Hyper MV -Algebras Defined by Bipolar-valued Fuzzy Sets

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Abstract. In this paper the notion of hyper MV -algebra is extended via bipolar-valued fuzzy set and some theorems about this concept are stated and proved. The concepts of strong positive t -cut and strong negative s -cut of bipolar-valued fuzzy set are defined and some of their properties are studied.

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1 Introduction

The concept of an MV -algebra was introduced by C. C. Chang in 1958 to prove the completeness theorem of infinite valued Łukasiewicz propositional calculus.

The hyper structure theory was introduced by F. Marty at 8th congress of Scandinavian Mathematicians in 1934. Since then many researches have worked in these areas. Recently in [3], Sh. Ghorbani et.al, applied the hyper structure to MV -algebras and introduced the concept of a hyper MV -algebra which is a generalization of an MV -algebra and investigated some related results.

In 1965, Zadeh [7] introduced the notion of a fuzzy subset of a set. Since then it has become a vigorous area of research in different domains. There have been a number of generalizations of this fundamental concept such as intuitionistic fuzzy sets, interval-valued fuzzy sets, vague sets, soft sets.

Lee [5] introduced the notion of bipolar-valued fuzzy sets. Bipolar-valued fuzzy sets are an extension of fuzzy sets whose membership degree range is enlarged from the interval $[0, 1]$ to $[-1, 1]$. In a bipolar-valued fuzzy set, the membership degree 0 means that elements are irrelevant to the corresponding property, the membership degree $(0, 1]$ indicates that elements somewhat satisfy the property, and the membership degree $[-1, 0)$ indicates that elements somewhat satisfy the implicit counter-property. Bipolar-valued fuzzy sets and intuitionistic fuzzy sets look similar each other. However, they are different each other (see [6]).

In this note, we introduce the notion of Bipolar-valued fuzzy sub-hyper MV -algebra of a hyper- MV -algebra and study their properties in detail.

2 Preliminaries

Definition 2.1. [3] *Let M be a nonempty set endowed with a hyperoperation “ $\oplus$ ”, an unary operation “ $*$ ” and a constant “ 0 ” satisfying the following axioms, for all $x, y, z \in M$:*

- a1) $x \oplus (y \oplus z) = (x \oplus y) \oplus z$,
- a2) $x \oplus y = y \oplus x$,
- a3) $(x^*)^* = x$,
- a4) $(x^* \oplus y^*) \oplus y = (y^* \oplus x)^* \oplus x$,
- a5) $0^* \in x \oplus 0^*$,
- a6) $0^* \in x \oplus x^*$,
- a7) $x \ll y$ and $y \ll x \Rightarrow x = y$; where $x \ll y$ means $0^* \in x^* \oplus y$.

For every subsets A and B of M we define:

$$A \ll B \Leftrightarrow (\exists a \in A), (\exists b \in B) \text{ s.t. } a \ll b.$$

$$A \oplus B = \bigcup_{a \in A, b \in B} a \oplus b$$

We also define $0^* = 1$ and $A^* = \{a^* \mid a \in A\}$.

Proposition 2.2. [3] *Every hyper- MV -algebra M satisfies the following conditions; where $x, y, z \in M$ and $A, B, C \subseteq M$:*

- b1) $A \oplus (B \oplus C) = (A \oplus B) \oplus C$,
- b2) $0 \ll x, x \ll 1$,
- b3) $x \ll x$,
- b4) $x \ll y \Rightarrow y^* \ll x^*$,
- b5) $A \ll B \Rightarrow B^* \ll A^*$,
- b6) $A \ll A$,
- b7) $A \subseteq B \Rightarrow A \ll B$,
- b8) $x \ll x \oplus y \ \& \ A \ll A \oplus B$,
- b9) $(A^*)^* = A$,
- b10) $0 \oplus 0 = \{0\}$,
- b11) $x \in x \oplus 0$,
- b12) $y \in x \oplus 0 \Rightarrow y \ll x$,
- b13) $y \oplus 0 = x \oplus 0 \Rightarrow x = y$,

Definition 2.3. [3] Let $(M, \oplus, *)$ be a hyper-MV-algebra and $S \subseteq M$ be a nonempty set containing “0”. if S is a hyper-MV-algebra with respect to “ $\oplus$ ” and “ $*$ ” then we say that S is a sub-hyper MV-algebra of M .

