Weighted Lipschitz Estimates for Multilinear Commutator of Integral Operator

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Abstract. In this paper, the weighted boundedness for the multilinear commutator associated to some integral operator and the weighted Lipschitz functions are obtained. The integral operator includes the Littlewood-Paley operator, Marcinkiewicz operator and Bochner-Riesz operator.

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1 Introduction

As the development of the singular integral operators, their commutators and multilinear operators have been well studied. In [8][18-19], the authors proved that the commutators and multilinear operators generated by the singular integral operators and BMO functions are bounded on $L^p(R^n)$ for $1 < p < \infty$. In [6][14][17], the boundedness for the commutators and multilinear operators generated by the singular integral operators and Lipschitz functions on $L^p(R^n)$ ($1 < p < \infty$) and Triebel-Lizorkin spaces are obtained. And the weighted boundedness for the commutators generated by the singular integral operators and BMO or Lipschitz functions on $L^p(R^n)$ ($1 < p < \infty$) spaces are obtained (see [3][13]). Inspired by it, we will discuss the weighted boundedness

for multilinear commutator associated to some integral operators and the weighted Lipschitz functions. The integral operator includes the Littlewood-Paley operator, Marcinkiewicz operator and Bochner-Riesz operator.

2 Notations and Theorems

Suppose b_j ($j = 1, \dots, m$) are the fixed locally integrable functions on R^n . Let $F_t(x, y)$ be defined on $R^n \times R^n \times [0, +\infty)$. Set

$$F_t(f)(x) = \int_{R^n} F_t(x, y) f(y) dy$$

and

$$F_t^{\vec{b}}(f)(x) = \int_{R^n} \prod_{j=1}^m (b_j(x) - b_j(y)) F_t(x, y) f(y) dy,$$

for every bounded and compactly supported function f . Let H be the Banach space $(H, \|\cdot\|)$ such that, for each fixed $x \in R^n$, $F_t(f)(x)$ and $F_t^{\vec{b}}(f)(x)$ may be viewed as the mappings from $[0, +\infty)$ to H . The multilinear commutator related to F_t is defined by

$$T_{\vec{b}}(f)(x) = \|F_{(\cdot)}^{\vec{b}}(f)(x)\|,$$

where F_t satisfies: for fixed $\varepsilon > 0$

$$\|F_t(x, y)\| \leq C|x - y|^{-n}$$

and

$$\|F_t(y, x) - F_t(z, x)\| + \|F_t(x, y) - F_t(x, z)\| \leq C|y - z|^\varepsilon |x - z|^{-n-\varepsilon},$$

if $2|y - z| \leq |x - z|$. We also define that $T(f)(x) = \|F_{(\cdot)}(f)(x)\|$.

Note that when $b_1 = \dots = b_m$, $T_{\vec{b}}$ is just the m order commutator (see [1][22]). It is well known that commutators are of great interest in harmonic analysis and have been widely studied by many authors (see [7][9][15][18][19]). Our main purpose is to establish the sharp inequality for the multilinear commutator.

To state our result, we first give some notions (see [1][10][16][19][21]).

In this paper, Q will denote a cube of R^n and $f_Q = |Q|^{-1} \int_Q f(x) dx$, we define the sharp function of f as

$$f^\#(x) = \sup_{Q \ni x} \frac{1}{|Q|} \int_Q |f(y) - f_Q| dy.$$

It is well-known that (see [10])

$$f^\#(x) \approx \sup_{Q \ni x} \inf_{c \in C} \frac{1}{|Q|} \int_Q |f(y) - c| dy.$$

Let M be the Hardy-Littlewood maximal operator defined by

$$M(f)(x) = \sup_{Q \ni x} |Q|^{-1} \int_Q |f(y)| dy.$$

For $1 \leq p < \infty$ and $0 \leq \eta < n$, we denote $M_{\eta,p}(f)$ by

$$M_{\eta,p}(f)(x) = \sup_{Q \ni x} \left(\frac{1}{|Q|^{1-p\eta/n}} \int_Q |f(y)|^p dy \right)^{1/p}.$$

The A_p weight is defined by (see [10])

$$A_p = \left\{ w : \sup_Q \left(\frac{1}{|Q|} \int_Q w(x) dx \right) \left(\frac{1}{|Q|} \int_Q w(x)^{-1/(p-1)} dx \right)^{p-1} < \infty \right\},$$

$$1 < p < \infty,$$

$$A_1 = \{w > 0 : M(w)(x) \leq Cw(x), a.e.\},$$

and $A_\infty = \cup_{p \geq 1} A_p$. We know, for $w \in A_1$, w satisfies the double condition, that is, for any cube Q ,

$$w(2Q) \leq Cw(Q).$$

The $A(p, q)$ weight is defined by (see [10])

$$A(p, q) = \{w > 0 :$$

$$\sup_Q \left(\frac{1}{|Q|} \int_Q w(x)^q dx \right)^{1/q} \left(\frac{1}{|Q|} \int_Q w(x)^{-p/(p-1)} dx \right)^{(p-1)/p} < \infty\}$$

for $1 < p, q < \infty$.

