Restricted Oriented Slack Convexity and Extremal Elements with Respect to a Set

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Abstract. In a real linear space V , a set of directions $\mathcal{O} \subset V$ and a set of representative vectors $M \subset V$ generate a double restricted concept of convexity: with directional and with particular punctual restrictions. A Krein-Milman type property for slack $\mathcal{O}$ -directionally convex sets with respect to M is derived in terms of double restricted extremal elements of a set.

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1 Introduction

More domains of applied mathematics use procedures restrained to representative points of a set. For example, the integer programming is interested in points having integer coordinates, which means that $\mathbb{Z}^n$ represents $\mathbb{R}^n$ in this context [4], [12]. The rational programming restricts the procedures to $\mathbb{Q}^n$ instead $\mathbb{R}^n$ [12]. Image processing, image analysis and pattern recognition restricts the plane $\mathbb{R}^2$ to a set of grid points $\mathbb{Z}^2(h) = \{(ih, jh) | i \in \mathbb{Z}, j \in \mathbb{Z}\}$, with $h > 0$, [1], [4], [17]. In a set of pixels isomorphic with $\mathbb{Z}^2(h)$, many tests are performed working on particular directions [17], [15], [14], [16]. The convexity property is a very useful one in all these domains of research and

applications. Let us remind that, in a space V in which a concept of straight-line segment $\langle x, y \rangle$ joining any two elements $x, y \in V$ is defined, a subset $X \subseteq V$ is said to be convex if it contains the entire straight-line segment $\langle a, b \rangle$, whenever $a \in X$ and $b \in X$. There are many attempts to build models of convexity, either in digital framework [2], [7], [5], [9], or restricted to a set of representative points of a space [3], [11], [13], or restricting the orientation of straight-line segments [5], [15], [14], [16]. These directions of research, together with some practical applications [6] we faced with, are at the origin of our idea of double restricting the elements used in order to define the convexity: the elements of a set and the straight-line segments taken into account.

The framework of this paper is a real linear space V , in which two subsets are chosen: a subset of directions $\mathcal{O} \subseteq V$ and a subset of representative points $M \subseteq V$. The concept of slack $\mathcal{O}$ -directionally convex set with respect to M , introduced in [5], is studied from the point of view of structural and algebraic properties. A representation theorem of Krein-Milman type [10], in a weaker form, is deduced in the framework of restricted directional convexity with respect to a given set. So, the facial structure [18] of sets with this kind of convexity is described in terms of $\mathcal{O}$ -extremal points with respect to M .

2 Slack $\mathcal{O}$ -directional convexity with respect to a set

Let $(V, +, \cdot)$ be a real linear space and $M \subseteq V$ a nonempty set of elements called representative points of V . Let $\mathcal{O} \subseteq V$ a set of vectors in V , called directions or orientations in V .

Remark 2.1. *If the vectors of $\mathcal{O}$ are supposed to be linearly independent then the directions of the space are distinct. If its cardinality $|\mathcal{O}| = \dim V$, then $\mathcal{O}$ contains the directions of a basis in V .*

Remark 2.2. *If $\dim V = n$ and $\mathcal{O}$ is an orthogonal basis in V , then $\mathcal{O}$ contains a set of orthogonal directions in space. This is generally used in computational geometry, where the horizontal and vertical directions are preferred in various computations and testing procedures (see [17]).*

Definition 2.1. *A nonempty subset $X \subseteq V$ is said to be slackly $\mathcal{O}$ -directionally convex with respect to M if*

$$\{tx + (1 - t)y | t \in [0, 1]\} \cap M \subseteq X, \quad (2.1)$$

whenever $x, y \in X \cap M$ such that there are $\alpha \in \mathbb{R}$ and $e \in \mathcal{O}$ having the property that $x - y = \alpha e$.

The symbol $\mathcal{C}_s(\mathcal{O}; M)$ is used, in the following, to denote the collection of all the subsets of V , which are slackly $\mathcal{O}$ -directionally convex with respect to M . Let us remind that, according to [8], a pair $(S, \mathcal{K})$ is said to be a convexity space if S is a nonempty set and $\mathcal{K}$ is a collection of subsets of S , which contains $\emptyset$ and S and is closed for arbitrary intersection. The collection $\mathcal{K}$ is called a convexity in S .