Lemma 2.4. [3] Let $(M, \oplus, *)$ be a hyper-MV-algebra and $S \subseteq M$ be a nonempty set containing “0”. Then S is a sub-hyper MV-algebra of M iff for all $x, y, z \in S$:

- 1) $x \oplus y \in S$
- 2) $x^* \in S$

Lemma 2.5. A fuzzy subset μ of M is called a fuzzy sub-hyper-MV-algebra of M if it satisfies:

- 1) $(\forall x \in M), \mu(x) \leq \mu(x^*)$,
- 2) $(\forall x, y \in M), \inf_{v \in x \oplus y} \mu(v) \geq \min\{\mu(x), \mu(y)\}$.

Definition 2.6. [5, 6] Let G be a nonempty set. A Bipolar-valued fuzzy set in G is an object having the form

$$B = \{(x, \mu^+(x), \mu^-(x)); x \in G\}$$

where $\mu^+ : G \rightarrow [0, 1]$ and $\mu^- : G \rightarrow [-1, 0]$ are mapping.

Note: In this paper we use the symbol $B = (\mu^+, \mu^-)$ for the Bipolar-valued fuzzy set $B = \{(x, \mu^+(x), \mu^-(x)); x \in G\}$

3 Bipolar-valued fuzzy sub-hyper MV-algebras

In what follows, let M denote a hyper-MV-algebra unless otherwise specified.

Definition 3.1. A Bipolar-valued fuzzy set $B = (\mu^+, \mu^-)$ is called a "Bipolar-valued fuzzy sub-hyper MV-algebra" (BFSH) of M , if for every $x, y \in M$ it satisfies:

- i) $\mu^+(x) \leq \mu^+(x^*)$,
- ii) $\mu^-(x) \geq \mu^-(x^*)$,
- iii) $\inf_{v \in x \oplus y} \mu^+(x) \geq \min\{\mu^+(x), \mu^+(y)\}$,
- iv) $\sup_{v \in x \oplus y} \mu^-(x) \leq \max\{\mu^-(x), \mu^-(y)\}$.

Remark 3.2. Note that by Definition 2.1(a3) we have:

$$\mu^+((x^*)^*) \geq \mu^+(x^*) \Rightarrow \mu^+(x) \geq \mu^+(x^*)$$

$$\mu^-(x^*) \leq \mu^-((x^*)^*) \Rightarrow \mu^-(x^*) \geq \mu^-(x)$$

Hence conditions (i), (ii) in Definition 3.1 can be written as:

- i) $\mu^+(x) = \mu^+(x^*)$,
- ii) $\mu^-(x) = \mu^-(x^*)$.

Example 3.3. Let $M = \{0, a, 1\}$ and define " $\oplus$ ", " $*$ " by the following tables:

$\oplus$	0	a	1
0	$\{0\}$	$\{0, a\}$	$\{0, 1\}$
a	$\{0, a\}$	$\{0, a, 1\}$	$\{0, a, 1\}$
1	$\{0, 1\}$	$\{0, a, 1\}$	$\{0, 1\}$

x	0	a	1
x^*	1	a	0

Now define μ^+ and μ^- as follow:

x	0	a	1
$\mu^+(x)$	0.7	0.2	0.7

x	0	a	1
$\mu^-(x)$	-1	-0.3	-1

Then $(M, \oplus, *)$ is a hyper-MV-algebra and $B = (\mu^+, \mu^-)$ is Bipolar-valued fuzzy sub-hyper MV-algebra of M .

Proposition 3.4. Let μ^+, μ^- be fuzzy sub-hyper MV-algebra of M . Then $B = (\mu^+, (-\mu^-))$ is a (BFSH) of M .

Conversely, if $B = (\mu^+, \mu^-)$ is a (BFSH) of M , then $\mu^+, (-\mu^-)$ are fuzzy sub-hyper MV-algebra of M .