Given a weight function w , for $1 < p < \infty$, the weighted Lebesgue space $L^p(w)$ is the space of functions f such that

$$\|f\|_{L^p(w)} = \left(\int_{R^n} |f(x)|^p w(x) dx \right)^{1/p} < \infty.$$

For $\beta > 0$ and $p > 1$, let $\dot{F}_p^{\beta,\infty}(w)$ be the weighted homogeneous Triebel-Lizorkin space. For $0 < \beta < 1$, the weighted Lipschitz space $Lip_\beta(w)$ is the space of functions f such that

$$\|f\|_{Lip_\beta(w)} = \sup_Q \frac{1}{w(Q)^{1+\beta/n}} \int_Q |f(y) - f_Q| dy < \infty.$$

Given some function $b_j \in Lip_\beta(w)$, $1 \leq j \leq m$, we denote by C_j^m the family of all finite subsets $\sigma = \{\sigma(1), \dots, \sigma(j)\}$ of $\{1, \dots, m\}$ of j different elements and $\sigma(i) < \sigma(j)$ when $i < j$. For $\sigma \in C_j^m$, set $\sigma^c = \{1, \dots, m\} \setminus \sigma$. For $\vec{b} = (b_1, \dots, b_m)$ and $\sigma = \{\sigma(1), \dots, \sigma(j)\} \in C_j^m$, set $\vec{b}_\sigma = (b_{\sigma(1)}, \dots, b_{\sigma(j)})$, $b_\sigma = \prod_{i=1}^j b_{\sigma(i)}$ and $\|\vec{b}_\sigma\|_{Lip_\beta(w)} = \prod_{i=1}^j \|b_{\sigma(i)}\|_{Lip_\beta(w)}$.

Now we shall state our theorems as following.

Theorem 1. *Let $b_j \in Lip_\beta(w)$ for $1 \leq j \leq m$, $0 < \beta < 1$ and $w \in A_1$. Suppose $1 < p < n/m\beta$ and $1/q = 1/p - m\beta/n$. Then T_b^m is bounded from $L^p(w)$ to $L^q(w^{1-m+(q-1)m\beta/n})$.*

Theorem 2. *Let $b_j \in Lip_\beta(w)$ for $1 \leq j \leq m$, $0 < \beta < 1$ and $w \in A_1$. Suppose $1 < p < \infty$. Then T_b^m is bounded from $L^p(w)$ to $\tilde{F}_p^{m\beta,\infty}(w^{1-m-m\beta/n})$.*

3 Proofs of Theorems

In order to prove the theorems, we need the following lemmas.

Lemma 1.(see [11][13]) *For $0 < \beta < 1$, $w \in A_1$, $b \in Lip_\beta(w)$ and $1 \leq p \leq \infty$, we have*

$$\|b\|_{Lip_\beta(w)} \approx \sup_Q w(Q)^{-\beta/n} \left(w(Q)^{-1} \int_Q |b(x) - b_Q|^p w(x)^{1-p} dx \right)^{1/p}.$$

Lemma 2.(see [10][13]) *For $0 < \beta < 1$, $w \in A_1$, $b \in Lip_\beta(w)$ and any cube Q , we have*

$$\sup_{x \in Q} |b(x) - b_Q| \leq C \|b\|_{Lip_\beta(w)} w(Q)^{1+\beta/n} |Q|^{-1}.$$

Lemma 3. *For $0 < \beta < 1$, $w \in A_1$, $b \in Lip_\beta(w)$, any cube Q and $\tilde{x} \in Q$, we have*

$$|b_{2^k Q} - b_Q| \leq C k w(\tilde{x}) w(2^k Q)^{\beta/n} \|b\|_{Lip_\beta(w)}.$$

Lemma 4.(see [12]) For $0 < \beta < 1$, $w \in A_1$, $1 < p < \infty$ and $m > 0$, we have

$$\begin{aligned} \|f\|_{\tilde{F}_p^{m\beta, \infty}(w)} &\approx \left\| \sup_{Q \ni \tilde{x}} |Q|^{-1-m\beta/n} \int_Q |f(x) - f_Q| dx \right\|_{L^p(w)} \\ &\approx \left\| \sup_{Q \ni \tilde{x}} \inf_{c \in C} |Q|^{-1-m\beta/n} \int_Q |f(x) - c| dx \right\|_{L^p(w)}. \end{aligned}$$

Lemma 5.(see [11]) Suppose that $0 \leq \eta < n$, $1 < s < p < n/\eta$, $1/q = 1/p - \eta/n$ and $w \in A(p, q)$. Then

$$\|M_{\eta, s}(f)\|_{L^q(w^q)} \leq C \|f\|_{L^p(w^p)}.$$

Proof of Theorem 1. In order to prove the theorem, we first prove a sharp function estimate for the multilinear commutator. We will prove that for any $1 < r < \infty$ and cube Q , there exists some constant C_0 such that

$$\frac{1}{|Q|} \int_Q |T_{\tilde{b}}(f)(x) - C_0| dx \leq C \|\tilde{b}\|_{Lip_\beta(w)} w(\tilde{x})^{m+m\beta/n} (M_{m\beta, r}(f)(\tilde{x}) + M_{m\beta, r}(T(f))(\tilde{x})).$$

Fix a cube $Q = Q(x_0, d)$ and $\tilde{x} \in Q$, we write $f(x) = f_1(x) + f_2(x)$ and $f_1 = f\chi_{2Q}$, $f_2 = f\chi_{(2Q)^c}$.