Theorem 2.1. *The pair $(V; \mathcal{C}_s(\mathcal{O}; M))$ is a convexity space.*

Proof. Obviously, $\emptyset \in \mathcal{C}_s(\mathcal{O}; M)$ and $V \in \mathcal{C}_s(\mathcal{O}; M)$. Let us suppose that $\mathcal{F} \subset \mathcal{C}_s(\mathcal{O}; M)$. Denote by $F = \bigcap_{A \in \mathcal{F}} A$ and we shall prove that F is slackly $\mathcal{O}$ -directionally convex with respect to M . Let $x \in F \cap M$ and $y \in F \cap M$ such that there are $k \in \mathbb{R}$ and $e \in \mathcal{O}$ such that $x - y = ke$. Then, $x \in A \cap M$ and $y \in A \cap M$, whenever $A \in \mathcal{F}$. Since A is slackly $\mathcal{O}$ -directionally convex with respect to M it follows that

$$\{tx + (1 - t)y | t \in [0, 1]\} \cap M \subseteq A,$$

this inclusion being true for any $A \in \mathcal{F}$. As consequence,

$$\{tx + (1 - t)y | t \in [0, 1]\} \cap M \subseteq \bigcap_{A \in \mathcal{F}} A = F,$$

which means that F is slackly $\mathcal{O}$ -directionally convex with respect to M . So, $\mathcal{C}_s(\mathcal{O}; M)$ is a convexity in V .

Q.E.D.

Example 2.1. Let $V = \mathbb{R}^2$, $M = \mathbb{Z}^2$, $\mathcal{O} = B_2 = \{e_1, e_2\}$, with $e_1 = (1, 0)$ and $e_2 = (0, 1)$. The set $X = ([1, 3] \times \{1\}) \cup (\{1\} \times [1, 2])$ is slackly B_2 -directionally convex with respect to $\mathbb{Z}^2$. It is also integer convex according to the definition of Lupşa [12]. Also, it is ortho-convex in the sense defined by Rawlins and Wood [15], meaning that the straight-line segments with endpoints in X lie in X whenever they are either horizontal or vertical.

Example 2.2. Let $V = \mathbb{R}^2$, $M = \mathbb{Z}^2$, $\mathcal{O} = B_2 = \{e_1, e_2\}$, with $e_1 = (1, 0)$ and $e_2 = (0, 1)$. The set $X = [1, 3] \times [1, 3] \setminus \{(2, 2)\}$ is not slackly B_2 -directionally convex with respect to $\mathbb{Z}^2$. Indeed, let $x = (1, 2) \in X$, $y = (3, 2) \in X$,

$$\langle x, y \rangle = \{tx + (1 - t)y | t \in [0, 1]\} = [1, 3] \times \{2\},$$

$$\langle x, y \rangle \cap M = \{(1, 2), (2, 2), (3, 2)\} \not\subseteq X$$

since $(2, 2) \notin X$.

Example 2.3. Let $V = \mathbb{R}^2$, $M = \mathbb{Z}^2$, $\mathcal{O} = B_2 = \{e_1, e_2\}$, with $e_1 = (1, 0)$ and $e_2 = (0, 1)$. The set $X = \{(0, 0), (0, 1), (0, 2), (1, 0), (2, 0), (\frac{3}{2}, 2)\}$ is slackly B_2 -directionally convex with respect to $\mathbb{Z}^2$. Let us remark that the convexity requirement that the straight line segment having the endpoints in X intersects $\mathbb{Z}^2$ in points belonging to X is not valid. Indeed, choosing the points $x = (0, 2)$ and $y = (\frac{3}{2}, 2)$ then $\langle x, y \rangle \cap M = \{(0, 2), (1, 2)\} \not\subset X$ since $(1, 2) \notin X$. As consequence, eliminating the condition $x, y \in X \cap M$ leads us to a different concept, stronger than that from definition 2.1.