Proof. Let $x, y \in M$. We know that:

$$\min\{-\mu(x), -\mu(y)\} = -\max\{\mu(x), \mu(y)\}$$

$$\max\{-\mu(x), -\mu(y)\} = -\min\{\mu(x), \mu(y)\}$$

$$\inf_{v \in x \oplus y} (-\mu(v)) = - \sup_{v \in x \oplus y} \mu(v)$$

$$\sup_{v \in x \oplus y} (-\mu(v)) = - \inf_{v \in x \oplus y} \mu(v)$$

Hence by definition of $(BF\mathcal{SH})$ and fuzzy sub-hyper MV-algebra the proof is clear.

The following lemma immediately follows from definition of $(BF\mathcal{SH})$:

Lemma 3.5. *If $B = (\mu^+, \mu^-)$ is a $(BF\mathcal{SH})$ of M , then for all $a \in x \oplus y$ we have:*

$$\mu^+(a) \geq \min\{\mu^+(x), \mu^+(y)\}$$

and

$$\mu^-(a) \leq \max\{\mu^-(x), \mu^-(y)\}$$

Lemma 3.6. *Let $B = (\mu^+, \mu^-)$ be a $(BF\mathcal{SH})$ of M . Then for all $x \in M$:*

$$\mu^+(0) \geq \mu^+(x) \quad \text{and} \quad \mu^-(0) \leq \mu^-(x)$$

Proof. By Proposition 2.2(b3) we have $x \ll x$. Hence $0^* \in x^* \oplus x$. By Lemma 2.4 this implies that:

$$\mu^+(0) = \mu^+(0^*) \geq \min\{\mu^+(x^*), \mu^+(x)\} = \min\{\mu^+(x), \mu^+(x)\} = \mu^+(x)$$

$$\mu^-(0) = \mu^-(0^*) \leq \max\{\mu^-(x^*), \mu^-(x)\} = \max\{\mu^-(x), \mu^-(x)\} = \mu^-(x)$$

Lemma 3.7. *Let B_1 and B_2 are $(BF\mathcal{SH})$ of M . Then $B_1 \cap B_2$ is a $(BF\mathcal{SH})$ of M .*

Proof. If $x, y \in B_1 \cap B_2$, then $x, y \in B_1$ and $x, y \in B_2$. Since B_1, B_2 are $(BF\mathcal{SH})$, hence:

$$i) \mu_{B_1 \cap B_2}^+(x) = \min\{\mu_{B_1}^+(x), \mu_{B_2}^+(x)\} = \min\{\mu_{B_1}^+(x^*), \mu_{B_2}^+(x^*)\} = \mu_{B_1 \cap B_2}^+(x^*)$$

$$ii) \mu_{B_1 \cap B_2}^-(x) = \min\{\mu_{B_1}^-(x), \mu_{B_2}^-(x)\} = \min\{\mu_{B_1}^-(x^*), \mu_{B_2}^-(x^*)\} = \mu_{B_1 \cap B_2}^-(x^*)$$

$$\begin{aligned} iii) \inf_{v \in x \oplus y} \mu_{B_1 \cap B_2}^+(v) &= \min\left\{\inf_{v \in x \oplus y} \mu_{B_1}^+(v), \inf_{v \in x \oplus y} \mu_{B_2}^+(v)\right\} \\ &\geq \min\{\min\{\mu_{B_1}^+(x), \mu_{B_1}^+(y)\}, \min\{\mu_{B_2}^+(x), \mu_{B_2}^+(y)\}\} \end{aligned}$$

$$\begin{aligned}
&= \min\{\mu_{B_1 \cap B_2}^+(x), \mu_{B_1 \cap B_2}^+(y)\} \\
iv) \quad \sup_{v \in x \oplus y} \mu_{B_1 \cap B_2}^-(x) &= \max\left\{\sup_{v \in x \oplus y} \mu_{B_1}^-(v), \sup_{v \in x \oplus y} \mu_{B_2}^-(v)\right\} \\
&\leq \max\{\max\{\mu_{B_1}^-(x), \mu_{B_1}^-(y)\}, \max\{\mu_{B_2}^-(x), \mu_{B_2}^-(y)\}\} \\
&= \max\{\mu_{B_1 \cap B_2}^-(x), \mu_{B_1 \cap B_2}^-(y)\}
\end{aligned}$$

Lemma 3.8. Let $B_\alpha; \alpha \in I$ be a family of (BFSH) of M . Then $\bigcap_{\alpha \in I} B_\alpha$ is a (BFSH) of M .