We first consider the **Case** $m = 1$. For $C_0 = T(((b_1)_{2Q} - b_1)f_2)(x_0)$, we write $F_t^{b_1}(f)(x) = (b_1(x) - (b_1)_{2Q})F_t(f)(x) - F_t((b_1 - (b_1)_{2Q})f_1)(x) - F_t((b_1 - (b_1)_{2Q})f_2)(x)$. Then

$$\begin{aligned} &|T_{b_1}(f)(x) - T(((b_1)_{2Q} - b_1)f_2)(x_0)| \\ &= \left| \|F_t^{b_1}(f)(x)\| - \|F_t(((b_1)_{2Q} - b_1)f_2)(x_0)\| \right| \\ &\leq \|F_t^{b_1}(f)(x) - F_t(((b_1)_{2Q} - b_1)f_2)(x_0)\| \\ &\leq \|(b_1(x) - (b_1)_{2Q})F_t(f)(x)\| + \|F_t((b_1 - (b_1)_{2Q})f_1)(x)\| \\ &\quad + \|F_t((b_1 - (b_1)_{2Q})f_2)(x) - F_t((b_1 - (b_1)_{2Q})f_2)(x_0)\| \\ &= A(x) + B(x) + C(x). \end{aligned}$$

For $A(x)$, by Hölder's inequality and lemma 2, we have

$$\begin{aligned} &\frac{1}{|Q|} \int_Q A(x) dx \\ &\leq \frac{1}{|Q|} \int_Q |b_1(x) - (b_1)_{2Q}| |T(f)(x)| dx \\ &\leq \frac{1}{|Q|} \left(\int_Q |b_1(x) - (b_1)_{2Q}|^{r'} dx \right)^{\frac{1}{r'}} \left(\int_Q |T(f)(x)|^r dx \right)^{\frac{1}{r}} \end{aligned}$$

$$\begin{aligned}
&\leq \frac{1}{|2Q|} \sup_{x \in 2Q} |b_1(x) - (b_1)_{2Q}| |Q|^{\frac{1}{r'}} \left(\int_Q |T(f)(x)|^r(x) dx \right)^{\frac{1}{r}} \\
&\leq \frac{C}{|Q|} \|b_1\|_{Lip_\beta(w)} w(Q)^{1+\beta/n} |Q|^{-1} |Q|^{1/r'} |Q|^{1/r-\beta/n} \cdot \\
&\quad \cdot \left(\frac{1}{|Q|^{1-r\beta/n}} \int_Q |T(f)(x)|^r dx \right)^{\frac{1}{r}} \\
&\leq C \|b_1\|_{Lip_\beta(w)} \left(\frac{w(Q)}{|Q|} \right)^{1+\beta/n} M_{\beta,r}(T(f))(\tilde{x}) \\
&\leq C \|b_1\|_{Lip_\beta(w)} w(\tilde{x})^{1+\beta/n} M_{\beta,r}(T(f))(\tilde{x}).
\end{aligned}$$

For $B(x)$, by the type (r, r) of T and Lemma 2, we obtain

$$\begin{aligned}
&\frac{1}{|Q|} \int_Q B(x) dx \\
&\leq \frac{1}{|Q|} \left(\int_{R^n} |T(((b_1)_Q - b_1)f_1)(x)|^r dx \right)^{\frac{1}{r}} |Q|^{\frac{1}{r'}} \\
&\leq \frac{C}{|Q|} \left(\int_{R^n} |(b_1)_Q - b_1(x)|^r |f_1(x)|^r dx \right)^{\frac{1}{r}} |Q|^{\frac{1}{r'}} \\
&\leq \frac{C}{|Q|} \sup_{x \in 2Q} |b_1(x) - (b_1)_{2Q}| \left(\int_{2Q} |f(x)|^r dx \right)^{\frac{1}{r}} |Q|^{\frac{1}{r'}} \\
&\leq \frac{C}{|Q|} \|b_1\|_{Lip_\beta(w)} w(2Q)^{1+\beta/n} |2Q|^{-1} |2Q|^{1/r-\beta/n} \cdot \\
&\quad \cdot \left(\frac{1}{|2Q|^{1-r\beta/n}} \int_{2Q} |f(x)|^r dx \right)^{\frac{1}{r}} |Q|^{\frac{1}{r'}} \\
&\leq C \|b_1\|_{Lip_\beta(w)} \left(\frac{w(2Q)}{|2Q|} \right)^{1+\beta/n} M_{\beta,r}(f)(x) \\
&\leq C \|b_1\|_{Lip_\beta(w)} w(\tilde{x})^{1+\beta/n} M_{\beta,r}(f)(\tilde{x}).
\end{aligned}$$

