}_{lms}(t_1, t_2, Ta) \leq M\mathcal{K}_{i(l)j(m)k(s)}(t_1, t_2, a)$$

and

$$r'_{lms}(Ta) \leq Mr_{i(l)j(m)k(s)}(a)$$

This means that T is continuous from $\mathcal{K}_\Phi(\vec{A})$ into $\mathcal{K}_\Phi(\vec{B})$ and the proof is complete.

Theorem 3.2. *Let A be an partly interpolation space with respect to $\vec{A} = (A_0, A_1, A_2)$ and $i, j, k \in \{0, 1, 2\}$ with $i \neq j \neq k$.*

a) *If $A \not\subset \overline{A_i + A_j}^{\Sigma(\vec{A})}$ then $A \supset A_k$.*

b) *If $A \not\subset \overline{A_i}^{\Sigma(\vec{A})}$ then $A \supset A_j \cap A_k$.*

Proof. It is enough to prove a) and b) in the case $i = 0, j = 1, k = 1$.

a) If $A \not\subset \overline{A_0 + A_1}^{\Sigma(\vec{A})}$, there exists a and φ , so that:

$$0 \neq a \in A, a \notin \overline{A_0 + A_1}^{\Sigma(\vec{A})}, \varphi \in \left(\Sigma(\vec{A})\right)^*, \varphi_{/\overline{A_0 + A_1}} = 0 \text{ and } \varphi(a) = 1.$$

Let b be an arbitrary fixed element of A_2 . The linear mapping $T(x) = \varphi(x)b$ belongs to $\mathcal{L}(\vec{A}, \vec{A})$ and $b = Ta \in A$, which finishes the proof of a).

b) If $A \subset \overline{A_0}^{\Sigma(\vec{A})}$ there exists a and φ , so that:

$$0 \neq a \in A, a \in \overline{A_0}^{\Sigma(\vec{A})}, \varphi \in \left(\Sigma(\vec{A})\right)^*, \varphi_{/\overline{A_0}} = 0 \text{ and } \varphi(a) = 1.$$

Let b be an arbitrary fixed element of $A_1 \cap A_2$. The linear mapping $Tx = \varphi(x)b$ belongs to $\mathcal{L}(\vec{A}, \vec{A})$ and $b = T(a) \in A$.

Theorem 3.3. *Let $\vec{A} = (A_0, A_1, A_2)$ be a locally convex triple. Let C be a subspace of $A_0 + A_1 + A_2$. Then, for every interpolation functor F , we have*

$$F(A_0 \cap C, A_1 \cap C, A_2 \cap C) \hookrightarrow F(\vec{A}) \cap C \quad (1)$$

The spaces $A_0 \cap C, A_1 \cap C, A_2 \cap C$ and $F(\vec{A}) \cap C$ are endowed with the topologies of A_0, A_1, A_2 and $F(\vec{A})$ respectively.

Proof. $A_0 \cap C \hookrightarrow C, A_1 \cap C \hookrightarrow C$ and $A_2 \cap C \hookrightarrow C$ show that $(A_0 \cap C, A_1 \cap C, A_2 \cap C)$ is a locally convex triple. If I denotes the embedding operator from C into $A_0 + A_1 + A_2$, then $I \in \mathcal{L}((A_0 \cap C, A_1 \cap C, A_2 \cap C), (A_0, A_1, A_2))$. Now, if J denotes the embedding operator from $A_0 \cap C + A_1 \cap C + A_2 \cap C$ into C , then $J \in \mathcal{L}((A_0 \cap C, A_1 \cap C, A_2 \cap C), (C, C, C))$. Then by the interpolation property, we obtain (1).

Now we give a sufficient condition for obtaining the reverse inclusion in (1).