Property 2.2. Let $\mathcal{O}_1 \subseteq V$, $\mathcal{O}_2 \subseteq V$ and $\mathcal{O}_1 \subset \mathcal{O}_2$. Then

$$\mathcal{C}_s(\mathcal{O}_2; M) \subset \mathcal{C}_s(\mathcal{O}_1; M).$$

Proof. If X is slackly $\mathcal{O}_2$ -directionally convex with respect to M then for any $x, y \in X \cap M$, such that there are $e \in \mathcal{O}_2$ and $\alpha \in \mathbb{R}$ satisfying $x - y = \alpha e$, $\langle x, y \rangle \cap M \subset X$. Let $x, y \in X \cap M$ and let us suppose that there is $f \in \mathcal{O}_1$ such that $x - y = \beta f$, for a number $\beta \in \mathbb{R}$. Since $\mathcal{O}_1 \subset \mathcal{O}_2$ it follows that $f \in \mathcal{O}_2$. So, due to the slack $\mathcal{O}_2$ -directional convexity of X with respect to M it follows that $\langle x, y \rangle \cap M \subset X$, which proves the slack $\mathcal{O}_1$ -directional convexity of X with respect to M . Q.E.D.

Property 2.3. Let $\mathcal{O} \subseteq V$, $M_1 \subseteq V$, $M_2 \subseteq V$ and $M_1 \subset M_2$. Then

$$\mathcal{C}_s(\mathcal{O}; M_2) \subset \mathcal{C}_s(\mathcal{O}; M_1).$$

Proof. If X is slackly $\mathcal{O}$ -directionally convex with respect to M_2 then for any $x, y \in X \cap M_2$, such that there are $e \in \mathcal{O}$ and $\alpha \in \mathbb{R}$ satisfying $x - y = \alpha e$, $\langle x, y \rangle \cap M_2 \subset X$. Let $x, y \in X \cap M_2$ and let us suppose that there is $f \in \mathcal{O}$ such that $x - y = \beta f$, for a number $\beta \in \mathbb{R}$. Since $M_1 \subset M_2$ and due to the slack $\mathcal{O}$ -directional convexity of X with respect to M_2 it follows that

$$\langle x, y \rangle \cap M_1 \subset \langle x, y \rangle \cap M_2 \subset X,$$

which proves the slack $\mathcal{O}$ -directional convexity of X with respect to M_1 . Q.E.D.

Let $\alpha \in \mathbb{R}^*$. Like in [4] we say that a set $M \subseteq V$ is an α -set if $\frac{1}{\alpha}x \in M$ for any $x \in M$.

Example 2.4. For example, every cone in a vectorial space is an α -set, whenever $\alpha \in \mathbb{R}_+^*$.

Example 2.5. The set $\mathbb{Z}$ of all integer numbers is a (-1) -set.

Example 2.6. The set $\mathbb{Q}$ of all rational numbers is an α -set, whenever $\alpha \in \mathbb{Q}^*$.

Property 2.4. *If $M \subseteq V$ is a nonempty α -set and $X \subseteq V$ is slackly $\mathcal{O}$ -directionally convex with respect to M then αX is slackly $\mathcal{O}$ -directionally convex with respect to M .*

Proof. Let us suppose that X is slackly $\mathcal{O}$ -directionally convex with respect to M . Let $x \in \alpha X \cap M$ and $y \in \alpha X \cap M$ such that $x - y = ke$ for a number $k \in \mathbb{R}$ and a vector $e \in \mathcal{O}$. Then there are $u \in X$ and $v \in X$ such that $x = \alpha u$ and $y = \alpha v$. Then one has

$$u - v = \frac{1}{\alpha}(x - y) = \frac{k}{\alpha}e$$

and, from the slack $\mathcal{O}$ -directional convexity of X with respect to M it follows that

$$\{tu + (1 - t)v | t \in [0, 1]\} \cap M \subseteq X.$$

Let us suppose that for a number $t_1 \in [0, 1]$, one has $t_1u + (1 - t_1)v \in M$. Since M is an α -set, it follows that $\alpha[t_1u + (1 - t_1)v] \in M$. But

$$\alpha[t_1u + (1 - t_1)v] = t_1(\alpha u) + (1 - t_1)\alpha v = t_1x + (1 - t_1)y \in M.$$

Since X is slackly $\mathcal{O}$ -directionally convex with respect to M one gets $t_1u + (1 - t_1)v \in X$. Then

$$\alpha[t_1u + (1 - t_1)v] = t_1x + (1 - t_1)y \in \alpha X.$$

Therefore, αX is slackly $\mathcal{O}$ -directionally convex with respect to M . Q.E.D.