4 Strong positive t-cut and strong negative s-cut

Definition 4.1. Let $B = (\mu^+, \mu^-)$ be a Bipolar-valued fuzzy set of M and $(s, t) \in [-1, 0] \times [0, 1]$. Then:

- i) The set $B_t^+ = \{x \in M; \mu^+(x) \geq t\}$ is called positive t -cut of B .
- ii) The set $B_s^- = \{x \in M; \mu^-(x) \leq s\}$ is called negative s -cut of B .
- iii) The set ${}^s B_t^+ = \{x \in M; \mu^+(x) > t\}$ is called strong positive s -cut of B .
- iv) The set ${}^s B_s^- = \{x \in M; \mu^-(x) < s\}$ is called strong negative s -cut of B .
- v) The set $M_B^{(t,s)} = \{x \in M; \mu^+(x) \geq t, \mu^-(x) \leq s\}$ is called (t, s) -cut of B .
- vi) The set ${}^s M_B^{(t,s)} = \{x \in M; \mu^+(x) > t, \mu^-(x) < s\}$ is called strong (t, s) -cut of B .

Note that $M_B^{(t,s)} = B_t^+ \cap B_s^-$.

Proposition 4.2. Let $B = (\mu^+, \mu^-)$ be a (BFSH) of M and $x \in M$. Then $\mu^+(x) = t$ and $\mu^-(x) = s$ iff $x \in M_B^{(t,s)}$ and $x \notin M_B^{(u,v)}$, for all $u > t, v < s$.

Proposition 4.3. Let $B = (\mu^+, \mu^-)$ be a (BFSH) of M . Then the set $M_B = \{x \in M; \mu^+(x) = \mu^+(0), \mu^-(x) = \mu^-(0)\}$ is a sub-hyper MV-algebra of M .

Proof. Let $x, y \in M_B$. Then $\mu^+(x) = \mu^+(0) = \mu^+(y)$ and $\mu^-(x) = \mu^-(0) = \mu^-(y)$. Thus:

- i) $\mu^+(x^*) = \mu^+(x) = \mu^+(0)$
- ii) $\mu^-(x^*) = \mu^-(x) = \mu^-(0)$ Hence $x^* \in M_B$.

Now let $a \in x \oplus y$. Then

- iii) $\inf_{v \in x \oplus y} \mu^+(v) \geq \min\{\mu^+(x), \mu^+(y)\} = \min\{\mu^+(0), \mu^+(0)\} = \mu^+(0)$

Hence $\mu^+(a) \geq \mu^+(0)$. (I)

$$\text{iv) } \sup_{v \in x \oplus y} \mu^-(v) \leq \max\{\mu^-(x), \mu^-(y)\} = \max\{\mu^-(0), \mu^-(0)\} = \mu^-(0)$$

Hence $\mu^-(a) \leq \mu^-(0)$. (II)

Now by (I), (II) and Lemma 3.6 we conclude that $\mu^+(a) = \mu^+(0)$ and $\mu^-(a) = \mu^-(0)$. Hence $a \in M_B$ and this follows that $x \oplus y \subseteq M_B$ and the proof is complete.

Proposition 4.4. *Let S be a subset of M , and $B = (\mu^+, \mu^-)$ be a Bipolar-valued fuzzy set defined as:*

$$\mu^+(x) = \begin{cases} k & \text{if } x \in S \\ l & \text{if } x \notin S \end{cases}$$

$$\mu^-(x) = \begin{cases} m & \text{if } x \in S \\ n & \text{if } x \notin S \end{cases}$$

Where $k, l \in [0, 1]$ and $m, n \in [-1, 0]$ with $k \geq l$, $m \leq n$. Then B is a (BFSH) of M iff S is a sub-hyper MV-algebra of M .

Proof. Let S be a sub-hyper MV-algebra of M . If $x, y \in M$ are arbitrary hence:

if $x \in S$, then $x^* \in S$. Hence $\mu^+(x) = \mu^+(x^*)$, $\mu^-(x) = \mu^-(x^*)$.

if $x \notin S$, then $x^* \notin S$. (if $x^* \in S$ then $x = (x^*)^* \in S$ a contradiction).