For $C(x)$, for any $x \in Q$, we have

$$\begin{aligned}
C(x) &= \left\| \int_{R^n} (b_1(y) - (b_1)_{2Q}) f_2(y) (F_t(x, y) - F_t(x_0, y)) dy \right\| \\
&\leq \int_{(2Q)^c} |b_1(y) - (b_1)_{2Q}| |f(y)| |F_t(x, y) - F_t(x_0, y)| dy \\
&\leq C \int_{(2Q)^c} |b_1(y) - (b_1)_{2Q}| |f(y)| \frac{|x_0 - x|^\varepsilon}{|x_0 - y|^{n+\varepsilon}} dy \\
&\leq C \sum_{k=1}^{\infty} \int_{2^{k+1}Q \setminus 2^kQ} |b_1(y) - (b_1)_{2Q}| |f(y)| \frac{|x_0 - x|^\varepsilon}{|x_0 - y|^{n+\varepsilon}} dy \\
&\leq C \sum_{k=1}^{\infty} 2^{-k\varepsilon} \frac{1}{|2^{k+1}Q|} \int_{2^{k+1}Q} |b_1(y) - (b_1)_{2^{k+1}Q}| |f(y)| dy \\
&\quad + C \sum_{k=1}^{\infty} 2^{-k\varepsilon} \frac{1}{|2^{k+1}Q|} \int_{2^{k+1}Q} |(b_1)_{2^{k+1}Q} - (b_1)_Q| |f(y)| dy \\
&\leq C \sum_{k=1}^{\infty} 2^{-k\varepsilon} \frac{1}{|2^{k+1}Q|} \left(\int_{2^{k+1}Q} |b_1(y) - (b_1)_{2^{k+1}Q}|^{r'} dy \right)^{\frac{1}{r'}} \cdot \\
&\quad \cdot \left(\int_{2^{k+1}Q} |f(y)|^r dy \right)^{\frac{1}{r}} + \\
&\quad + C \sum_{k=1}^{\infty} 2^{-k\varepsilon} \frac{1}{|2^{k+1}Q|} |(b_1)_{2^{k+1}Q} - (b_1)_Q| \left(\int_{2^{k+1}Q} |f(y)|^r dy \right)^{\frac{1}{r}} |2^{k+1}Q|^{\frac{1}{r'}} \\
&\leq C \sum_{k=1}^{\infty} 2^{-k\varepsilon} \sup_{y \in 2^{k+1}Q} |b_1(y) - (b_1)_{2^{k+1}Q}| |2^{k+1}Q|^{-\beta/n} \cdot \\
&\quad \cdot \left(\frac{1}{|2^{k+1}Q|^{1-r\beta/n}} \int_{2^{k+1}Q} |f(y)|^r dy \right)^{\frac{1}{r}} + \\
&\quad + C \sum_{k=1}^{\infty} 2^{-k\varepsilon} |(b_1)_{2^{k+1}Q} - (b_1)_{2Q}| |2^{k+1}Q|^{-\beta/n} \cdot \\
&\quad \cdot \left(\frac{1}{|2^{k+1}Q|^{1-r\beta/n}} \int_{2^{k+1}Q} |f(y)|^r dy \right)^{\frac{1}{r}} \\
&\leq C \sum_{k=1}^{\infty} 2^{-k\varepsilon} \|b_1\|_{Lip_\beta(w)} w(2^{k+1}Q)^{1+\beta/n} |2^{k+1}Q|^{-1} |2^{k+1}Q|^{-\beta/n} M_{\beta,r}(f)(\tilde{x}) \\
&\quad + C \sum_{k=1}^{\infty} 2^{-k\varepsilon} kw(\tilde{x}) w(2^kQ)^{\beta/n} \|b_1\|_{Lip_\beta(w)} |2^{k+1}Q|^{-\beta/n} M_{\beta,r}(f)(\tilde{x}) \\
&\leq C \|b_1\|_{Lip_\beta(w)} \sum_{k=1}^{\infty} 2^{-k\varepsilon} \left(\frac{w(2^{k+1}Q)}{|2^{k+1}Q|} \right)^{1+\beta/n} M_{\beta,r}(f)(\tilde{x})
\end{aligned}$$

$$\begin{aligned}
& +C||b_1||_{Lip_\beta(w)} \sum_{k=1}^{\infty} k2^{-k\varepsilon} w(\tilde{x}) \left(\frac{w(2^k Q)}{|2^{k+1} Q|} \right)^{\beta/n} M_{\beta,r}(f)(\tilde{x}) \\
& \leq C||b_1||_{Lip_\beta(w)} w(\tilde{x})^{1+\beta/n} M_{\beta,r}(f)(\tilde{x}),
\end{aligned}$$

thus, we can obtain

$$\frac{1}{|Q|} \int_Q C(x) dx \leq C||b_1||_{Lip_\beta(w)} w(\tilde{x})^{1+\beta/n} M_{\beta,r}(f)(\tilde{x}).$$

Now, we consider the **Case** $m \geq 2$. For $b = (b_1, \dots, b_m)$, we have,

$$\begin{aligned}
F_t^{\vec{b}}(f)(x) &= \int_{R^n} \left[\prod_{j=1}^m (b_j(x) - b_j(y)) \right] F_t(x, y) f(y) dy \\
&= \int_{R^n} \prod_{j=1}^m [(b_j(x) - (b_j)_{2Q}) - (b_j(y) - (b_j)_{2Q})] F_t(x, y) f(y) dy \\
&= \sum_{j=0}^m \sum_{\sigma \in C_j^m} (-1)^{m-j} (b(x) - (b)_{2Q})_\sigma \int_{R^n} (b(y) - (b)_{2Q})_\sigma F_t(x, y) f(y) dy \\
&= (b_1(x) - (b_1)_{2Q}) \cdots (b_m(x) - (b_m)_{2Q}) F_t(f)(x) \\
&\quad + (-1)^m F_t((b_1 - (b_1)_{2Q}) \cdots (b_m - (b_m)_{2Q}) f)(x) \\
&\quad + \sum_{j=1}^{m-1} \sum_{\sigma \in C_j^m} (-1)^{m-j} (b(x) - (b)_{2Q})_\sigma \int_{R^n} (b(y) - b(x))_{\sigma^c} F_t(x, y) f(y) dy \\
&= (b_1(x) - (b_1)_{2Q}) \cdots (b_m(x) - (b_m)_{2Q}) F_t(f)(x) \\
&\quad + (-1)^m F_t((b_1 - (b_1)_{2Q}) \cdots (b_m - (b_m)_{2Q}) f)(x) \\
&\quad + \sum_{j=1}^{m-1} \sum_{\sigma \in C_j^m} (-1)^{m-j} (b(x) - (b)_{2Q})_\sigma F_t^{\vec{b}_{\sigma^c}}(f)(x),
\end{aligned}$$