3 Slack $\mathcal{O}$ -directional convex hull with respect to M

As it is well known from the classical theory of convexity ([7] and [18]) the convex hull of a set $S \subset V$ is given, using the outer characterization, by the intersection of all the convex sets including S . The inner geometric approach describes the convex hull of S as the union of all the straight-line segments joining points of S . Similarly one may introduce the slack $\mathcal{O}$ -directionally convex hull with respect to M of a set $S \subset V$.

Definition 3.1. If $S \subset V$ then the set

$$\mathcal{O}_M^{(s)}\text{conv}(S) = \bigcap \{P \subset V \mid S \subseteq P \text{ and } P \in \mathcal{C}_s(\mathcal{O}; M)\} \quad (3.1)$$

is called the slack $\mathcal{O}$ -directionally convex hull with respect to M of S .

If $A \subset V$, let us denote by $A|_{\mathcal{O}}$ the set of all those points $x \in A$ that have the property that there is at least one point $y \in A$ such that $y - x = ke$ for some $k \in \mathbb{R}$ and $e \in \mathcal{O}$. It means that each point of $A|_{\mathcal{O}}$ is an extremity of a straight-line segment lying in A and having a direction from $\mathcal{O}$.

Theorem 3.1. A set $X \subseteq V$ is slackly $\mathcal{O}$ -directionally convex with respect to M if and only if

$$M \cap \mathcal{O}_M^{(s)}\text{conv}(X) = M \cap X. \quad (3.2)$$

Proof . Necessity. Let us suppose that X is slackly $\mathcal{O}$ -directionally convex with respect to M . From definition 2.1 one gets that

$$M \cap \mathcal{O}_M^{(s)}\text{conv}(X) \subseteq X.$$

But $M \cap \mathcal{O}_M^{(s)}\text{conv}(X) \subseteq M$, so it follows that

$$M \cap \mathcal{O}_M^{(s)}\text{conv}(X) \subseteq M \cap X. \quad (3.3)$$

Since $X \cap M \subseteq \mathcal{O}_M^{(s)}\text{conv}(X)$, one has

$$M \cap X \subseteq M \cap \mathcal{O}_M^{(s)}\text{conv}(X). \quad (3.4)$$

From (3.3) and (3.4) one gets

$$M \cap X = M \cap \mathcal{O}_M^{(s)}\text{conv}(X).$$

Sufficiency Let $X \cap M \subseteq \mathcal{O}_M^{(s)}\text{conv}(X)$. It follows that $X \cap M$ is slackly $\mathcal{O}$ -directionally convex with respect to M . Then, from definition 2.1, one gets that

$$M \cap [\text{conv}(M \cap X)]|_{\mathcal{O}} \subseteq X \cap M.$$

This means that

$$M \cap [\text{conv}(M \cap X)]|_{\mathcal{O}} \subseteq X \cap M \subseteq X,$$

therefore X is slackly $\mathcal{O}$ -directionally convex with respect to M . Q.E.D

4 Extremal elements of slack $\mathcal{O}$ -directionally convex sets with respect to M

Let V be a vectorial space, $\mathcal{O} \subseteq V$ a set of directions in V and a set $M \subseteq V$, $\mathcal{O} \neq \emptyset$, $M \neq \emptyset$.

Definition 4.1. An element $x \in X \cap M$ is said to be an $\mathcal{O}$ -extremal point of X with respect to M if there are no two distinct points $y_1, y_2 \in (X \setminus \{x\}) \cap M$, with $y_1 - y_2 = \lambda e$ for a number $\lambda \in \mathbb{R}$ and a direction $e \in \mathcal{O}$, and there is no number $t \in [0, 1]$ such that the following conditions simultaneously hold:

$$ty_1 + (1 - t)y_2 \in M \quad (4.1)$$

$$ty_1 + (1 - t)y_2 = x \quad (4.2)$$

Corollary 4.1. An element $x \in X \cap M$ is an $\mathcal{O}$ -extremal point of X with respect to M if

$$x \notin M \cap [\text{conv}((X \setminus \{x\}) \cap M)]|_{\mathcal{O}}. \quad (4.3)$$

Let us denote by $\mathcal{O}_M \text{Err}(X)$ the set of all $\mathcal{O}$ -extremal points of the set X with respect to M .