Hence $\mu^+(x) = \mu^+(x^*)$, $\mu^-(x) = \mu^-(x^*)$. We consider the following cases:

Case 1) Let $x, y \in S$. Then $x \oplus y \subseteq S$ so:

$$\inf_{v \in x \oplus y} \mu^+(v) = k = \min\{k, k\} = \min\{\mu^+(x), \mu^+(y)\}$$

and

$$\sup_{v \in x \oplus y} \mu^-(v) = m = \max\{m, m\} = \max\{\mu^-(x), \mu^-(y)\}$$

Case 2) Let $x, y \notin S$. Then:

$$\mu^+(x) = l = \mu^+(y) \text{ \& } \mu^-(x) = n = \mu^-(y)$$

so:

$$\min\{\mu^+(x), \mu^+(y)\} = \min\{l, l\} = l \leq \inf_{v \in x \oplus y} \mu^+(v)$$

and

$$\max\{\mu^-(x), \mu^-(y)\} = \max\{n, n\} = n \geq \sup_{v \in x \oplus y} \mu^-(v)$$

Case 3) Let $x \in S$ and $y \notin S$. Then:

$$\mu^+(x) = k, \mu^+(y) = l, \mu^-(x) = m, \mu^-(y) = n$$

Now:

$$\min\{\mu^+(x), \mu^+(y)\} = \min\{k, l\} = l \leq \inf_{v \in x \oplus y} \mu^+(v)$$

and

$$\max\{\mu^-(x), \mu^-(y)\} = \max\{m, n\} = n \geq \sup_{v \in x \oplus y} \mu^-(v)$$

Hence B is a $(BF\mathcal{SH})$ of M .

Conversely, let $x, y \in S$. Since B is a $(BF\mathcal{SH})$:

$\mu^+(x) = k = \mu^+(x^*)$. Hence $x^* \in S$. Now let $a \in x \oplus y$. By hypothesis we have

$$\inf_{v \in x \oplus y} \mu^+(v) \geq \min\{\mu^+(x), \mu^+(y)\} = \min\{k, k\} = k$$

and

$$\sup_{v \in x \oplus y} \mu^-(v) \leq \max\{\mu^-(x), \mu^-(y)\} = \max\{m, m\} = m$$

Thus $\mu^+(a) = k$ and $\mu^-(a) = m$. This follows that $a \in S$, hence $x \oplus y \subseteq S$ and this proves that S is a sub-hyper MV -algebra of M .

Proposition 4.5. *If Bipolar-valued fuzzy set $B = (\mu^+, \mu^-)$ is a $(BF\mathcal{SH})$ of M , then for all $(s, t) \in [-1, 0] \times [0, 1]$ the nonempty strong positive t -cut of B and the nonempty strong negative s -cut of B are sub-hyper MV -algebras of M .*

Proof. Let $B = (\mu^+, \mu^-)$ be a $(BF\mathcal{SH})$ of M and assume that ${}^sB_t^+$ and ${}^sB_s^-$ are nonempty for all $(s, t) \in [-1, 0] \times [0, 1]$. Let $l, m \in {}^sB_t^+$ and $p, q \in {}^sB_s^-$. Then:

$$\mu^+(l), \mu^+(m) > t, \mu^-(p), \mu^-(q) < s$$

Now:

$$\inf_{v \in l \oplus m} \mu^+(v) \geq \min\{\mu^+(l), \mu^+(m)\} > t$$

which implies that $l \oplus m \subseteq {}^sB_t^+$. Now, let $x \in {}^sB_t^+$. Since B is a $(BF\mathcal{SH})$ we have $\mu^+(x^*) = \mu(x) > t$. Hence $x^* \in {}^sB_t^+$. So ${}^sB_t^+$ is a sub-hyper MV -algebra of M .