thus

$$\begin{aligned}
& |T_{\vec{b}}(f)(x) - T(((b_1)_{2Q} - b_1) \cdots ((b_m)_{2Q} - b_m)) f_2)(x_0)| \\
&= \left| \|F_t^{\vec{b}}(f)(x)\| - \|F_t(((b_1)_{2Q} - b_1) \cdots ((b_m)_{2Q} - b_m) f_2)(x_0)\| \right|
\end{aligned}$$

$$\begin{aligned}
&\leq \|F_t^{\vec{b}}(f)(x) - F_t(((b_1)_{2Q} - b_1) \cdots ((b_m)_{2Q} - b_m)f_2)(x_0)\| \\
&\leq \|(b_1(x) - (b_1)_{2Q}) \cdots (b_m(x) - (b_m)_{2Q})F_t(f)(x)\| \\
&+ \sum_{j=1}^{m-1} \sum_{\sigma \in C_j^m} \|(\vec{b}(x) - (b_m)_{2Q})_{\sigma} F_t^{\vec{b}_{\sigma^c}}(f)(x)\| \\
&+ \|F_t((b_1 - (b_1)_{2Q}) \cdots (b_m - (b_m)_{2Q})f_1)(x)\| \\
&+ \|F_t(\prod_{j=1}^m ((b_j - (b_j)_{2Q})f_2)(x) - F_t(\prod_{j=1}^m (b_j - (b_j)_{2Q})f_2)(x_0))\| \\
&= A_1(x) + B_1(x) + C_1(x) + D_1(x).
\end{aligned}$$

For $A_1(x)$, by Hölder's inequality and lemma 2, we get,

$$\begin{aligned}
&\frac{1}{|Q|} \int_Q A_1(x) d\mu(x) \\
&\leq \frac{1}{|Q|} \int_Q \left| \prod_{j=1}^m (b_j(x) - (b_j)_{2Q}) \right| |T(f)(x)| dx \\
&\leq \frac{1}{|2Q|} \prod_{j=1}^m \left(\int_{2Q} |b_j(x) - (b_j)_{2Q}|^{r_j} dx \right)^{\frac{1}{r_j}} \left(\int_Q |T(f)(x)|^r dx \right)^{\frac{1}{r}} \\
&\leq \frac{1}{|2Q|} \prod_{j=1}^m \left(\sup_{x \in 2Q} |b_j(x) - (b_j)_{2Q}| |2Q|^{\frac{1}{r_j}} \right) \left(\int_Q |T(f)(x)|^r dx \right)^{\frac{1}{r}} \\
&\leq C \frac{1}{|Q|} \prod_{j=1}^m (\|b_j\|_{Lip_{\beta}(w)} w(2Q)^{1+\beta/n} |2Q|^{-1}) |Q|^{(1-\frac{1}{r})+(\frac{1}{r}-m\beta/n)} \\
&\quad \cdot \left(\frac{1}{|Q|^{1-rm\beta/n}} \int_Q |T(f)(x)|^r dx \right)^{\frac{1}{r}} \\
&\leq C \|\vec{b}\|_{Lip_{\beta}(w)} w(2Q)^{m+m\beta/n} |2Q|^{-m-m\beta/n} M_{m\beta,r}(T(f))(\tilde{x}) \\
&\leq C \|\vec{b}\|_{Lip_{\beta}(w)} \left(\frac{w(2Q)}{|2Q|} \right)^{m+m\beta/n} M_{m\beta,r}(T(f))(\tilde{x}) \\
&\leq C \|\vec{b}\|_{Lip_{\beta}(w)} w(\tilde{x})^{m+m\beta/n} M_{m\beta,r}(T(f))(\tilde{x}).
\end{aligned}$$

For $B_1(x)$, similar to $B(x)$, using the boundness of T and Lemma 2, we get

$$\begin{aligned}
& \frac{1}{|Q|} \int_Q B_1(x) dx \\
& \leq \frac{1}{|Q|} \left(\int_{R^n} |T(\prod_{j=1}^m (b_j - (b_j)_{2Q}) f_1)(x)|^r dx \right)^{\frac{1}{r}} |Q|^{\frac{1}{r'}} \\
& \leq \frac{C}{|Q|} \left(\int_{R^n} |\prod_{j=1}^m (b_j(x) - (b_j)_{2Q}) f_1(x)|^r dx \right)^{\frac{1}{r}} |Q|^{\frac{1}{r'}} \\
& \leq \frac{C}{|Q|} \prod_{j=1}^m \sup_{x \in 2Q} |b_j(x) - (b_j)_{2Q}| \left(\int_{2Q} |f(x)|^r dx \right)^{\frac{1}{r}} |Q|^{\frac{1}{r'}} \\
& \leq \frac{C}{|Q|} \left(\prod_{j=1}^m \|b_j\|_{Lip_\beta(w)} w(2Q)^{1+\beta/n} |2Q|^{-1} \right) |2Q|^{\frac{1}{r'} + \frac{1}{r} - m\beta/n} \cdot \\
& \quad \cdot \left(\frac{1}{|2Q|^{1-rm\beta/n}} \int_{2Q} |f(x)|^r dx \right)^{\frac{1}{r}} \\
& \leq C \|\vec{b}\|_{Lip_\beta(w)} \left(\frac{w(2Q)}{|2Q|} \right)^{m+m\beta/n} M_{m\beta,r}(f)(\tilde{x}) \\
& \leq C \|\vec{b}\|_{Lip_\beta(w)} w(\tilde{x})^{m+m\beta/n} M_{m\beta,r}(f)(\tilde{x}).
\end{aligned}$$