Theorem 3.4. *Let $\vec{A} = (A_0, A_1, A_2)$ be a locally convex triple. Let C be a subspace of $A_0 + A_1 + A_2$. If there is an operator $R \in \mathcal{L}(\vec{A}, \vec{A})$ so that:*

$$i) R(A_0 + A_1 + A_2) = C$$

ii) the restriction of R to C is the identity

then, for every interpolation functor F , we have

$$F(A_0 \cap C, A_1 \cap C, A_2 \cap C) = F(\vec{A}) \cap C \quad (2)$$

Proof. We have

$$C = A_0 \cap C + A_1 \cap C + A_2 \cap C$$

So, $R \in \mathcal{L}((A_0, A_1, A_2), (A_0 \cap C, A_1 \cap C, A_2 \cap C))$. Then, by the interpolation property, the restriction of R to $F(\vec{A})$ is continuous from $F(\vec{A})$ into $F(A_0 \cap C, A_1 \cap C, A_2 \cap C)$. Since the restriction of R to C is the identity we obtain

$$F(\vec{A}) \cap C \hookrightarrow F(A_0 \cap C, A_1 \cap C, A_2 \cap C)$$

Corollary 3.4.1. *Let $\mathcal{C}$ be a complemented subspace of $A_0 + A_1 + A_2$, whose projection $P \in \mathcal{L}(\vec{A}, \vec{A})$. Then (2) holds.*

Theorem 3.5. *Let $\vec{A} = (A_0, A_1, A_2)$, $\vec{B} = (B_0, B_1, B_2)$ be a locally convex triples and $T \in \mathcal{L}(\vec{A}, \vec{B})$. If there is an operator $S \in \mathcal{L}(\vec{B}, \vec{A})$ so that $TS = I_d \vec{B}$ (the identity map) then, for every interpolation functor F , we have*

$$F(A_{0T}, A_{1T}, A_{2T}) = F(A_0, A_1, A_2)_T$$

where A_T denotes $\{a \in A \hookrightarrow A_0 + A_1 + A_2 : Ta = 0\}$ endowed with the A -topology.

Proof. We put

$$C = \{a \in A_0 + A_1 + A_2 : Ta = 0\}$$

$$R = I_d \vec{A} - ST$$

and then apply theorem 3.4.

4 Application

Let A be a Banach space, and let $G(t)$ ($t > 0$) be an strongly continuous semi-group of operators on A . $\mathbb{T}$ denotes the torrus $\mathbb{R}/2\pi\mathbb{Z}$ and $\mathcal{C}^\infty(\mathbb{T}, A)$ the space of all infinitely differentiable functions from $\mathbb{T}$ into A . Let $(M_k)_{k \in \mathbb{N}}$ be a positive sequence of real numbers and:

$$A_{M_k} = A_{M_k}(\mathbb{T}, A) = \left\{ f \in \mathcal{C}^\infty(\mathbb{T}, A); \exists L > 0 : \sup_{k \in \mathbb{N}} \frac{\|f^{(k)}\|_L^2(\mathbb{T}, A)}{L^k M_k} < \infty \right\}$$

For $1 \leq s_0 \leq s_1 \leq s_2 < \infty$ and $M_k = s_{0k}!$, $N_k = s_{1k}!$, $W_k = s_{2k}!$ we denote by $\dot{A}_{M_k} = \dot{A}_{M_k}(\mathbb{T}, A) = \{f \in A_{M_k} : f(0) = 0\}$ and by analogy $\dot{A}_{N_k}$, $\dot{A}_{W_k}$. We define the morphism

$$\mathcal{R} : (A_{M_k}, A_{N_k}, A_{W_k}) \longrightarrow (\dot{A}_{M_k}, \dot{A}_{N_k}, \dot{A}_{W_k})$$

by $\mathcal{R}f = f - (G \circ \varphi)f(0)$, where $\varphi(t) = \sin t$, $t \in \mathbb{T}$. Then there is an morphisms $\mathcal{S} : (\dot{A}_{M_k}, \dot{A}_{N_k}, \dot{A}_{W_k}) \longrightarrow (A_{M_k}, A_{N_k}, A_{W_k})$ defined by $\mathcal{S}f = f$, such that $\mathcal{R}\mathcal{S} = I_d$. Consequently, by Theorema 3.5. we obtain, for every interpolation functor F ,

$$F(A_{M_k\mathcal{R}}, A_{N_k\mathcal{R}}, A_{W_k\mathcal{R}}) = F(A_{M_k}, A_{N_k}, A_{W_k})_{\mathcal{R}}$$

Remark 4.1. *In the case of couples, this example is given in [6].*

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