Example 4.1. If $V = \mathbb{R}$, $M = Q$, $\mathcal{O} = \{1\}$ and $X = \{x \in Q \mid -\sqrt{2} \leq x \leq \sqrt{2}\}$ then it is obvious that $\mathcal{O}_M \text{Err}(X) = \emptyset$.

Example 4.2. If $V = \mathbb{R}$, $M = Q$, $\mathcal{O} = \{1\}$ and $X = \{0, 1, 3\}$ then $\mathcal{O}_M \text{Err}(X) = \{0, 3\}$.

Example 4.3. If $V = \mathbb{R}^2$, $M = \mathbb{Z}^2$, $\mathcal{O} = \{(0, 1), (1, 0), (1, 1), (-1, 1)\}$ and $X = \{(0, 2), (-1, 1), (1, 0), (-0, 1)\}$ then $\mathcal{O}_M \text{Err}(X) = X$.

Theorem 4.2. If $X \subseteq V$, $\mathcal{O} \subseteq V$, $M \subseteq V$, $M \neq \emptyset$, and the set $\mathcal{O} \neq \emptyset$, X is slackly $\mathcal{O}$ -directionally convex with respect to M then the following sentences are equivalent:

1. $x \in \mathcal{O}_M \text{Err}(X)$;
2. if there are two elements $y_1, y_2 \in X \cap M$, with $y_1 - y_2 = \lambda e$, for a $\lambda \in \mathbb{R}$ and $e \in \mathcal{O}$, such that

$$x \in M \cap \mathcal{O}_M^s \text{conv}\{y_1, y_2\},$$

then it follows that $y_1 = y_2 = x$;

3. the set $X \setminus \{x\}$ is slackly $\mathcal{O}$ -directionally convex with respect to M .

Proof . 1) $\Rightarrow$ 2) Let us suppose that sentence 1) is true and sentence 2) is false. Then there are two distinct elements $y_1, y_2 \in X \cap M$, situated on one of the directions from $\mathcal{O}$ and there is a number $t \in [0, 1]$ such that

$$x = ty_1 + (1 - t)y_2.$$

- If $y_1 = y_2 = x$ then 2) is true.
- Otherwise, it is possible that one of the points y_1 or y_2 coincides with x . Let us suppose that $y_1 = x$ and $y_2 \neq x$. Then

$$(1 - t)x = (1 - t)y_2$$

and dividing by $(1 - t)$ one gets $y_2 = x$, which is a contradiction. So, sentence 2) is true.

- It is possible that both $y_1 \neq x$ and $y_2 \neq x$. In this case, let us chose $z = x$. One gets the elements y_1 and z , with $y_1 \neq x, z = x$ and there is $t \in [0, 1]$ such that $tz + (1 - t)y_1 = x$, which reduces to the previous one. Analogously, if one chooses $z = x$ and $y_2 \neq x$.

As consequence, proposition 2) is true.

2) $\Rightarrow$ 3) Let us suppose that proposition 2) is true and 3) is false. Then there are $y_1, y_2 \in (X \setminus \{x\}) \cap M$, situated on one of the directions from $\mathcal{O}$ and there is a number $t \in [0, 1]$ such that

$$u = ty_1 + (1 - t)y_2 \in M$$

and $u \notin X \setminus \{x\}$. From the property of slackly $\mathcal{O}$ -directional convexity with respect to M of set X it follows that $u \in X$. As consequence, from 2) one deduces that $u = x = y_1 = y_2$, which represents a contradiction with the condition $y_1, y_2 \in X \setminus \{x\}$. Therefore, 3) is true.