For $C_1(x)$, by Hölder's inequality and lemma 2, we get

$$\begin{aligned}
& \frac{1}{|Q|} \int_Q C_1(x) dx \\
& \leq C \sum_{j=1}^{m-1} \sum_{\sigma \in C_j^m} \frac{1}{|2Q|} \int_{2Q} |(b_j(x) - (b_j)_{2Q})_\sigma| |T(((b_j - (b_j)_{2Q})_{\sigma^c}) f)(x)| dx \\
& \leq C \sum_{j=1}^{m-1} \sum_{\sigma \in C_j^m} \frac{1}{|2Q|} \left(\int_{2Q} |(b_j(x) - (b_j)_{2Q})_\sigma|^{r'} dx \right)^{\frac{1}{r'}} \cdot \\
& \quad \cdot \left(\int_{2Q} |T(((b_j - (b_j)_{2Q})_{\sigma^c}) f)(x)|^r dx \right)^{\frac{1}{r}} \\
& \leq C \sum_{j=1}^{m-1} \sum_{\sigma \in C_j^m} \frac{1}{|2Q|} \sup_{x \in 2Q} |(b_j(x) - (b_j)_{2Q})_\sigma| |2Q|^{\frac{1}{r'}} \sup_{x \in 2Q} |(b_j(x) - (b_j)_{2Q})_{\sigma^c}| \cdot \\
& \quad \cdot \left(\int_{2Q} |T(f)(x)|^r dx \right)^{\frac{1}{r}}
\end{aligned}$$

$$\begin{aligned}
&\leq C \sum_{j=1}^{m-1} \sum_{\sigma \in C_j^m} \frac{1}{|2Q|} \|\vec{b}_\sigma\|_{Lip_\beta(w)} w(2Q)^{j+\beta/n} |2Q|^{-j} \|\vec{b}_{\sigma^c}\|_{Lip_\beta(w)} \cdot \\
&\quad \cdot w(2Q)^{(m-j)+(m-j)\beta/n} |2Q|^{\frac{1}{r'} + \frac{1}{r} - m\beta/n - (m-j)} \cdot \\
&\quad \cdot \left(\frac{1}{|2Q|^{1-rm\beta/n}} \int_{2Q} |T(f)(x)|^r dx \right)^{\frac{1}{r}} \\
&\leq C \|\vec{b}\|_{Lip_\beta(w)} \left(\frac{w(2Q)}{|2Q|} \right)^{m+m\beta/n} M_{m\beta,r}(T(f))(\tilde{x}) \\
&\leq C \|\vec{b}\|_{Lip_\beta(w)} w(\tilde{x})^{m+m\beta/n} M_{m\beta,r}(T(f))(\tilde{x}).
\end{aligned}$$

For $D_1(x)$, similar to the methods above, we have

$$\begin{aligned}
D_1(x) &= \|F_t((b_1 - (b_1)_{2Q}) \cdots (b_m - (b_m)_{2Q}) f_2)(x) - F_t((b_1 - (b_1)_{2Q}) \cdots \\
&\quad (b_m - (b_m)_{2Q}) f_2)(x_0)\| \\
&\leq \int_{R^n} \left| \prod_{j=1}^m (b_j(y) - (b_j)_{2Q}) \right| |f(y) \chi_{(2Q)^c}(y)| \|F_t(x, y) - F_t(x_0, y)\| dy \\
&\leq C \int_{(2Q)^c} \left| \prod_{j=1}^m (b_j(y) - (b_j)_{2Q}) \right| |f(y)| \frac{|x - x_0|^\varepsilon}{|x_0 - y|^{n+\varepsilon}} dy \\
&\leq C \sum_{k=1}^{\infty} \int_{2^{k+1}Q \setminus 2^kQ} |x - x_0|^\varepsilon |x_0 - y|^{-(n+\varepsilon)} \left| \prod_{j=1}^m (b_j(y) - (b_j)_{2Q}) \right| |f(y)| dy \\
&\leq C \sum_{k=1}^{\infty} 2^{-k\varepsilon} \frac{1}{|2^kQ|} \\
&\quad \int_{2^{k+1}Q} \left| \prod_{j=1}^m ((b_j(y) - (b_j)_{2^{k+1}Q}) + ((b_j)_{2^{k+1}Q} - (b_j)_{2Q})) \right| |f(y)| dy \\
&\leq C \sum_{k=1}^{\infty} 2^{-k\varepsilon} \frac{1}{|2^kQ|} \prod_{j=1}^m \left[\int_{2^{k+1}Q} |(b_j(y) - (b_j)_{2^{k+1}Q})| |f(y)| dy \right. \\
&\quad \left. + |(b_j)_{2^{k+1}Q} - (b_j)_{2Q}| \int_{2^{k+1}Q} |f(y)| dy \right]
\end{aligned}$$