In the other hand:

$$\sup_{v \in p \oplus q} \mu^-(v) \leq \max\{\mu^-(p), \mu^-(q)\} < s$$

hence $p \oplus q \subseteq {}^s B_s^-$. If $x \in {}^s B_s^-$ since B is a $(BF\mathcal{S}H)$, then $\mu^-(x^*) = \mu^-(x) < s$. Hence $x^* \in {}^s B_s^-$ and this proved that ${}^s B_s^-$ is a sub-hyper MV-algebra of M .

Corollary 4.6. *If Bipolar-valued fuzzy set $B = (\mu^+, \mu^-)$ is a $(BF\mathcal{S}H)$ of M . Then for all $(s, t) \in [-1, 0] \times [0, 1]$ the nonempty strong positive (s, t) -cut of B is a sub-hyper MV-algebra of M .*

Proof. We have ${}^s M_B^{(t, s)} = {}^s B_t^+ \cap {}^s B_s^-$, proof follows by Lemma 3.7 and Proposition 4.5.

Let $B = (\mu^+, \mu^-)$ be a $(BF\mathcal{S}H)$ of M . Then the set of all $(t, s) \in \text{Im}(\mu^+) \times \text{Im}(\mu^-)$ is known as image of B .

Proposition 4.7. *Let $B = (\mu^+, \mu^-)$ be a $(BF\mathcal{S}H)$ of M . Then for all $(t, s) \in \text{Im}(\mu^+) \times \text{Im}(\mu^-) \setminus (t_0, s_0)$ where $t_0 = \text{l.u.b}$ of $\text{Im}(\mu^+)$ and $s_0 = \text{g.l.b}$ of $\text{Im}(\mu^-)$, the nonempty strong (t, s) -cut of B is a sub-hyper MV-algebra of M .*

Proof. By Corollary 4.6 it is clear.

5 Bipolar-valued fuzzy characteristic sub-hyper MV-algebras of hyper-MV-algebras

Definition 5.1. *Let X, Y be hyper-MV-algebras. A map $f : X \rightarrow Y$ is called a hyper homomorphism if:*

- 1) $f(0) = 0$
- 2) $f(x \oplus y) = f(x) \oplus f(y)$
- 3) $f(x^*) = (f(x))^*$

Lemma 5.2. *Let $f : X \rightarrow Y$ be a hyper homomorphism of hyper-MV-algebras. If $x \ll y$, then $f(x) \ll f(y)$.*

Proof.) Since $x \ll y$ therefore $0^* \in x^* \oplus y$, so:

$$f(0^*) \in f(x^* \oplus y) \Rightarrow f(0)^* \in f(x^* \oplus y) \Rightarrow 0^* \in f(x)^* \oplus f(y) \Rightarrow f(x) \ll f(y)$$

Lemma 5.3. *Let $f : X \rightarrow Y$ be a hyper homomorphism of hyper-MV-algebras. If $A \ll B$, then $f(A) \ll f(B)$.*

Proof. Let A, B be subsets of X such that $A \ll B$. If $y \in f(A)$, then there exists $a \in A$ such that $y = f(a)$ and since $A \ll B$, there exists $b \in B$ such that $a \ll b$.

From Lemma 5.2 we have $f(a) \ll f(b)$. Hence we get that $f(A) \ll f(B)$.

Definition 5.4. *Let f be a hyper endomorphism of M , and $B = (\mu^+, \mu^-)$ be a Bipolar-valued fuzzy set of M . Define a new Bipolar-valued fuzzy set*

$$B^f = (\mu^{+f}(x), \mu^{-f}(x)) = (\mu^+(f(x)), \mu^-(f(x))) \quad \forall x \in M$$

Definition 5.5. *A Bipolar-valued fuzzy sub-hyper MV-algebra $B = (\mu^+, \mu^-)$ of M is said to be Bipolar-valued fuzzy characteristic if:*

$$B^f = (\mu^{+f}(x), \mu^{-f}(x)) = (\mu^+(f(x)), \mu^-(f(x))) = B \quad \forall x \in M$$

Lemma 5.6. *Let f be a hyper endomorphism of M . If $B = (\mu^+, \mu^-)$ is a (BFSH) of M , then so is B^f .*

Proof. Since $\mu^{+f}(a) = \mu^+(a)$ and $\mu^{-f}(a) = \mu^-(a)$ for all $a \in M$, the proof is clear.