3) $\Rightarrow$ 1) If the set $X \setminus \{x\}$ is slackly $\mathcal{O}$ -directionally convex with respect to M then the element x satisfies the conditions of an $\mathcal{O}$ -extremal point of X with respect to M , in an obvious manner. Q.E.D

Let us consider a subset $S \subset V$. The set $(\mathcal{O}, M) - Core(S) = \{\bigcup \{< x, y > | x \in S \cap M, y \in S \cap M, \exists k \in \mathbb{R}, \exists e \in \mathcal{O}, y - x = ke\}\} \cap S$ is called the slack $\mathcal{O}$ -directional core of S with respect to M .

Lemma 4.3. *If $X \subseteq V$, $\mathcal{O} \subseteq V$, $M \subseteq V$, and $\mathcal{O} \neq \emptyset$ and $M \neq \emptyset$, than*

$$\mathcal{O}_M \text{Ext}(X) \supset \mathcal{O}_M \text{Ext}((\mathcal{O}, M) - core(X)).$$

Proof . Let $x \in \mathcal{O}_M \text{Err}((\mathcal{O}, M) - \text{core}(X))$. It means that there are not two points $y_1, y_2 \in (X \setminus \{x\}) \cap M$ such as there are $\lambda \in \mathbb{R}$ and $e \in \mathcal{O}$ with $y_1 - y_2 = \lambda e$ and there is not $t \in [0, 1]$ such as simultaneously $ty_1 + (1-t)y_2 \in M$ and $ty_1 + (1-t)y_2 = x$. It means that $x \in \mathcal{O}_M \text{Err}(X)$. Q.E.D

Remark 4.1. *The converse inclusion does not hold, since every element of $X \cap M$ that does not belong to $(\mathcal{O}, M) - \text{core}(X)$ is an $\mathcal{O}$ -extremal point with respect to M of X .*

Theorem 4.4. *If $X \subset V$ is slackly $\mathcal{O}$ -directionally convex with respect to M then*

$$X \cap M = \{\mathcal{O}_M^s \text{conv}(\mathcal{O}_M \text{Err}(X))\} \cap M.$$

Proof . Let us suppose that $x \in X \cap M$. Then two cases are possible:

- $x \in (\mathcal{O}, M) - \text{core}(X)$, which means that $x \in \{\mathcal{O}_M^s \text{conv}(\mathcal{O}_M \text{Err}((\mathcal{O}, M) - \text{core}(X)))\} \cap M \subset \{\mathcal{O}_M^s \text{conv}(\mathcal{O}_M \text{Err}(X))\} \cap M$, according to lemma 4.3.
- $x \notin (\mathcal{O}, M) - \text{core}(X)$, which means that x is an $\mathcal{O}$ -extremal point of X with respect to M and then $x \in \{\mathcal{O}_M^s \text{conv}(\mathcal{O}_M \text{Err}(X))\} \cap M$. It follows that $x \cap M \subseteq \{\mathcal{O}_M^s \text{conv}(\mathcal{O}_M \text{Err}(X))\} \cap M$.

Conversely, let us suppose that $x \in \{\mathcal{O}_M^s \text{conv}(\mathcal{O}_M \text{Err}(X))\} \cap M$. It follows that there are two elements $y_1, y_2 \in \mathcal{O}_M \text{Err}(X)$ and $z_1, z_2 \in \mathcal{O}_M \text{Err}(X)$ and two elements $a_1 \in \langle z_1, z_2 \rangle \cap M$ and $a_2 \in \langle z_1, z_2 \rangle \cap M$ such as $x \in \langle a_1, a_2 \rangle \cap M$ (more similar steps may be necessary). But X is slackly $\mathcal{O}$ -directionally convex with respect to M , implying that $a_1 \in X \cap M$, $a_2 \in X \cap M$ and, as consequence, $x \in X \cap M$. It implies that $\{\mathcal{O}_M^s \text{conv}(\mathcal{O}_M \text{Err}(X))\} \cap M \subseteq X \cap M$. The theorem is completely proved. Q.E.D

Remark 4.2. *Theorem 4.4 is a weaker result than the classical Krein-Milman theorem [10]. It is the best possible result of Krein-Milman type in the framework of the slack $\mathcal{O}$ -directional convexity with respect to a given set M .*

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