$$\begin{aligned}
&\leq C \sum_{k=1}^{\infty} 2^{-k\varepsilon} \frac{1}{|2^k Q|} \prod_{j=1}^m \left[\left(\int_{2^{k+1}Q} |(b_j(y) - (b_j)_{2^{k+1}Q})_{\sigma^c}|^{r'} dy \right)^{\frac{1}{r'}} \left(\int_{2^{k+1}Q} |f(y)|^r dy \right)^{\frac{1}{r}} \right. \\
&\quad \left. + |((b_j)_{2^{k+1}Q} - (b_j)_{2Q})| |2^{k+1}Q|^{\frac{1}{r'} + (\frac{1}{r} - m\beta/n)} \left(\frac{1}{|2^{k+1}Q|^{1-rm\beta/n}} \int_{2^{k+1}Q} |f(y)|^r dy \right)^{\frac{1}{r}} \right] \\
&\leq C \|\vec{b}\|_{Lip_{\beta}(w)} w(\tilde{x})^j \left(\frac{w(2^{k+1}Q)}{|2^{k+1}Q|} \right)^{(m-j)+m\beta/n} M_{m\beta,r}(f)(\tilde{x}) \\
&\leq C \|\vec{b}\|_{Lip_{\beta}(w)} w(\tilde{x})^{m+m\beta/n} M_{m\beta,r}(f)(\tilde{x}).
\end{aligned}$$

Thus, we get

$$\frac{1}{|Q|} \int_Q D_1(x) dx \leq C \|\vec{b}\|_{Lip_{\beta}(w)} w(\tilde{x})^{m+m\beta/n} M_{m\beta,r}(f)(\tilde{x}).$$

Combining all the estimates above, we get

$$(T_{\vec{b}}(f))^{\#}(\tilde{x}) \leq C \|\vec{b}\|_{Lip_{\beta}(w)} w(\tilde{x})^{m+m\beta/n} (M_{m\beta,r}(f)(\tilde{x}) + M_{m\beta,r}(T(f))(\tilde{x})).$$

Now, choose $1 < r < p$ and by lemma 5, we have

$$\begin{aligned}
&\|T_{\vec{b}}(f)\|_{L^q(w^{1-m+(q-1)m\beta/n})} \\
&\leq C \|M(T_{\vec{b}}(f))\|_{L^q(w^{1-m+(q-1)m\beta/n})} \\
&\leq C \|(T_{\vec{b}}(f))^{\#}\|_{L^q(w^{1-m+(q-1)m\beta/n})} \\
&\leq C \|\vec{b}\|_{Lip_{\beta}(w)} (\|w^{m+m\beta/n} M_{m\beta,r}(f)\|_{L^q(w^{1-m+(q-1)m\beta/n})} \\
&\quad + \|w^{m+m\beta/n} M_{m\beta,r}(T(f))\|_{L^q(w^{1-m+(q-1)m\beta/n})}) \\
&\leq C \|\vec{b}\|_{Lip_{\beta}(w)} (\|M_{m\beta,r}(f)\|_{L^q(w^{\frac{q}{p}})} + \|M_{m\beta,r}(T(f))\|_{L^q(w^{\frac{q}{p}})}) \\
&\leq C \|\vec{b}\|_{Lip_{\beta}(w)} (\|f\|_{L^p(w)} + \|T(f)\|_{L^p(w)}) \\
&\leq C \|\vec{b}\|_{Lip_{\beta}(w)} \|f\|_{L^p(w)}.
\end{aligned}$$

This completes the proof of Theorem 1.

Proof of Theorem 2. Similar to Theorem 1, for any $1 < r < \infty$ and cube Q , there exists some constant C_0 such that for any $\tilde{x} \in Q$,

$$\frac{1}{|Q|^{1+m\beta/n}} \int_Q |T_{\vec{b}}(f)(x) - C_0| dx \leq C \|\vec{b}\|_{Lip_{\beta}(w)} w(\tilde{x})^{m+m\beta/n} (M_r(f)(\tilde{x}) + M_r(T(f))(\tilde{x})).$$

Further, we have

$$\sup_{\tilde{x} \in Q} \inf_{c \in C} \frac{1}{|Q|^{1+m\beta/n}} \int_Q |T_{\vec{b}}(x) - c| dx \leq C \|\vec{b}\|_{Lip_{\beta}(w)} w(\tilde{x})^{m+m\beta/n} (M_r(f)(\tilde{x}) + M_r(T(f))(\tilde{x})).$$

Choose $1 < r < p$ and by lemma 4, we obtain

$$\begin{aligned}
& \|T_{\vec{b}}(f)\|_{\dot{F}_p^{m\beta, \infty}(w^{1-m-m\beta/n})} \approx \left\| \sup_{\vec{x} \in Q} \inf_{c \in C} |Q|^{-1-m\beta/n} \int_Q |T_{\vec{b}}(f)(x) - c| dx \right\|_{L^p(w^{1-m-m\beta/n})} \\
& \leq C \|\vec{b}\|_{Lip_\beta(w)} (\|w^{m+m\beta/n} M_r(f)\|_{L^p(w^{1-m-m\beta/n})} + \|w^{m+m\beta/n} M_r(T(f))\|_{L^p(w^{1-m-m\beta/n})}) \\
& \leq C \|\vec{b}\|_{Lip_\beta(w)} (\|M_r(f)\|_{L^p(w)} + \|M_r(T(f))\|_{L^p(w)}) \\
& \leq C \|\vec{b}\|_{Lip_\beta(w)} (\|f\|_{L^p(w)} + \|T(f)\|_{L^p(w)}) \\
& \leq C \|\vec{b}\|_{Lip_\beta(w)} \|f\|_{L^p(w)}.
\end{aligned}$$

This completes the proof of Theorem 2.

4 Applications

Now we give some applications of the theorems in this paper.