Proposition 5.7. *Let $B = (\mu^+, \mu^-)$ be a (BFSH) of M . Then the following are equivalent:*

- i) B is Bipolar-valued fuzzy characteristic.
- ii) Each (t, s) -cut of B is characteristic.

Proof. Assume that B is Bipolar-valued fuzzy characteristic and $(t, s) \in \text{Im}(\mu^+) \times \text{Im}(\mu^-)$. If $x \in M_B^{(t,s)}$, then:

$$\mu^+(f(x)) = \mu^{+f}(x) = \mu^+(x) \geq t \text{ \& } \mu^-(f(x)) = \mu^{-f}(x) = \mu^-(x) \leq s.$$

This shows that $f(x) \in M_B^{(t,s)}$ i.e. $f(M_B^{(t,s)}) \subseteq M_B^{(t,s)}$.

Now suppose that $x \in M_B^{(t,s)}$ and let $x = f(y)$; $y \in M$. Then:

$$\begin{aligned} \mu^+(y) &= \mu^{+f}(x) = \mu^+(f(x)) = \mu^+(x) \geq t \\ \mu^-(y) &= \mu^{-f}(x) = \mu^-(f(x)) = \mu^-(x) \leq s \end{aligned}$$

This shows that $y \in M_B^{(t,s)}$, hence $x \in f(M_B^{(t,s)})$. Therefore $M_B^{(t,s)} \subseteq f(M_B^{(t,s)})$, hence $M_B^{(t,s)} = f(M_B^{(t,s)})$.

Conversely, assume that each (t, s) -cut of B is characteristic and $x \in M$, $f \in \text{End}(M)$.

Define $t := \mu^+(x)$, $s := \mu^-(x)$. It follows from hypothesis that:

$$f(x) \in f(M_B^{(t,s)}) = M_B^{(t,s)}$$

So that:

$$\mu^{+f}(x) = \mu^+(f(x)) \geq t \text{ \& } \mu^{-f}(x) = \mu^-(f(x)) \leq s$$

Now let $u := \mu^{+f}(x)$ and $v := \mu^{-f}(x)$. Hence $u \geq t$ and $v \leq s$.

Assume that $u > t$ and $v < s$. Then

$$\mu^{+f}(x) = \mu^+(f(y)) = u \text{ \& } \mu^{-f}(x) = \mu^-(f(y)) = v$$

we have $f(x) \in M_B^{(u,v)} = f(M_B^{(u,v)})$ and this by injectivity of f implies that $x \in M_B^{(u,v)}$. By Proposition 4.2 this is a contradiction. Hence:

$$\mu^{+f}(x) = \mu^+(f(x)) = t = \mu^+(x) \text{ and } \mu^{-f}(x) = \mu^-(f(x)) = t = \mu^-(x).$$

And this shows that B is Bipolar-valued fuzzy characteristic.

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References

- [1] **A. Borumand Saeid**, Bipolar-valued fuzzy BCK/BCI -algebras, *World Applied Sci. J.* 7, **No.11**, (2009), 1404-1411.
- [2] **R. A. Borzooei, A. Hasankhani, M. M. Zahedi, and Y. B. Jun**, On hyper- K -algebras, **52(1)**, (2000), 113-121.
- [3] **Sh. Ghorbani, A. Hasankhani, and E. Eslami**, Hyper MV -Algebras, *Set-valued Math. and Appl.*, **1 No.2**, (2008), 205-222.
- [4] **Y. B. Jun, Min Su Kang, and Hee Sik Kim**, Fuzzy structures of hyper- MV -deductive systems in hyper- MV -algebras, *Computers and Math. with Appl.*, **59**, (2010), 2982-2989.
- [5] **K. M. Lee**, Bipolar-valued fuzzy sets and their operations, *Proc. Int. Conf. on Intelligent Technologies*, (2000), 307-312.
- [6] **K. M. Lee**, Comparison of interval-valued fuzzy sets, intuitionistic fuzzy sets, and bipolar-valued fuzzy sets, *J. Fuzzy Logic Intelligent Systems*, **14, No. 2**, (2004), 125-129.

- [7] **L. A. Zadeh**, Fuzzy sets, Inform. and Control, **8**, (1965), 338–353.

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