Application 1. Littlewood-Paley operator.

Fixed $\varepsilon > 0$. Let ψ be a fixed function which satisfies the following properties:

- (1) $\int_{R^n} \psi(x) dx = 0$,
- (2) $|\psi(x)| \leq C(1 + |x|)^{-(n+1)}$,
- (3) $|\psi(x+y) - \psi(x)| \leq C|y|^\varepsilon(1 + |x|)^{-(n+1+\varepsilon)}$ when $2|y| < |x|$.

The Littlewood-Paley multilinear commutator is defined by

$$g_{\vec{\psi}}^{\vec{b}}(f)(x) = \left(\int_0^\infty |F_t^{\vec{b}}(f)(x)|^2 \frac{dt}{t} \right)^{1/2},$$

where

$$F_t^{\vec{b}}(f)(x) = \int_{R^n} \prod_{j=1}^m (b_j(x) - b_j(y)) \psi_t(x - y) f(y) dy$$

and $\psi_t(x) = t^{-n} \psi(x/t)$ for $t > 0$. Set $F_t(f)(y) = f * \psi_t(y)$. We also define that

$$g_\psi(f)(x) = \left(\int_0^\infty |F_t(f)(x)|^2 \frac{dt}{t} \right)^{1/2},$$

which is the Littlewood-Paley operator(see [23]). Let H be the space

$$H = \left\{ h : \|h\| = \left(\int_0^\infty |h(t)|^2 dt/t \right)^{1/2} < \infty \right\},$$

then, for each fixed $x \in R^n$, $F_t^{\vec{b}}(f)(x)$ may be viewed as the mappings from $[0, +\infty)$ to H , and it is clear that

$$g_{\psi}^{\vec{b}}(f)(x) = \|F_t^{\vec{b}}(f)(x)\|, \quad g_{\psi}(f)(x) = \|F_t(f)(x)\|.$$

It is easily to see that g_{ψ} satisfies the conditions of Theorems 1 and 2 (see [11-13]), thus Theorems 1 and 2 hold for $g_{\psi}^{\vec{b}}$.

Application 2. Marcinkiewicz operator.

Fixed $0 < \gamma \leq 1$. Let Ω be homogeneous of degree zero on R^n with $\int_{S^{n-1}} \Omega(x') d\sigma(x') = 0$. Assume that $\Omega \in Lip_{\gamma}(S^{n-1})$. The Marcinkiewicz multilinear commutator is defined by

$$\mu_{\Omega}^{\vec{b}}(f)(x) = \left(\int_0^{\infty} |F_t^{\vec{b}}(f)(x)|^2 \frac{dt}{t^3} \right)^{1/2},$$

where

$$F_t^{\vec{b}}(f)(x) = \int_{|x-y| \leq t} \prod_{j=1}^m (b_j(x) - b_j(y)) \frac{\Omega(x-y)}{|x-y|^{n-1}} f(y) dy.$$

Set

$$F_t(f)(x) = \int_{|x-y| \leq t} \frac{\Omega(x-y)}{|x-y|^{n-1}} f(y) dy;$$

We also define that

$$\mu_{\Omega}(f)(x) = \left(\int_0^{\infty} |F_t(f)(x)|^2 \frac{dt}{t^3} \right)^{1/2},$$

which is the Marcinkiewicz operator(see [24]). Let H be the space

$$H = \left\{ h : \|h\| = \left(\int_0^{\infty} |h(t)|^2 dt/t^3 \right)^{1/2} < \infty \right\}.$$

Then, it is clear that

$$\mu_{\Omega}^{\vec{b}}(f)(x) = \|F_t^{\vec{b}}(f)(x)\|, \quad \mu_{\Omega}(f)(x) = \|F_t(f)(x)\|.$$

It is easily to see that μ_{Ω} satisfies the conditions of Theorems 1 and 2 (see [14][24]), thus Theorems 1 and 2 hold for $\mu_{\Omega}^{\vec{b}}$.

Application 3. Bochner-Riesz operator.

Let $\delta > (n-1)/2$, $B_t^{\delta}(f)(\xi) = (1 - t^2|\xi|^2)_{+}^{\delta} \hat{f}(\xi)$ and $B_t^{\delta}(z) = t^{-n} B^{\delta}(z/t)$ for $t > 0$. Set

$$F_{\delta,t}^{\vec{b}}(f)(x) = \int_{R^n} \prod_{j=1}^m (b_j(x) - b_j(y)) B_t^{\delta}(x-y) f(y) dy.$$

The maximal Bochner-Riesz multilinear commutator is defined by

$$B_{\delta,*}^{\vec{b}}(f)(x) = \sup_{t>0} |B_{\delta,t}^{\vec{b}}(f)(x)|.$$

We also define that

$$B_{\delta,*}(f)(x) = \sup_{t>0} |B_t^{\delta}(f)(x)|$$

which is the maximal Bochner-Riesz operator (see [17]). Let H be the space $H = \{h : \|h\| = \sup_{t>0} |h(t)| < \infty\}$, then

$$B_{\delta,*}^{\vec{b}}(f)(x) = \|B_{\delta,t}^A(f)(x)\|, \quad B_*^{\delta}(f)(x) = \|B_t^{\delta}(f)(x)\|.$$

It is easily to see that $B_{\delta,*}^{\vec{b}}$ satisfies the conditions of Theorems 1 and 2 (see [16][25]), thus Theorems 1 and 2 hold for $B_{\delta,*}^{\vec{b}}$.

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